A fixed point theorem for the hyperspace of a snake-like continuum

by

J. Segal (Seattle, Wash.)

Introduction. If $X$ is a metric continuum, $C(X)$ denotes the space of subcontinua of $X$ with the finite topology. As a partial answer to question 186 (due to B. Knaster 4/29/52) of the New Scottish Book it is shown that $C(X)$ has fixed point property if $X$ is a snake-like continuum. This is done by showing that $C(X)$ is a quasi-complex and since $C(X)$ is acyclic (see [9]) it has fixed point property by the Lefschetz Fixed Point Theorem.

Definition 1. If $G$ is a finite collection of open sets of $X$ let $\Omega(G)$ denote $\{K \in C(X) \mid K \supseteq G \}$ for each $G \subset \mathcal{O}(G)$. The finite topology on $C(X)$ is the one generated by open sets of the form $\Omega(G)$. (See [8], pp. 133.) If $U$ is a finite open covering of $X$ define $U^*$ to be $\Omega(G)$ where $G$ is a finite subset of $U$.

Lemma 1. If $U$ is a finite open covering of $X$, then $U^*$ is a finite open covering of $C(X)$.

Proof. The elements of $U^*$ are open by the definition of the finite topology, and since $U$ is finite, $U^*$ is finite. If $A \in C(X)$, there is a subcollection $G$ of $U$ which irreducibly covers $A$, so $A \in \Omega(G)$. Hence $U^*$ covers $C(X)$.

Lemma 2. If $U$ is a finite collection of open sets, then $\text{mesh } U^* \leq \text{mesh } U$.

Proof. Suppose that $G$ is a subcollection of $U$ and $X$ and $L$ are elements of $\Omega(G)$. If $x \in K$, there is an element $y_0$ of $G$ containing $x$. Given $L \supseteq G$ and $\text{diam} \supseteq \text{mesh } U$, there is a point $y$ of $L$ such that $d(x, L) \leq \text{mesh } U$. Hence for each $x \in K$, $d(x, L) \leq \text{mesh } U$. Therefore since $d^*(K, L) = \max \{ \text{diam}(x, L) \}$ and $d^*(K, L) \leq \text{mesh } U$, and hence $\text{diam} \supseteq \text{mesh } U$.

Lemma 3. If $(T_n)$ is a cofinal sequence of open coverings of $X$, then $(U^*_n)$ is a cofinal sequence of open coverings of $C(X)$.

Proof. A sequence $(U_n)$ of open coverings of a compact space $X$ is cofinal (in the set of all open coverings of $X$) if and only if $\text{mesh } U_n \to 0$. By Lemma 2 if mesh $U_n \to 0$ then mesh $U^*_n \to 0$. 

References

A fill point theorem

If \( V(i, j) \in V \), then \( i \leq i_{\text{max}} \) and \( j \geq j_{\text{min}} \) so \( U_{i_{\text{max}}} \cup \ldots \cup U_{j_{\text{min}}} \subseteq U_i \cup \ldots \cup U_j \) and hence \( K \subseteq U_i \cup \ldots \cup U_j \). Also, \( i \geq i_{\text{min}} \) and \( j \leq j_{\text{max}} \) so the sequence \( U_{i_{\text{min}}} \ldots \cup U_{i_{\text{max}}} \) intersects each of \( U_i, \ldots, U_j \). Hence \( K \subseteq \Omega(U(i, j)) \subseteq V(i, j) \).

(b) Suppose that \( V \) contains two elements \( V(i_1, j_1) \) and \( V(i_2, j_2) \) which are not \( A \)-related. Then either \( |i_1 - i_2| > 2 \) or \( |j_1 - j_2| > 2 \). If \( i_1 < i_2 < 2 \) then \( U_i \cup \ldots \cup U_j \) does not intersect \( U_k \) and hence no continuum lying in \( U_i \cup \ldots \cup U_j \) can intersect \( U_k \); consequently, no element of \( V(i_1, j_1) \) belongs to \( V(i_2, j_2) \). The other cases are similar.

(c) Suppose \( V \subseteq U^* \) and \( \bigcap V \neq \emptyset \). Let \( \delta_i = \min\{\delta \} \) for some \( j \), \( V(i_1, j_1) \cap V \) and \( j_0 = \min\{\delta \} \) for \( V(k_0, j_0) \cap V \). Then if \( V(i, j) \cap V \), then \( i \geq \delta_i \) or \( i + 1 \) and \( j = j_0 \) for \( j_0 + 1 \). The only possible elements of \( V \) are \( V(i_0, j_0) \), \( V(i_1, j_1) \), \( V(i_2, j_1 + 1) \), \( V(k_0, j_0 + 1) \) and \( V(k_0, j_0 + j_1) \).

DEFINITION 3. The nerve of a finite collection \( U \) of sets (denoted by \( N(U) \)) is an abstract complex \( G \) whose vertices are in 1-1 correspondence with the elements of \( U \) and which is such that a subset of the vertices of \( G \) is the set of vertices of a simplex of \( G \) if and only if the intersection of the corresponding elements of \( U \) is non-empty.

Remark. Let \( R \) denote the set of all lattice points of the plane lying in the region bounded by the lines \( x = 1, y = x, y = n \); two points of \( R \) will be said to be \( A \)-related if neither of them nor their absciss differ by more than 1.

If \( U \) is a chain covering \( X \) with \( n \) elements, a 1-1 correspondence between the elements of \( U^* \) and the points of \( R \) is obtained by letting the element \( V(i, j) \) of \( U^* \) correspond to the point \( (i, j) \) of \( R \). Hence \( R \) may be considered as the set of vertices \( \{a\} \) of \( N(U^*) \).

A subset \( K \) of \( R \) has the property that every two of its elements are \( A \)-related if and only if \( K \) is a subset of the vertices of a unit square in the plane; hence a "topological realization" of \( N(U) \) can be obtained by adjoining to \( R \) the solid triangle bounded by the lines \( x = 1, y = x, y = n \) and \( y = n \) together with a collection \( \mathcal{C} \) of "topological tetrahedrons" (i.e., closed 3-cells) given that each element of \( K \) contains a solid unit square with vertices in \( R \) and every such square is contained in an element of \( G \), and such that no two elements of \( G \) have a point in common not in the \( xy \)-plane.

The following is a special case of the Mayer-Vietoris Theorem.

LEMMA 5. ([3], p. 39) If \( X, A, B \) are simplicial complexes such that \( X = A \cup B \) and \( A, B, A \cap B \) are acyclic, then \( K \) is acyclic.

THEOREM 1. \( N(U^*) \) is acyclic.

Proof. If \( U \) has one element if \( N(U) \) is acyclic being just one vertex. Assume that the theorem is true for \( V \) which has \( k \) elements. Suppose

**Fundamenta Mathematicae, T. L. (1984)**
that $U$ has $k+1$ elements. Then $N(U^*) = N(V^*) \cup M$ where $M$ is the simplicial complex composed of those simplices added to $N(V^*)$ to obtain $N(U^*)$. So $M$ is composed of $k-1$ tetrahedrons and one triangle each joined to the next along one edge. So $|M|$ is contractible and hence $M$ is acyclic. $|N(U^*) \cap M|$ is the union of $k-1$ segments and is homeomorphic to an arc and so $N(U^*) \cap M$ is acyclic. Applying Lemma 5 we have $N(U^*)$ is acyclic.

**Definition of a quasi-complex.** The following definition is in [6], p. 322.

Let $X$ be a compact space, $\mathcal{U} = \{U_i\}$ a cofinal set of open coverings of $X$. For each pair of $i, j \in M$ such that $j > i$ let $\pi_{ij} : N(U_i) \rightarrow N(U_j)$ be one of the projections induced by the inclusion relations associated with the refinement of $U_i$ by $U_j$.

Further for each $j \in M$ there exists an $i \in M$ and one or more chain mappings $o_{ij} : N(U_j) \rightarrow N(U_i)$, called antiprojections, such that

1. $o_{ij} \circ \pi_{ij} \sim 1$,
2. if $o_{ij}$, $o_{ij}$ are antiprojections, then so is $o_{ij} \circ o_{ij}$,
3. if $o_{ij}$ and $o_{ij}$ are antiprojections, then $o_{ij} \sim o_{ij}$.

If $\alpha$ is a simplex of $N(U_i)$, let $\alpha_{ij}$ denote the kernel of $\alpha$, that is, the intersection of the sets of $U_j$ corresponding to vertices of $\alpha$. Further, if $\sigma$ is a chain of $N(U_i)$, let $|\sigma|$ denote the union of the kernels of simplices in the carrier of $\sigma$.

(d) all indices are understood in $M$, for every $i$ there is a $j, j > i$, and for every $k$ an $m, m > k, j$ (depending on $i$ and $j$) such that $o_{i, j}$ satisfies (a), (b), (c), and if the simplex $o_{ij} \circ (N(U_j))$, then $\alpha_{ij} \circ \sigma$ is contained in a set of $U_i$.

The collection $(X; \{U_i\}; (\pi_{ij}); (o_{ij}))$ defines a quasi-complex $X$.

**Definition of $\omega$.** The following definition of $\omega$ is in [2], p. 666. If $a$ and $b$ are arc-like finite simplicial complexes and $\pi$ is a simplicial mapping of $b$ onto $a$, there exists a chain mapping of $\pi$ onto $\beta$ which is defined as follows. Let $a_1, a_2, \ldots, a_n$ denote the vertices of $a$ ordered as on $a$. There is a subarc $\beta$ of $\beta$ such that $\pi(\beta') = a$ and there is no proper subarc $\gamma$ of $\beta'$ such that $\pi(\gamma) = a$. Let $b_1$ denote the vertex of $\beta'$ such that $\pi(b_1) = a_1$ and let $b_2, b_3, \ldots, b_n$ denote the vertices of $\beta'$ ordered as on $\beta'$. There is a subsequence $b_1, b_2, b_3, \ldots, b_n$ of $b_1, b_2, b_3, \ldots, b_n$ such that

1. $\pi(b_1) = a_1$ and $\pi(b_n) = a_n$,
2. if $\pi(b_1) = a_1$ and $\pi(b_n) = a_n$, then $|p - q| \leq 1$; and
3. for each $i, k$, $|k|$, the greatest integer $j$ such that
   a. $k_i < j < k_{i+1}$
   b. if $k_i < j < k_{i+1}$, $\pi(b_{k_i}) = \pi(b_{k_{i+1}})$.

Define $\omega(1 \cdot a)_{\pi_{ij}}$ to be $\sum_{i=1}^{n} X^i b_i$ where $X^i = 0$ if $\pi(b_i) \neq \pi(a)$, and

\[
\begin{align*}
\omega(1 \cdot a)_{\pi_{ij}} &= \pi(a_{k_{i+1}}) = a_{k_{i+1}}, \text{ then } X^i = +1, \\
\omega(1 \cdot a)_{\pi_{ij}} &= \pi(a_{k_i}) = a_{k_i}, \text{ then } X^i = -1, \\
\end{align*}
\]

Define $\omega(1 \cdot (a_{\pi_{ij}}))$ to be $\sum_{i=1}^{n} X^i b_i$, where $X^i = 0$ if $k_i < j < k_{i+1}$, and for all $k_i, X^i = +1$; otherwise $X^i = 0$.

**Definition of $\alpha_{ij}$.** Let $o_{ij}$ denote the chain mapping of $N(U_i)$ onto $N(U_j)$ defined for $\pi_{ij}$ in the preceding definition. $o_{ij}$ and finite products $o_{ij} \circ o_{ij} \circ o_{ij}$, where $i_1 < i_2 < \cdots < i_n$, are antiprojections and satisfy (a), (b), (c), and (d) (see Definition of quasi-complex). Moreover, $o_{ij}$ is an algebraic map.

**Definition of $\pi_{ij}$.** Since $X$ is a snake-like continuum, we have by [2], p. 666-667 that $(X; \{U_i\}; (\pi_{ij}); (o_{ij}))$ defines a quasi-complex $X$. For simplicity we now write $o_{ij}$ as $o_{ij}$ and use $i, j$ as indexes on the coverings. We define an extension of $\pi_{ij}, \alpha_{ij} : C^p[N(U_i)] \rightarrow C^p[N(U_j)]$. For each member of $U_i$ we select a member of $U_j$ containing it. This gives a simplicial mapping of $N(U_i)$ in $N(U_j)$ which is a projection. First we specify a particular projection $\pi_{ij}$ as follows. Let $\psi$ be the subscript of the vertex in $N(U_j)$ which is the image of $b_i$ under $\pi_{ij}$, i.e. $\pi_{ij}(b_i) = a_{\psi}$.

Let $\eta(q, s) = \max \{\psi(q) \mid q \leq \xi \leq s\}$ and $\mu(q, s) = \max \{q(q) \mid q \leq \xi \leq s\}$. Now we define a simplicial set transformation $\pi_{ij} : N(U_i) \rightarrow N(U_j)$ by

\[
\pi_{ij}(b_1) = a_{\eta(q, s)} \quad \text{and} \quad \pi_{ij}(b_n) = a_{\mu(q, s)},
\]

If $K \cap \bigcup_{q \leq q} U_j$ is connected in $N(U_j)$ and $K \cap U_j = \emptyset$ for $q \leq \xi \leq s$.

Since $\pi_{ij}(b_i) = o_{ij} b_i \subset U_j^{(\xi)}$ for each $\xi$. Hence $K \cap \bigcup_{q \leq q} U_j^{(\xi)} \subset \bigcup_{q \leq q} U_j^{(\xi)}$, and $\psi(U_j^{(\xi)}, U_j^{(\xi)}) \subset \bigcup_{q \leq q} U_j^{(\xi)}$, and $U_j^{(\xi)}$. Therefore $\pi_{ij}$ is induced by one of the inclusion relations associated with the refinement of $U_j$ by $U_i$. $\pi_{ij}$ induces a chain mapping $\pi_{ij}$ on $N(U_i)$, i.e. for a simplex $b_1^{(\xi)}, b_n^{(\xi)}$, we have

\[
\pi_{ij}(b_1^{(\xi)}, b_n^{(\xi)}) = \begin{cases} a_{\mu(q, s)}, \ldots, a_{\mu(q, s)}, & \text{if the } b_i^{(\xi)} \text{'s are distinct} \\ 0 & \text{otherwise} \end{cases}
\]

So $\pi_{ij}$ is actually an algebraic map of $N(U_i)$ onto $N(U_j)$ which is an extension of $\pi_{ij}$ and the carrier of $\pi_{ij}$ is $\pi_{ij}$ (see [6], p. 146, (9.13)).

**Definition 4.** If $v$ is a vertex of a simplicial complex $K$, then the $\overline{S}_V(b)$ is the subcomplex of $K$ consisting of all simplices having $v$ as a vertex and all faces of such simplices.
Definition 5. If $U = (U^i)$ is a finite covering of a continuum $X$ and $v_i$ is the vertex associated with $U^i$ in $N(U)$, then the $S_k(U^i) = \bigcup \{ \delta U^i : U^i \subset U \}$ and $v_i \in S_k(U^i)$.

Notation. In the following $K = K(U)$, $L = L(U)$, $K' = K(U_1)$, and $L' = L(U_1)$, where $U_1$ is finer than $U_2$ so that $\pi_{U_2}: C^k(N(U_2)) \to C^k(N(U))$ and $\omega_{U_1}: C^k(N(U_1)) \to C^k(N(U_2))$.

DEFINITION of $c^\alpha$. Define $c$ mapping simplexes of $L$ onto simplexes of $L'$ by $c(a) = \bigcup (S_k(h_0) \mid I(p) \leq q \leq R(p))$ where $I(p) = \min \{ k \mid \pi(b_2) = a_0 \}$, $R(p) = \max \{ k \mid \pi(b_2) = a_0 \}$, the $S_k$ is taken in $L'$ and the $a_0$s are vertices of $L$ and the $b_0$s are vertices of $L'$. Also $c(a_{p,p+1}) = c(a_{p}) \cup c(a_{p+1})$.

Lemma 6. $c$ is a carrier of $a$.

Proof. $c(a) = \bigcup \{S_k(h_0) \mid I(p) \leq q \leq R(p)\} \cap \bigcap_{i=1}^{k} X_i b_i = c(a_{p})$, since $X_i b_i = 0$ if $\pi(b_2) \neq a_0$ [see the definition of $a$).

If $c(a) = c(a_{p}) \cup c(a_{p+1}) \cap \bigcap_{i=1}^{k} X_{b_i} = 0$, then $c(a_{p})$ or $c(a_{p+1})$ contains $S_k(h_0)$. Hence $c(a) = c(a_{p}) \cup c(a_{p+1})$. Therefore $c(a) = c(a_{p}) \cup c(a_{p+1})$.

DEFINITION of $c^\alpha$. $c^\alpha$ maps the simplexes of $K$ into the simplexes of $K'$ by $c^\alpha : L \to c$ and is defined as follows on the rest of $K$: $c^\alpha(a) = \bigcup (S_k(h) \mid I(p) \leq q \leq R(p)) \cap S_k(h_0)$ where $I$ and $R$ are defined above and $S_k(h_0)$ is taken in $K'$, and $c^\alpha$ of a simplex is the union of the images of its vertices under $c^\alpha$.

Lemma 7. $c^\alpha$ is an acyclic carrier function.

Proof. We need to show that if $t$ is a simplex of $K$, then $c^\alpha(t)$ is a simplex of $K'$ and if $t \subset t'$ then $c^\alpha(t) \subset c^\alpha(t')$ and $c^\alpha(t)$ is acyclic. By definition, $c^\alpha(t)$ is a simplex of $K'$ and if $t' \subset t$ then $c^\alpha(t') \subset c^\alpha(t)$, so that $c^\alpha(t)$ is a carrier function. The images of simplexes of $L$ are clearly acyclic, so we consider simplexes in $K$ and not in $L$.

Suppose $c^\alpha(a)$ is $N(V^*)$, where $V = \{ U^{p+1} \cdots, U^{(p+1)}, \cdots \}$, $V = \{ U^{p+1} \cdots, U^{(p+1)} \}$. $\delta = 0$ if $I = 0$, and $\delta = 1$ otherwise. $\delta = 0$ if $R = u$ and $\delta = 1$ otherwise. These $U'$s are elements of the covering of which $U$ is the nerve and there are $u$ elements in this covering. In each case $c^\alpha(a)$ is acyclic, so $c^\alpha$ of any 0-simplex is acyclic. $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1} a_{3}^{1}) = c^\alpha(a_{0}) \cup c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$, which is $N(V^*) \cup N(W^*)$ where $V$ is as above and $W = \{ U^{p+1} \cdots, U^{(p+1)}, \cdots \}$, $N(V^*) \cap N(W^*) = N(V \cap W^*)$, where $V \cap W = \{ U^{p+1} \cdots, U^{(p+1)}, \cdots \}$.

So that $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ is acyclic and therefore since $N(V^*)$ and $N(W^*)$ are acyclic, by the Mayer-Vietoris Theorem we have that $N(V^*) \cap N(W^*)$ is acyclic. Therefore $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ is acyclic. Likewise if $a_{0} a_{1}^{1} a_{2}^{1}$ (i.e., the open sets associated with these vertices are $\delta$-related), then $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) = c^\alpha(a_{0}) \cup c^\alpha(a_{0} a_{1}^{1})$ which are each acyclic and $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) = c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$, which is acyclic. Therefore, by the Mayer-Vietoris Theorem, $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ is acyclic so that $c^\alpha$ of any 1-simplex is acyclic.

If each pair of vertices of $a_{0} a_{1}^{1} a_{2}^{1}$ are $\delta$-related, then $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) = c^\alpha(a_{0}) \cup c^\alpha(a_{0} a_{1}^{1}) \cup c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ which are each acyclic and the intersection of any two is $\delta$-acyclic. Moreover $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) \cap c^\alpha(a_{0}) = c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$, which is acyclic. Therefore, the Mayer-Vietoris Theorem, $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ is acyclic and so $c^\alpha$ of any 2-simplex is $\delta$-acyclic.

If each pair of vertices of $a_{0} a_{1}^{1} a_{2}^{1}$ are $\delta$-related then $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) = c^\alpha(a_{0}) \cup c^\alpha(a_{0} a_{1}^{1}) \cup c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ which are each acyclic and the intersection of any two is $\delta$-acyclic. Moreover $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1}) \cap c^\alpha(a_{0}) = c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$, which is acyclic. Therefore, $c^\alpha(a_{0} a_{1}^{1} a_{2}^{1})$ is $\delta$-acyclic and so $c^\alpha$ of any 3-simplex is $\delta$-acyclic. Hence $c^\alpha$ is an acyclic carrier function.

Lemma 8. (13) p. 171, Theorem 5.5. Let $K$ and $K'$ be simplicial complexes, let $c^\alpha$ be an acyclic carrier function defined on $K$ with values in $K'$, and let $L$ be a subcomplex of $K$. Any algebraic map $L \to K'$ with carrier $c^\alpha$ can be extended to an algebraic map $K \to K'$ with carrier $c^\alpha$. If $f, g : K \to K'$ are algebraic maps with carrier $c^\alpha$, then any algebraic homotopy between $fL$ and $gL$ with carrier $c^\alpha$ can be extended to an algebraic homotopy between $f$ and $g$ with carrier $c^\alpha$.

DEFINITION of $c^\alpha$. Let $a_{0}$ and $b_{0}$ be vertices of $N(U)$ and $N(U')$ respectively and $\gamma(p) = \max \{ k_0 \mid X_{b_0} \neq 0 \}$ and $\tau(p) = \min \{ k_0 \mid X_{a_0} \neq 0 \}$ and $k_0 \geq \gamma(p)$ (see definition of $a_0$). Following the construction of the extension in Lemma 8 we extend $c_{\alpha}$ to $c_{\alpha}$ where $c_{\alpha}(N(U)) = c_{\alpha}(N(U'))$ and in $N(U) - N(U')$ on 0-chains we have $c_{\alpha}(N(U)) = c_{\alpha}(N(U'))$. In $N(U) - N(U')$ on 1-chains we have

$$c_{\alpha}(1_{a_{0} a_{1}^{1} a_{2}^{1}}) = \sum_{i=0}^{d_{\alpha}} (b_{a_0} b_{a_1} b_{a_2}^{d_{\alpha}}) = \sum_{i=0}^{d_{\alpha}} (b_{a_0} b_{a_1} b_{a_2}^{d_{\alpha}})$$

where $d_{\alpha} = \{ 1 \mid p \neq r \}$ and $b_{a_0} b_{a_1} b_{a_2}^{d_{\alpha}}$ are $b_{a_0}$'s such that $X_{b_{a_0}} \neq 0$ and $k_0 < k_{a_0}$.
By part 2 of Lemma 8 the algebraic homotopy between $o\sigma$ and 1 can be extended to an algebraic homotopy between $o\sigma^\tau a_0$ and $1^r$ with carrier $\sigma$. Hence $o\sigma^\tau a_0 \sim 1_0^r$, i.e., condition (a) is satisfied.

**Lemma 10.** Condition (b) for a quasi-complex is satisfied.

**Proof.** If $\phi_1, \phi_2$ are antiprojections we wish to show $o\phi_1 o\phi_2$ is also, $o\phi_1 o\phi_2$ is a chain map from $N(U_1)$ to $N(U_2)$. So we need to show that $o\phi_1 o\phi_2 \sim 1_1^r$. First $o\phi_1 o\phi_2 \sim 1_1^r$ and

$$o\phi_1 o\phi_2(t) = o\phi_1 o\phi_2(t) = o\phi_1 o\phi_2(t),$$

where $t'=1^r$ since $\sigma^t$ only makes the domain of $o\phi_2$ larger. Hence by part 2 of Lemma 8 the algebraic homotopy between $o\phi_1 o\phi_2 a_0$ and $1_1^r$ can be extended to an algebraic homotopy between $o\phi_1 o\phi_2 a_0$ and $1_1^r$. Hence $o\phi_1 o\phi_2$ is also an antiprojection, i.e., condition (b) is satisfied.

**Lemma 11.** Condition (c) for a quasi-complex is satisfied.

**Proof.** If $\phi_1$ and $\phi_2$ are antiprojections, then since by Theorem 1 $N(U_1)$ is acyclic, we have $\phi_1 \sim \phi_2$, i.e., condition (c) is satisfied.

**Notation.** The star of a simplex $\sigma$, $St(\sigma)$, is the union of open sets corresponding to the vertices of $\sigma$, $U_1$ is a star refinement of $U_1$, if the star of every vertex corresponding to elements of $U_1$ is contained in some element of $U_1$.

**Lemma 12.** Condition (d) for a quasi-complex is satisfied.

**Proof.** For any $i$ we choose $j$ sufficiently large so that $U_1$ is a star-refinement of $U_1$ (see [3], p. 324). Then for any $U_1$ let $U_1$ be the one of the two $U_1$ and $U_1$ which is a refinement of both. Now we show that condition (d) is satisfied, i.e., for any $i$ there exists $j > i$, and for any $k$ an $m > k$, $j$ (depending on $i$ and $k$) such that $o\phi_1 o\phi_2$ exists, satisfies (a), (b), and (c) and if $\sigma \in N(U_1)$, then $[\sigma]_h \cup [\sigma_m a_0]$ is contained in a set of $U_1$. We have that if $\sigma_1 = o\phi_1 o\phi_2 a_0$, then condition (d) holds since $\phi_2$, $\phi_2$ defines a quasi-complex $X$ (see [3], p. 667).

In the following a with subscripts and superscripts will denote a vertex of $N(U_1)$ and $\bar{h}$ with subscripts and superscripts will denote a vertex of $N(U_1)$. We show for any simplex $\sigma$ that $[\sigma] \cup [\sigma_m a_0]$ is contained in the star of some vertex of $X(U_1)$. Since $U_1$ is a star refinement of $U_1$, we have that the star of a vertex of $X(U_1)$ is contained on some element of $U_1$ and hence condition (d) will be satisfied. We need only consider vertices whose images are non-zero.

$0$-chains:

$$[a_0] \cup [a_0 a_0] = [a_0] \cup [a_0 a_0] \cup [a_0 a_0] = \Omega(U_1, \ldots, U_1) \cup \Omega(U_2, \ldots, U_2) \cup \Omega(U_3, \ldots, U_3, \ldots, U_4, \ldots) = \Omega(U_1, \ldots, U_4) \cup \Omega(U_1, \ldots, U_4) \cup \Omega(U_1, \ldots, U_4) \cup \Omega(U_1, \ldots, U_4, \ldots) \cup \ldots$$

$p < q < r$. 

**J. Segal**

244
1-chains: If $p \neq r$ then
\[ [a^p_{a_1}a^p_{a_2}] \cup [a^r_{a_1}a^r_{a_2}] = [a^p_{a_1}a^p_{a_2}] \cup \left[ \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right] \]
\[ = [\Omega(U^p_1, ..., U^p_r) \cap \Omega(U^r_1, ..., U^r_r)] \cup \left( \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]
\[ \cap (U^{p+1}_{m_1}, ..., U^{p+1}_{m_2})] \subset \text{St}(a^p_{a_1}) \cap \text{St}(a^r_{a_2}) \]

since if $\tau(p, r) \leq \xi \leq \tau(p, r+1)$ then
\[ \Omega(U^p_1, ..., U^p_r) \subset \Omega(U^r_1, ..., U^r_r) \]

If $p = r$ then
\[ [a^p_{a_1}a^p_{a_2}] \cup [a^p_{a_1}a^p_{a_2}] \]
\[ = [a^p_{a_1}a^p_{a_2}] \cup \left[ \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right] \]
\[ = [\Omega(U^p_1, ..., U^p_r)] \cup \left( \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]
\[ \subset \Omega(U^p_1, ..., U^p_r) \cup \left( \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]
\[ \subset \Omega(U^p_1, ..., U^p_r) \cup \left( \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]
\[ \subset \text{St}(a^p_{a_1}) \cup \left( \sum_{\sigma = (p,r)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]

since if $\gamma(p) \leq \lambda \leq \gamma(p+1)$ then
\[ \Omega(U^p_{m_1}, ..., U^p_{m_2}) \subset \Omega(U^{p+1}_{m_1}, ..., U^{p+1}_{m_2}) \]

and if $\tau(p, r) \leq \xi \leq \tau(p+1, r)$ then
\[ \Omega(U^{p+1}_{m_1}, ..., U^{p+1}_{m_2}) \subset \Omega(U^{p+1}_{m_1}, ..., U^{p+1}_{m_2}) \]

2-chains:
\[ [a^{p+1}_{a_1}a^{p+1}_{a_2}] \cup [a^p_{a_1}a^p_{a_2}] \]
\[ = [a^{p+1}_{a_1}a^{p+1}_{a_2}] \cup \left[ \sum_{\xi = (p+1)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right] \]
\[ = [\Omega(U^{p+1}_{m_1}, ..., U^{p+1}_{m_2}) \cap \Omega(U^{p+1}_{m_1}, ..., U^{p+1}_{m_2})] \]
\[ \subset [\text{St}(a^{p+1}_{a_1}) \cup \left( \sum_{\xi = (p+1)} \binom{y^{(p+1)-1}}{x^{(p+1)}} \right) \]

Hence condition (d) is satisfied.

From lemmas 9, 10, 11, and 12 follows

**Theorem 2.** If $O(X)$ is acyclic and since it is connected, it is a zero-cyclic quasi-complex. By [10], p. 326, (36.4), a zero-cyclic quasi-complex has fixed point property.

**Proof.** By [9] $O(X)$ is acyclic and since it is connected, it is a zero-cyclic quasi-complex. By [10], p. 326, (36.4), a zero-cyclic quasi-complex has fixed point property.

**Remark.** For any continuum $Y$, $O(Y)$ is an absolute retract if and only if $Y$ is locally connected (see [5], Theorem 4.4). Hence if $Y$ is locally connected, $O(Y)$ has the fixed point property. It follows from a theorem of Lefschetz ([7], p. 46) that if $O(Y)$ is an absolute neighborhood retract, then it has the fixed point property. By the following theorem if $O(Y)$ is finite dimensional (in particular, if $Y$ is snake-like), then $O(Y)$ is an absolute neighborhood retract only in case $Y$ is locally connected; hence neither of the above results applies when $Y$ is a non-locally connected snake-like continuum.

**Theorem 4.** If $O(Y)$ is a finite dimensional absolute neighborhood retract then it is an absolute retract.

**Proof.** Fox [4] has shown that any $m$-dimensional absolute neighborhood retract which is simply connected and acyclic in all dimensions
Sur la représentation topologique des graphes
par
Á. Csávzsár (Budapest)

1. Nous entendons par graphe (abstrait) le système \((S, A, e)\) composé d'un ensemble \(S\) (dont les éléments sont appelés sommets du graphe), d'un ensemble \(A\) (dont les éléments s'appellent arêtes du graphe) et d'une application \(e\) qui fait correspondre à chaque arête \(a \in A\) un ensemble \(e(a) = \{a_1, a_2\} \subset S\) composé de deux sommets distincts, appelés extrémités de l'arête \(a\) (*)

Nous disons qu'un graphe \((S, A, e)\) est fini si les ensembles \(S\) et \(A\) sont finis, et qu'il est dénombrable si \(S\) et \(A\) sont dénombrables (†). Le graphe \((S, A, e)\) est dit connexe si, deux sommets distincts \(s, s' \in S\), \(s \neq s'\) étant donnés, on peut toujours trouver une suite finie d'arêtes \(a_1, ..., a_n\) telles que \(s \in e(a_1), s' \in e(a_n)\) et que \(e(a_i) \cap e(a_{i+1}) \neq \emptyset\) pour \(1 \leq i \leq n-1\).

2. On a l'habitude de représenter un graphe fini \((S, A, e)\) par un sous-ensemble \(\mathcal{G}\) de l'espace euclidien \(\mathbb{E}\), composé de certains points \(s_1, ..., s_n\) et de certains arcs \(s'_1, ..., s'_m\) (où \(m\) est égal à la puissance de l'ensemble \(S\) et \(n\) à celle de l'ensemble \(A\)), de manière que les points \(s_i\) correspondent binormalement aux sommets \(s_i \in S\) et les arcs \(s'_i\) aux arêtes \(a_i \in A\), l'arc \(s'_i\) ayant pour extrémités les points \(s_i\) et \(s_{i+1}\) si et seulement si l'arête correspondante \(a_i\) a pour extrémités les sommets \(s_i\) et \(s_{i+1}\) qui correspondent à \(s'_i\) et \(s'_{i+1}\) respectivement, et deux arcs \(s'_i\) et \(s'_j\) n'ayant d'aucun point commun que leurs extrémités au plus. Beaucoup de propriétés du graphe \((S, A, e)\) peuvent être formulées au moyen des propriétés topologiques de l'ensemble \(\mathcal{G}\); p.ex. le graphe \((S, A, e)\) est connexe si et seulement si l'ensemble \(\mathcal{G}\) est connexe (au sens topologique).

Pour un graphe \((S, A, e)\) quelconque (fini ou non), on peut définir une représentation topologique analogue de la façon suivante. Considérons un ensemble \(G\) dont les éléments sont d'une part les sommets \(s \in S\) du graphe, de l'autre les couples \((a, s)\) formés par une arête \(a \in A\)

(*) D'après la terminologie adoptée par D. König ([3], pp. 1 et 2), il faudrait encore postuler que, pour \(s \in S\), il existe au moins un \(a \in A\) tel que \(s \in e(a)\); d'après C. Berge ([1], p. 27), on devrait dire multi-graphes au lieu de graphe. Cependant, la terminologie que nous venons d'introduire conviendra mieux à nos buts.

(†) C'est-à-dire finis ou dénombrablement infinis.