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Metric property of linear sets

by

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DEFINITION. A decreasing sequence $\{h_i\}$ of positive numbers is called a *progression* if there exists a positive integer k such that $h_{i+k}/h_i < \frac{1}{2}$ for every i. Denote by $f(x, a, b)$ the *characteristic function of the interval* (a, b) and write

$$s_j(x) = \sum_{i=1}^{j} \frac{1}{h_i} f(x, h_i - h_j, h_i)$$

THEOREM 1. *A necessary and sufficient condition for the sequence h_j to be a progression is that the integral $\int_0^{h_1} s_j(x)\,dx$ be bounded.*

We have

$$\int_0^{h_1} s_j(x)\,dx = \left(\frac{1}{h_1} + \frac{1}{h_2} + \ldots + \frac{1}{h_j}\right) h_j .$$

(i) Suppose that there exists an integer $k > 0$ such that for all j

$$\left(\frac{1}{h_1} + \frac{1}{h_2} + \ldots + \frac{1}{h_j}\right) h_j < \frac{k}{2} .$$

For $j > k$ we have

$$\frac{k}{2} > \left(\frac{1}{h_{j-k}} + \ldots + \frac{1}{h_j}\right) h_j > k\frac{h_j}{h_{j-k}} \quad \text{or} \quad \frac{h_j}{h_{j-k}} < \frac{1}{2} .$$

Hence $\{h_j\}$ is a progression.

(ii) Conversely if h_{i+k}/h_i is always $< \frac{1}{2}$ then

$$\int_0^{h_1} s_j\,dx = \left(\frac{1}{h_1} + \frac{1}{h_2} + \ldots + \frac{1}{h_j}\right) h_j < h_j\left(\frac{1}{h_j} + \frac{1}{h_{j-k}} + \frac{1}{h_{j-2k}} + \ldots\right) + h_{j-1}\left(\frac{1}{h_{j-1}} + \frac{1}{h_{j-1-k}} + \frac{1}{h_{j-1-2k}} + \ldots\right) + h_{j-k+1}\left(\frac{1}{h_{j-k+1}} + \frac{1}{h_{j-2k+1}} + \ldots\right) < 2k$$

that is, the integral is bounded, and the theorem is proved.

Introduce the *truncated function* $\bar{s}_j(x)$ by the conditions

$$\bar{s}_j(x) = \begin{cases} s_j(x) & \text{when} \quad s_j(x) \leqslant \dfrac{1}{h_j}, \\ \dfrac{1}{h_j} & \text{when} \quad s_j(x) > \dfrac{1}{h_j}. \end{cases}$$

DEFINITION. A sequence $\{h_i\}$ is called a *generalized progression* if the integral $\int_0^{h_1} \bar{s}_j(x)\,dx$ is bounded.

N. Fine [1] has established a very interesting metric property of all linear sets of points. Denoting by ϱ the distance on a straight line, he has proved

THEOREM 2. *Given an arbitrary linear set E and a progression $\{h_i\}$* [2] *the series*

$$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i}$$

converges at almost all points of E.

Theorem 2 represents a metric property of linear sets, being a sharpening of Lebesgue Theorem on existence of density equal to 1 at points of the set. From this point of view the theorem is very interesting and it is important to see what is the most general type of a sequence $\{h_i\}$ for which the theorem holds. The answer will be given in Theorems 3 and 4.

As $\varrho(x, E) = \varrho(x, \bar{E})$ where $\bar{E}$ is the closure of E, it is sufficient to prove the theorem only for closed sets.

THEOREM 3. *For any bounded closed set E and for a generalized progression $\{h_i\}$ the series*

$$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i}$$

converges at almost all points of E.

With the notation introduced above we have

$$\text{(1)} \qquad \int_0^{h_1} \bar{s}_j(x)\,dx < k$$

[1] *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. of Sciences 41 (1955), pp. 588-591.

[2] Actually, conditions on the sequence $\{h_i\}$ are given by N. Fine in the form: $\sum_{h_j \leqslant \delta} h_j \leqslant M\delta$, $\sum_{h_j > \delta} h_j \leqslant M/\delta$ for a constant M and all $\delta > 0$. It follows from the first inequality that the sequence is a progression.

for all j. Denote by (a_n, b_n), $n = 1, 2, \ldots$, all the complementary intervals of E and write $b_n - a_n = l_n$. If we drop from our series terms equal 0, then we may write

$$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i} = \sum_{n=1}^{\infty} \sum_{a_n < x-h_i < b_n} \frac{\varrho(x-h_i, E)}{h_i}.$$

We shall assume that for every i

$$\text{(2)} \qquad h_i < 2h_{i+1}.$$

Later on we shall get rid of this condition very easily. Consider the sum

$$\sum_{a_n < x-h_i < b_n} \frac{\varrho(x-h_i, E)}{h_i}$$

for values of x outside the interval $(b_n, b_n + l_n)$ that is for $x > b_n + l_n$. Then only those terms of the series are different from 0 for which h_i is greater than l_n. Let j_n be the largest i for which $h_i > l_n$. We have

$$\text{(3)} \qquad \sum_{a_n < x-h_i < b_n} \frac{\varrho(x-h_i, E)}{h_i} \leqslant \sum_{i=1}^{j_n} \frac{\varrho(x-h_i, E)}{h_i} f(x-b_n, h_i - h_{j_n}, h_i).$$

For, whenever $a_n < x - h_i < b_n$, $f(x-b_n, h_i-h_{j_n}, h_i) = 1$ and on the left hand side the summation is extended only on values of $i \leqslant j_n$. For all members of the sum

$$\varrho(x-h_i, E) \leqslant \tfrac{1}{2} l_n < \tfrac{1}{2} h_{j_n},$$

whence

$$\text{(4)} \qquad \sum_{a_n < x-h_i < b_n} \frac{\varrho(x-h_i, E)}{h_i} < \tfrac{1}{2} h_{j_n} s_{j_n}(x-b_n) < l_n s_{j_n}(x-b_n).$$

Denote by U_n the set of values of x for which $s_{j_n}(x-b_n) \geqslant 1/h_{j_n}$. For all points of U_n, $\bar{s}_{j_n}(x-b_n) = 1/h_{j_n}$.

Hence by (1) and (2)

$$\text{(5)} \qquad mU_n < kh_{j_n} < 2kl_n.$$

Given $\varepsilon > 0$ define N so that

$$\text{(6)} \qquad \sum_{n>N} l_n < \frac{\varepsilon}{2(2k+1)}$$

and consider convergence of our series on the set

$$E' = E - \sum_{n=1}^{N} (b_n, b_n + \varepsilon/2N) - \sum_{n=N+1}^{\infty} (b_n, b_n + l_n) - \sum_{n=N+1}^{\infty} U_n.$$

By (5) and (6)

$$mE' > mE - \varepsilon. \tag{7}$$

On the set E' each of the sums $\sum\limits_{a_n<x-h_i<b_n} (\varrho(x-h_i, E)/h_i)$ has a finite number of terms and thus convergence of the full series depends on the convergence of the series

$$\sum_{n=N+1}^{\infty} \sum_{a_n<x-h_i<b_n} \frac{\varrho(x-h_i, E)}{h_i}. \tag{8}$$

By (4) we have outside U_n

$$\sum_{a_n<x-h_i<b_n} \frac{\varrho(x-h_i, E)}{h_i} < l_n \bar{s}_{j_n}(x-b_n).$$

Thus on the set E'

$$\sum_{n=N+1}^{\infty} \sum_{a_n<x-h_i<b_n} \frac{\varrho(x-h_i, E)}{h_i} < \sum_{n=N+1}^{\infty} l_n \bar{s}_{j_n}(x-b_n). \tag{9}$$

But by (1)

$$\int_E \sum_{n=N+1}^{\infty} l_n \bar{s}_{j_n}(x-b_n)\,dx < \sum_{n=N+1}^{\infty} kl_n < \infty$$

that is, the series on the right hand side of (9) converges at almost all points of E'. Hence the series on the left hand side does, and so does the full series, from which the theorem follows. We get rid of the condition (2) in the following way. If $\{h_i\}$ does not satisfy it we take the sequence S of numbers $(\frac{3}{2})^{-n}$. The theorem holds for the sequence S and for $\{h_i\}+S$ and consequently it holds for the difference, that is for $\{h_i\}$.

THEOREM 4. *If the sequence $\{h_i\}$ is not a generalized progression then there exists a set E, $mE > 0$, at almost all points of which the series*

$$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i} \tag{10}$$

diverges.

It is more convenient for us to define a set E not on a straight line but on the unit circle Γ. We shall mark an origin and the coordinate x of a point on Γ will be the length of the arc between the origin and the point.

The sequence h_i being not a generalized progression, the sequence of integrals

$$A(j) = \int_0^{2\pi} \bar{s}_j(x)\,dx \leqslant 2\pi h_j^{-1} \tag{11}$$

is not bounded. We can choose a sequence of integers $j_1 < j_2 < \ldots$ so that the sequence

$$[\sqrt{A(j_1)}] = k_1, \quad [\sqrt{A(j_2)}] = k_2, \ \ldots \tag{12}$$

increases as rapidly as we like. Integers $j_1, j_2, \ldots$ will be defined more precisely later. We shall also need integers $k_1', k_2', \ldots$ such that $k_1 < k_1' < k_2 < k_2' < \ldots$, also to be defined more precisely later.

LEMMA. *Given a bounded function $s(x) = s(x+2\pi)$ on the unit circle Γ and a set E also on Γ with $\theta_E(x)$ as its characteristic function, then*

$$\int_0^{2\pi} dt \int_0^{2\pi} s(x-t)\,\theta_E(x)\,dx = mE \int_0^{2\pi} s(x)\,dx. \tag{13}$$

(13) is received by the change of order of integration.

A sequence $\{(a_n, b_n)\}$ of non-overlapping arcs on Γ of total length $<2\pi$ will be defined so that the set $E = \Gamma - \sum\limits_{n=1}^{\infty} (a_n, b_n)$ will satisfy the theorem. Writing

$$a_n' = \frac{2a_n+b_n}{3}, \quad b_n' = \frac{a_n+2b_n}{3}$$

we have

$$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i} = \sum_{n=1}^{\infty} \sum_{a_n<x-h_i<b_n} \frac{\varrho(x-h_i, E)}{h_i} \geqslant \sum_{n=1}^{\infty} \sum_{a_n'<x-h_i<b_n'} \frac{\varrho(x-h_i, E)}{h_i}. \tag{14}$$

We shall define now $b_n - a_n$ by formulae

$$b_n - a_n = 3h_{j_1} \quad \text{for} \quad n \leqslant k_1'$$

and

$$b_n - a_n = 3h_{j_s} \quad \text{for} \quad k_{s-1}' < n \leqslant k_s', \quad s > 1.$$

As the interval a_n', b_n' is the interior third of (a_n, b_n) we have $b_n' - a_n' = h_{j_s}$ and for $a_n' < x-h_i < b_n'$ $\varrho(x-h_i, E) > h_{j_s}$, so that

$$\sum_{a_n'<x-h_i<b_n'} \frac{\varrho(x-h_i, E)}{h_i} > h_{j_s} \sum_{a_n'<x-h_i<b_n'} \frac{1}{h_i} \geqslant h_{j_s} \sum_{\substack{a_n'<x-h_i<b_n' \\ i\leqslant j_s}} \frac{1}{h_i} \tag{15}$$

and as in (3) and (4)

$$\sum_{\substack{a_n'<x-h_i<b_n' \\ i\leqslant j_s}} \frac{1}{h_i} = \sum_{1\leqslant i\leqslant j_s} \frac{1}{h_i} f(x-b_n', h_i - h_{j_s}, h_i) = s_{j_s}(x-b_n') \geqslant \bar{s}_{j_s}(x-b_n'). \tag{16}$$

Thus defining s as a function of n, equal to j_1 for $n \leqslant j_1$ and for $n > j_1$ by the inequalities $j_{s-1} < n \leqslant j_s$ we shall have by (14), (15), (16)

(17) $$\sum_{i=1}^{\infty} \frac{\varrho(x-h_i, E)}{h_i} \geqslant \sum_{i=1}^{\infty} h_{j_s}\bar{s}_{j_s}(x-b'_n).$$

We shall now show that by a proper choice of numbers $j_1, j_2, \ldots$ and $b'_1, b'_2, \ldots$ we can have the series on the right hand side of (17) to diverge at almost all points of E. First we define j_1 so that $k_1 = [\sqrt{A(j_1)}]$ be large, say $>10^{10}$. Writing

$$S_m(x) = \sum_{n=1}^{m} h_{j_s}\bar{s}_{j_s}(x-b'_n)$$

denote by E_n, for $n \leqslant k'_1$, the set of values of x for which $S_n(x) < k_1$. Remembering that $\bar{s}_{j_s}(x-b_n)$ is always $\leqslant h_{j_s}^{-1}$ we see that $S_m(x) \leqslant m$ for all x. Hence, for $n < k_1$, $E_n = (0, 2\pi)$. Numbers b_n and j_s will be defined by induction. For $n \leqslant k_1$ the intervals (a_n, b_n), $b_n - a_n = 3h_{j_1}$, are subject only to the condition that they do not overlap. By (11) and (12) $k_1^2 h_{j_1} \leqslant 2\pi$ so that the total length of intervals (a_n, b_n), $n \leqslant k_1$ is small. The consecutive definitions of b'_n for $n < k'$ will aim at reducing mE_n to a small value. This will be achieved by choosing b'_{n+1} so that the integral

$$\int_{E_n} \bar{s}_{j_1}(x-b'_{n+1})\,dx = \int_0^{2\pi} \theta_{E_n}(x)\bar{s}_{j_1}(x-b'_{n+1})\,dx$$

has as large value as possible.

We have by the Lemma

(18) $$\int_0^{2\pi} dt \int_0^{2\pi} \theta_{E_n}(x)\bar{s}_{j_1}(x-t)\,dx = mE_n \int_0^{2\pi}\bar{s}_{j_1}(x)\,dx.$$

But we want for b'_{n+1} such value of t for which the interval (a_{n+1}, b_{n+1}) does not overlap with any interval previously defined, that is a value

$$t \in (0, 2\pi) - \sum_{i=1}^{n}(a_i - h_{j_1}, b_i + 2h_{j_1}) = I, \qquad mI = 2\pi - 6nh_{j_1}.$$

By (18)

$$\int_I dt \int_0^{2\pi} \theta_{E_n}(x)\bar{s}_{j_1}(x-t)\,dx \geqslant (mE_n - 6nh_{j_1})\int_0^{2\pi}\bar{s}_j(x)\,dx.$$

Hence there exists $t \in I$ such that

$$\int_0^{2\pi} \theta_{E_n}(x)\bar{s}_{j_1}(x-t)\,dt \geqslant \frac{mE_n - 6nh_{j_1}}{2\pi - 6nh_{j_1}}\int_0^{2\pi}\bar{s}_{j_1}(x)\,dx > \frac{mE_n-6nh_{j_1}}{2\pi}\int_0^{2\pi}\bar{s}_{j_1}(x)\,dx.$$

We define b'_{n+1} by one of such values:

(19) $$\int_{E_n}\bar{s}_{j_1}(x-b'_{n+1})\,dt > \frac{mE_n-6nh_{j_1}}{2\pi}\int_0^{2\pi}\bar{s}_{j_1}(x)\,dx > \frac{mE_n - 6nh_{j_1}}{2\pi}k_1^2.$$

For any $n < k'_1$ we have

(20) $$\int_{E_n - E_{n+1}} S_{n+1}(x)\,dx < (k_1+1)m(E_n - E_{n+1}).$$

For

$$S_{n+1}(x) = S_n(x) + h_{j_1}\bar{s}_{j_1}(x-b'_{n+1}) < k_1 + 1$$

on E_n. Hence

(21) $$U_n = \int_{E_{k_1-1}-E_{k_1}} S_{k_1}(x)\,dx + \ldots + \int_{E_k - E_{k+1}} S_{k+1}(x)\,dx + \ldots + \int_{E_{n-1}-E_n} S_n(x)\,dx + \int_{E_n} S_{n+1}(x)\,dx < (k_1+1)2\pi.$$

But

(22) $$U_n = \int_0^{2\pi} S_{k_1}(x)\,dx + \int_{E_{k_1}} h_{j_1}\bar{s}_{j_1}(x-b'_{k_1+1})\,dx + \int_{E_{k_1+1}} h_{j_1}\bar{s}_{j_1}(x-b'_{k_1+2})\,dx + \ldots + \int_{E_n} h_{j_1}\bar{s}_{j_1}(x-b_{n+1})\,dx$$

and by (19)

(23) $$U_n > \sum_{k=k_1}^{n}\frac{mE_k - 6kh_{j_1}}{2\pi}k_1^2 h_{j_1}.$$

By (21) and (23)

(24) $$\sum_{k=k_1}^{n}\frac{mE_k-6kh_{j_1}}{2\pi}k_1^2h_{j_1} < (k_1+1)2\pi.$$

Writing $[2\pi h_{j_1}^{-1}] = N$ we have, by (11), $k_1^2 \leqslant N$. mE_k cannot remain $\geqslant 14\pi k_1^{-1/3}$ for all $k \leqslant Nk_1^{-1/3}$. For if it did we would have

$$6kh_{j_1} < 6Nk_1^{-1/3}h_{j_1} < 12\pi k_1^{-1/3},$$
$$\frac{mE_k - 6kh_{j_1}}{2\pi}k_1^2h_{j_1} > k_1^{5/3}h_{j_1},$$

and taking for n the largest value, that is $\leqslant Nk_1^{-1/3}$, we shall have

$$\sum_{k=k_1}^{n}\frac{mE_k-6kh_{j_1}}{2\pi}k_1^2h_{j_1} > (Nk_1^{-1/3}-k_1)k_1^{5/3}h_{j_1} > \tfrac{3}{4}Nk_1^{4/3}h_{j_1} > \pi k_1^{4/3}$$

which is impossible by (24).

Denote by k_1' the smallest integer for which $mE_k < 14\pi k_1^{-1/3}$ and suppose that b_n' has been defined for all $n \leqslant k_1'$. We have $S_{k_1'}(x) > k_1$ at all points of $(0, 2\pi)$ except at points of the set $E_{k_1'}$ of measure $< 14 k_1^{-1/3}$. Thus by the above definition of b_k' for $k = 1, 2, \ldots, k_1'$ we have achieved that the series (17) is $> k_1$ except at a set of measure $< 14\pi k_1^{-1/3}$. We define now j_2 so that $k_2 = [\sqrt{A(j_2)}]$ be greater than k_1^2 and k_1'. There will be further a fixed value of $k_2' > k_2$. We define, for $k_1' < n \leqslant k_2'$, $b_n' - a_n' = h_{j_2}$ and b_n will be defined by an induction similar to the first one. First for $k' < n \leqslant k_1$ the intervals (a_n, b_n) are subject only to the condition not to overlap with the intervals already defined. For these values of n we denote by E_n the set of those values of x for which $S_n(x) < k_2$ so that $E_n = (0, 2\pi)$ for all $k' < n < k_2$. After b_n' has been defined, for an $n \geqslant k_2$, b_{n+1}' can be defined so that the interval (a_{n+1}, b_{n+1}) does not overlap with the intervals that have already been defined and that

$$\int_{E_n} s_{j_2}(x - b_{n+1}')\, dx > \frac{mE_n - 6nh_{j_2}}{2\pi} \int_0^{2\pi} \bar{s}_{j_2}(x)\, dx .$$

As before we shall arrive at the value k_2' of n such that $mE_{k_2'} < 14\pi k_2^{-1/3}$. Having defined b_n' for all $n \leqslant k_2'$ we shall have the value of the series (17) $> k_2$ except at a set of points of measure $< 14 k_2^{-1/3}$. Continuing in this way we arrive at a proof of the theorem.

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Orderable spaces

by

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Introduction. A topological space E will be called *orderable* if there exists a total order relation R on E such that the interval topology of the totally ordered set (E, R) coincides with the topology of the space E. Any total order relation R on an orderable space E which has this property will be called an *order* of the space E. For any relation R on a set E, the dual relation will be denoted by σR. It is clear that if R is one of the orders of an orderable space E then σR is also an order of the space E. The pair $(R, \sigma R)$ will then be called an *order class* of the space. The basic theorem concerning connected orderable spaces is due to Eilenberg ([2]): A connected space E is orderable exactly if it is locally connected and the subset $E \times E - D$ of the product space $(D = \{(x, x) \mid x \in E\}$, the diagonal in $E \times E)$ is not connected; in this case there is exactly one order class of E given by the closures of the components of $E \times E - D$.

This note will mainly be concerned with densely orderable spaces, i.e. with orderable spaces possessing orders R such that the ordered set (E, R) is dense in itself. It will be shown that there is a one-to-one correspondence between the order classes $(R, \sigma R)$ with dense order R of such a space E and its connected orderable compact extension spaces which is, in one direction, given by the passage to the Dedekind completion $\delta(E, R)$ by cuts of the ordered set (E, R). Also, if R is a dense order of a space E, then $\delta(E, R)$ will be described as the completion of E with respect to a certain uniform structure which is defined by means of R. Then, the uniform structures of a densely orderable space arising in this way out of a dense ordering of E will be characterized by a number of properties; this constitutes a criterion for the existence of dense orders on a space in terms of uniform structures. The application of these ideas to topological groups is shown to lead to a characterization of the dense subgroups of the additive group of reals. Finally, the existence of dense orders of a locally connected space is considered.

All concepts of general topology are taken in the sense of N. Bourbaki. The same goes for notions and notations related to totally ordered sets.