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Structure maps in group theory

by

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Introduction and notation. In [3] the authors elaborate a theory of structure maps in general categories. If the category $\mathcal{C}$ admits direct products the notion of group structure (¹) and the group axioms may be formulated entirely in terms of the maps of the category; and this enables us to carry over to categories other than categories of sets and element-maps certain classical notions of group theory. Moreover the definitions and results may be dualized and applied to categories admitting free products. We also discuss in [3] various generalizations of the classical notions of group theory, for example, the notions of unions and intersections of subgroups.

Among the concrete categories to which the notions of [3] may be applied is the category $\mathcal{G}$ of groups and homomorphisms; indeed the notions and terminology of [3] were in part inspired by the category $\mathcal{G}$. The present paper consists of a fairly detailed discussion of the application of the notions of [3] to the category $\mathcal{G}$. In the course of this discussion we naturally find ourselves introducing ideas and adopting arguments peculiar to the category of groups. Thus, unlike [3], we claim here no generality for our results which are all group-theoretic, and the present paper is intended to be, more or less, readable independently of [3], owing to [3] merely its motivation.

If $\mathcal{C}$ is a category of the type considered in [3] and if $X_1, \ldots, X_n$ are n objects of $\mathcal{C}$ then there is a natural (self-dual) map

$$\varkappa\colon\; X_1 * \ldots * X_n \to X_1 \times \ldots \times X_n$$

in $\mathcal{C}$ from the free product of $X_1, \ldots, X_n$ to the direct product. An important general construction described in [3] consists of the two (dual) factorizations of the map $\varkappa$, namely,

$$\text{(F)}\qquad X_1 * \ldots * X_n = X^n \xrightarrow{\varkappa^{n-1}} X^{n-1} \to \ldots \to X^{p+1} \xrightarrow{\varkappa^p} X^p \to \ldots \xrightarrow{\varkappa^1} X^1 = X_1 \times \ldots \times X_n\,,$$

$$\text{(F')}\qquad X_1 * \ldots * X_n = {}^1X \xrightarrow{{}^1\varkappa} \ldots \to {}^qX \xrightarrow{{}^q\varkappa} {}^{q+1}X \to \ldots \to {}^{n-1}X \xrightarrow{{}^{n-1}\varkappa} {}^nX = X_1 \times \ldots \times X_n\,,$$

(¹) And, of course, semigroup structure.

where X^p, ${}^pX$, $1<p<n$, are defined as certain generalized intersections and unions. The factorizations (F) and (F′) were used in [3] to define the two dual concepts of length, $l(X)$ and $l'(X)$, of an object X of $\mathcal{C}$. Precisely, let us single out the map $\varkappa^{n-1}$ in (F) and write σ for $\varkappa^{n-1}$, $T = T(X_1, \ldots, X_n)$ for X^{n-1}. Then if $X_1 = \ldots = X_n = X$ we have a folding map $d\colon X^n \to X$ and we define $l(X) < n$ if there exists a *structure map* $\mu\colon T \to X$ such that $\mu\sigma = d$. A dual procedure yields the definition of $l'(X)$.

It turns out (Theorem 1.1) that, in the category $\mathcal{G}$, the factorization (F′) is trivial for $q>1$ in the sense that ${}^q\varkappa$ is an isomorphism if $q>1$; it follows therefore that $l'(G)<3$ for all $G \in \mathcal{G}$. On the other hand the factorization (F) coincides, in $\mathcal{G}$, with that described in [5], Proposition 2.3, where it is shown that each $\varkappa^p$ is an epimorphism. Thus in testing the length $l(G)$ of $G \in \mathcal{G}$ it is sufficient to look at the kernel of $\sigma = \varkappa^{n-1}$ and see whether the map d annihilates this kernel. If so, the map induced by d on the quotient group $G^n/\ker\sigma \cong T$ is the corresponding structure map. Let $X_1 * \ldots * \hat{X}_i * \ldots * X_n$ be the free product of the X_j with $j \neq i$ and let $\pi_i\colon X^n \to X_1 * \ldots * \hat{X}_i * \ldots * X_n$ be the projection off X_i, $i = 1, \ldots, n$. Then $\ker \varkappa^{n-1} = \ker\sigma$ is just the intersection of the kernels of the projections π_i, so that the study of the length $l(G)$ in $\mathcal{G}$ leads naturally to a consideration of the projection kernels $\ker\pi_i$ and their intersections.

We recall that in a general category $\mathcal{C}$ of the type considered in [3] the property $l(X) \leqslant 1$ is equivalent to the statement that X admits an H-structure; for an H-structure map on X is a map

$$\mu\colon X \times X \to X \quad \text{such that} \quad \mu\iota_1 = \mu\iota_2 = 1,$$

where ι_1, ι_2 embed X as the first, second factor in $X \times X$; and $\mu\iota_1 = \mu\iota_2 = 1$ if and only if $\mu\varkappa = d$ where $\varkappa\colon X * X \to X \times X$. Dually $l'(X) \leqslant 1$ if and only if X admits an H'-structure. Section 1 of this paper is devoted to proving the triviality of (F′) in $\mathcal{G}$ for $q>1$ and studying H- and H'-structures in $\mathcal{G}$; it turns out that $l(G) \leqslant 1$ if and only if G is abelian and $l'(G) \leqslant 1$ if and only if G is free. Section 2 is devoted to a detailed study of the structure of arbitrary intersections of projection kernels in $\mathcal{G}$. Our general theorems lead in particular to a description of a canonical system of generators for $\ker\sigma$ and the proof that $l(G)$ is just the nilpotency class of G (Theorem 2.17; see also [1]). We also prove a theorem (Theorem 2.19) which generalizes the result that the free group $[G_1, G_2] \subseteq G_1 \cap G_2$ is freely generated by commutators $[g_1, g_2]$, $g_1 \in G_1$, $g_2 \in G_2$, $g_1 \neq e$, $g_2 \neq e$.

The notations of sections 1 and 3 are desciibed either in the foregoing introduction or explicitly in those sections. The notations of sections 2 are those of [5] appearing in the foregoing introduction augmented as follows. If I is an r-string (ordered subset of r elements of the ordered set $1, 2, \ldots, n$) then, if $j \in I$, I_j is the $(r-1)$-string obtained from I by removing j, and,

if $j \notin I$, I^j is the $(r+1)$-string obtained from I by adjoining j. We adopt the notational convention that the *left-normed* commutator

$$\big[g_1, [g_2, \ldots, [g_{k-1}, g_k]\ldots]\big]$$

is written

$$[g_1, \ldots, g_k].$$

We also write a^b for $b^{-1}ab$, where a, b are group elements.

The authors wish to acknowledge the benefit of correspondence with T. Ganea in developing the ideas of this paper. It is hoped to publish later a joint paper with Ganea discussing the notion of *n-mean* (see [2]) in general categories, and in special categories including $\mathcal{G}$ and categories appropriate to algebraic topology.

1. Dual lengths in $\mathcal{G}$. To define the dual lengths in $\mathcal{G}$ we must construct the groups $T = T(G_1, \ldots, G_n)$ and $T' = T'(G_1, \ldots, G_n)$ and the natural homomorphisms

$$\sigma\colon G_1 * \ldots * G_n \to T,$$
$$\sigma'\colon T' \to G_1 \times \ldots \times G_n.$$

Certainly $T(G_1) = T'(G_1) = (e)$; also $T(G_1, G_2) = G_1 \times G_2$, $T'(G_1, G_2) = G_1 * G_2$ and, for $n=2$, $\sigma = \sigma'$ is the natural map $\varkappa$ from the free product $G_1 * G_2$ to the direct product (2) $G_1 \times G_2$. In fact, quite generally, σ' coincides with the map ${}^{n-1}\varkappa$ in the factorization

$$(\mathrm{F}') \qquad G_1 * \ldots * G_n = {}^1G \xrightarrow{{}^1\varkappa} {}^2G \to \ldots \to {}^qG \xrightarrow{{}^q\varkappa} {}^{q+1}G \to \ldots \to {}^{n-1}G \xrightarrow{{}^{n-1}\varkappa} {}^nG = G_1 \times \ldots \times G_n$$

of the natural map $\varkappa\colon G_1 * \ldots * G_n \to G_1 \times \ldots \times G_n$. We now prove

THEOREM 1.1. *For $q>1$ the map ${}^q\varkappa$ is an isomorphism: ${}^qG \to {}^{q+1}G$; in particular $\sigma'\colon T' \cong G_1 \times \ldots \times G_n$ if $n>2$.*

Proof. It is plainly sufficient to prove that ${}^q\lambda = \lambda\colon {}^qG \cong {}^nG$ is an isomorphism, where ${}^q\lambda = {}^{n-1}\varkappa \circ \ldots \circ {}^q\varkappa$, $q>1$. Since $\varkappa$ is onto so is λ and it remains to construct a left inverse of λ. To do this we first give a description of the construction of ${}^qG$ and λ appropriate to the category $\mathcal{G}$.

For any string I let $G^I = \underset{i \in I}{\times} G_i$. If $J \subseteq I$ there is a natural embedding $i_I^J\colon G^J \to G^I$; let $i^J\colon G^J \to {}^nG$ also be the embedding map. We form $\underset{|I|=q}{*} G^I$ and impose the relations

$$i_I^J(x) = i_{I'}^J(x), \quad x \in G^J, \quad J \subseteq I \cap I'.$$

The resulting group is ${}^qG$. The inclusions i^I together yield a map $\underset{|I|=q}{*} G^I \to {}^nG$ which respects the defining relations for ${}^qG$ and thus induces a map ${}^qG \to {}^nG$ which is precisely the map λ.

(2) These remarks follow from the general theory; see [3].

Let us write $x \equiv y$, where $x, y \in \underset{|I|=q}{\ast} G^I$, to indicate that x and y represent the same element of ${}^qG$ and let us write $\{x\}$ for that element; let us also write $(z)^I$ for the image, in $\ast\, G^I$, of $z \in G^I$. Finally, we will agree to identify $z \in G^J$ with $i_I^J z \in G^I$ if no confusion is to be feared. With these notational conventions we observe first that, if $g'_i \in G_i$, $g_j \in G_j$, $i \neq j$, then

$$(g'_i)^I (g_j)^J \equiv (g_j)^J (g'_i)^I\,, \qquad i \in I\,,\ j \in J\,. \tag{1.2}$$

For let K be any q-string containing i and j; then $(g'_i)^I \equiv (g'_i)^K$, $(g_j)^J \equiv (g_j)^K$, so that

$$(g'_i)^I (g_j)^J \equiv (g'_i)^K (g_j)^K = (g_j)^K (g'_i)^K \equiv (g_j)^J (g'_i)^I\,.$$

Now if we adjoin a finite number of trivial groups to the collection $G_1, \ldots, G_n$ we affect neither their direct product nor ${}^qG$; thus there is no real loss of generality in supposing that $q|n$, say $n = kq$. We suppose this and understand by $I(r)$ the q-string $rq+1, rq+2, \ldots, (r+1)q$, for $r = 0, \ldots, k-1$. We may now define $\tau\colon {}^nG \to {}^qG$ by

$$\tau(g_1, \ldots, g_n) = \{(g_1, \ldots, g_q)^{I(0)} (g_{q+1}, \ldots, g_{2q})^{I(1)} \ldots (g_{n-q+1}, \ldots, g_n)^{I(k-1)}\}\,.$$

We must first prove that τ is a homomorphism; to do this it is clearly sufficient to establish that $\{(g'_{rq+1}, \ldots, g'_{(r+1)q})^{I(r)}\}$ commutes with $\{g_{sq+1}, \ldots, g_{(s+1)q})^{I(s)}\}$ if $r \neq s$. But then $I(r)$ and $I(s)$ are disjoint and $(g'_{rq+1}, \ldots, g'_{(r+1)q})^{I(r)} = (g'_{rq+1})^{I(r)} \ldots (g'_{(r+1)q})^{I(r)}$, $(g_{sq+1}, \ldots, g_{(s+1)q})^{I(s)} = (g_{sq+1})^{I(s)} \ldots (g_{(s+1)q})^{I(s)}$. Thus the required commutativity relation follows from (1.2) and τ is a homomorphism.

Finally we show that $\tau\lambda = 1$. Let $p_r\colon G^I \to G^{I \cap I(r)}$ be the natural projection, $r = 0, \ldots, k-1$; if $I \cap I(r) = \emptyset$, $p_r(G^I) = e$. Then, if $p_r(x) = x_r$, $(x_r)^{I(r)} \equiv (x_r)^I$. Now to show that $\tau\lambda = 1$ it is sufficient to prove that $\tau\lambda\{x\} = \{x\}$ for $x \in G^I$; or, in the more elaborate notation, $\tau\lambda\{(x)^I\} = \{(x)^I\}$. But $\lambda\{(x)^I\} = i^I(x)$, so plainly $\tau\lambda\{(x)^I\} = \{(x_0)^{I(0)} \ldots (x_{k-1})^{I(k-1)}\} = \{(x_0)^I \ldots (x_{k-1})^I\} = \{(x)^I\}$; and the theorem is proved.

Corollary 1.3. *For any group G, $l'(G) < 3$.*

Now only the trivial group has length 0. It remains then to consider the groups of length $l' = 1$. These are precisely the non-trivial groups G admitting a map $\mu\colon G \to G \ast G$ such that $\sigma\mu = \Delta\colon G \to G \times G$, where Δ is the diagonal map and σ the natural map $\varkappa\colon G \ast G \to G \times G$. We refer to the pair (G, μ) as an *H'-object* and to μ as an *H'-structure* on G, and prove

Theorem 1.4. *The group G admits an H'-structure if and only if it is free. Thus $l'(G) = 1$ if and only if G is free and non-trivial.*

Proof. Let G be free and let $\{g_i\}$ be a set of free generators of G. Let $g_i \to g'_i$, $g_i \to g''_i$ copy G into the first and second factors of $G \ast G$ respectively; then the map $\mu\colon G \to G \ast G$ defined by $\mu(g_i) = g'_i g''_i$ is easily seen to be an H'-structure.

We now prove the converse. Consider, for any group G, the subgroup $F = \sigma^{-1}(\Delta G)$ of $G \ast G$. Grant for the moment that F is free. Then if μ is an H'-structure, μ maps G monomorphically into F, so that G is itself free.

It remains to show that F is free. This may be done by an application of the Kuros subgroup theorem, but a more direct argument is to hand [3].

Namely, we show that if $\{\nu_i\}$ ranges over the elements of G different from e, then the elements $\{\nu'_i\nu''_i\}$ constitute a free generating set for F. It is obvious that they are free. Now any element of F, being also an element of $G \ast G$, is expressible as

$$x = \alpha'_1 \beta''_1 \ldots \alpha'_n \beta''_n\,, \tag{1.5}$$

where $\alpha_i, \beta_i \in G$. We argue by induction on the 'length' n that x belongs to the group generated by the elements $\nu'_i\nu''_i$. If $n = 1$, then $\sigma x = (\alpha_1, \beta_1)$ so that $\alpha_1 = \beta_1$ and $x = e$ or x is itself one of the proposed generators. Now suppose the assertion proved for elements of length $< n$, and consider (1.5). Then if $y = \alpha_1''^{-1} \alpha_1'^{-1} x \beta_n''^{-1} \beta_n'^{-1}$, it is clear that y^{-1} is of length $< n$. It follows that y is expressible in terms of the chosen generators and so therefore is x. This completes the proof.

In considering H'-structures in a general category we may investigate associativity and commutativity. An associative H'-structure is a *comonoid structure* in the sense of [7] and we have Kan's theorem ([7], Theorem 3.10).

Theorem 1.6. *Let (G, μ) be an associative H'-object. Then G is a free group and those elements $g \in G$, different from e, such that $\mu(g) = g'g''$ constitute a set of free generators of G.*

For the sake of completeness we sketch briefly the proof of Kan's theorem. It is already clear from the proof of Theorem 1.4 that the elements $g \neq e$ such that $\mu(g) = g'g''$ constitute a free set of elements, and it remains to show that they generate G if μ is associative. Let $y \in G$ and let $\mu(y) = \alpha'_1\beta''_1 \ldots \alpha'_n\beta''_n$ (compare (1.5)). Then it is not difficult to show that the associativity of μ implies that $\mu(\alpha_1) = \alpha'_1\alpha''_1$, $\mu(\beta_n) = \beta'_n\beta''_n$, and Theorem 1.6 then follows by induction with respect to the integer n.

Theorems 1.4 and 1.6 show that, within the category of groups, we may characterize both free groups and their sets of free generators without any reference to the elements of the groups in question, but simply in terms of the maps of the category.

Certainly any group admitting an H'-structure admits an associative H'-structure; for these groups are just the free groups and the H'-structure described in the proof of theorem 1.4 is clearly associative. Theorem 1.6 shows that every associative H'-structure is, in fact, of the described form.

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[3] This argument was suggested to us by M. Kneser.

However there do exist non-associative H'-structures. Thus if G is free cyclic generated by g and if c is an element of $[G', G'']$ different (4) from e or $[g'', g']$, then $\mu(g) = g'g''c$ is a non-associative H'-structure. Thus non-trivial free groups admit infinitely many H'-structures; we stress this because of the contrast with H-structures below.

We now prove

THEOREM 1.7. *An H'-structure on a non-trivial free group is non-commutative.*

Proof. Let g belong to a free generating set for the free group G and let $\mu\colon G \to G \ast G$ be an H'-structure. Then there exists $c \in [G', G'']$ such that

$$\mu g = g'g''c\,.$$

Let $\tau\colon G\ast G \to G \ast G$ interchange the factors. Then τ maps $[G', G'']$ into itself and

$$\tau\mu g = g''g'\bar{c}\,,$$

where $\bar{c} = \tau c$. Thus if μ is commutative, $g'g''c = g''g'\bar{c}$, or

$$[g', g''] = \bar{c}c^{-1}\,. \tag{1.8}$$

Now $[G', G'']$ is a free group freely generated by elements $[\alpha', \beta'']$ where $\alpha, \beta \in G$, $\alpha \neq e$, $\beta \neq e$; and $\bar{c}$, c clearly have the same length with respect to this generating set. Thus (1.8) purports to be a relation between members of a free generating set which is impossible. It follows that μ is not commutative.

Recall that, given two H'-objects (G, μ) and (K, ν), a *primitive* map $\Phi\colon (G, \mu) \to (K, \nu)$ is a map $\Phi\colon G \to K$ such that $(\Phi \ast \Phi)\circ\mu = \nu\circ\Phi$. Now by Theorem 1.6 an associative H'-object is just a free group with a preferred free generating set. We may thus assert

THEOREM 1.9. *Let (G, μ), (K, ν) be two associative H'-objects. Then a map $\Phi\colon G \to K$ is primitive if and only if Φ maps each preferred generator of G to e or to a preferred generator of K.*

Proof. The sufficiency is obvious. Conversely let Φ be primitive and let g be a preferred generator of G. Then if $\Phi g = h$, $\nu h = (\Phi\ast\Phi)\mu g = (\Phi\ast\Phi)g'g'' = (\Phi g)'(\Phi g)'' = h'h''$. Thus $h = e$ or h is a preferred generator of K.

COROLLARY 1.10. *Let (G, μ) be an H'-object and let $G\ast G$ be given the induced H'-structure. Then if G is non-trivial $\mu\colon G \to G\ast G$ is not primitive*

Proof. By the general theory μ is primitive if and only if it is associative and commutative. Thus the corollary follows from Theorem 1.7 or from Theorem 1.9.

(4) We use G', G'' for the first and second factors, respectively, of $G \ast G$, considered as subgroups of $G \ast G$.

Again we may point out the contrast with H-structures in $\mathcal{G}$; see Corollary 1.14. In fact we now turn our attention to the dual story. Then the group $T(G_1, \ldots, G_n)$ is just the group G^{n-1} considered in [5] and the natural homomorphism σ is the epimorphism

$$\sigma = \varkappa^{n-1}\colon G^n \to G^{n-1}\,.$$

Now consider the case $G_1 = \ldots = G_n = G$; then $l(G) < n$ if and only if there is a map $\mu\colon G^{n-1} \to G$ such that $d = \mu\sigma\colon G^n \to G$, where d is the homomorphism $G^n \to G$ which is the identity on each factor of G^n. However, since σ is an epimorphism, we have

PROPOSITION 1.11. *Let $[G^n]$ be the kernel of σ. Then $l(G) < n$ if and only if $d[G^n] = (e)$. Moreover if $l(G) < n$ the map $\mu\colon G^{n-1} \to G$ is uniquely determined.*

Again $l(G) = 0$ if and only if G is trivial. We study groups G with $l(G) > 1$ in the next section and direct our main attention here to groups G with $l(G) = 1$. The situation is very simple and is described in the following theorem.

An H-object is a pair (G, μ) consisting of a group G and a map $\mu\colon G \times G \to G$ such that $d = \mu\sigma\colon G \ast G \to G$, and $l(G) = 1$ if and only if G is non-trivial and admits an H-structure. Then

THEOREM 1.12. *The group G admits an H-structure if and only if it is abelian. The H-structure is then uniquely determined and is associative and commutative.*

Proof. Let G be an abelian group (5). Then the homomorphism $\mu\colon G\times G \to G$, given by

$$\mu(g, h) = gh \tag{1.13}$$

is plainly an H-structure. Conversely let $\mu\colon G\times G \to G$ be an H-structure and let $g, h \in G$. Then $\mu(g, e) = g$, $\mu(e, h) = h$. But $(g, e)(e, h) = (e, h)(g, e)$ so that $gh = hg$ and G is abelian.

It now follows from Proposition 1.11 that (1.13) is the unique H-structure on G if G is abelian. It is manifestly associative and commutative.

COROLLARY 1.14. *If (G, μ) is an H-object and $G\times G$ is given the induced H-structure then $\mu\colon G\times G \to G$ is primitive. Indeed, for any two abelian groups G, K every homomorphism $G \to K$ is primitive.*

This last result illustrates a general theorem on categories in which the natural map σ is an epimorphism. Confining our attention still to the category $\mathcal{G}$, suppose $l(G) < n$ and $\mu\colon G^{n-1} \to G$ is the structure map. Suppose also $l(K) < n$ with structure map $\nu\colon K^{n-1} \to K$; then a homomorphism

(5) Which we will *not* write additively.

$\Phi\colon G\to K$ is primitive if $\Phi\mu = \nu\Phi^{n-1}$ where $\Phi^{n-1}\colon G^{n-1}\to K^{n-1}$ is induced by Φ.

PROPOSITION 1.15. *Every homomorphism $\Phi\colon G\to K$ is primitive.*

Proof. Consider the diagram

$$\begin{array}{ccccc} G^n & \xrightarrow{\sigma} & G^{n-1} & \xrightarrow{\mu} & G \\ \downarrow{\scriptstyle\Phi^n} & & \downarrow{\scriptstyle\Phi^{n-1}} & & \downarrow{\scriptstyle\Phi} \\ K^n & \xrightarrow{\sigma} & K^{n-1} & \xrightarrow{\nu} & K \end{array}$$

Then $\Phi\mu\sigma = \Phi d = d\Phi^n = \nu\sigma\Phi^n = \nu\Phi^{n-1}\sigma$; but $\sigma\colon G^n\to G^{n-1}$ is onto so $\Phi\mu = \nu\Phi^{n-1}$.

2. The projection kernels [6]. Let $I = (i_1, \ldots, i_r)$ be an r-string, that is, an ordered subset of the ordered set $(1, 2, \ldots, n)$ containing r elements. We define an *I-commutator* in $G^n = G_1 * \ldots * G_n$ inductively with respect to $|I| = r$. Thus if $|I| = 1$ and $I = (i)$ then an I-commutator is just an element of $\bar{G}_i$, the normal closure of G_i in G^n. If $|I| > 1$ then an *I-commutator* is a commutator $[g_J, g_K]$ where (J, K) is a partition of I and g_J is a J-commutator, g_K is a K-commutator. Further a *left-normed* I-commutator is defined inductively in the obvious way: every I-commutator is left-normed if $|I| = 1$ and, if $|I| > 1$, a left-normed I-commutator is an I-commutator $[g_J, g_K]$ such that $|J| = 1$ and g_K is left-normed. Let $N(I) = N(i_1, \ldots, i_r)$ be the subgroup of G^n generated by all I-commutators and let $N_l(I)$ be the subgroup of G^n generated by all left-normed I-commutators. A simple inductive argument shows that a conjugate of a (left-normed) I-commutator is a (left-normed) I-commutator, so that $N(I)$ and $N_l(I)$ are normal subgroups of G^n. We first prove some elementary propositions about I-commutators.

PROPOSITION 2.1. *Every right-normed I-commutator (defined in the obvious way) is equal to a left-normed I-commutator.*

Proof. We use the commutator identity

$$[a, b] = [b^{-1}, a^b] \tag{2.2}$$

to establish the induction on $|I|$. For if x is a right-normed I-commutator then $x = [y, z]$ where y is a right-normed I_j-commutator and $z \in \bar{G}_j$ for some $j \in I$. By the inductive hypothesis $y = y'$, where y' is a left-normed I_j-commutator so

$$x = [y, z] = [y', z] = [z^{-1}, y'^z]$$

and thus x is equal to a left-normed I-commutator.

Since the inverse of a left-normed I-commutator is a right-normed I-commutator, it follows from Proposition 2.1 that every element of $N_l(I)$ is expressible as a product of left-normed I-commutators.

[6] See the introduction for an explanation of the notations used.

PROPOSITION 2.3. *If (J, K) is a partition of I then*

$$[N_l(J), N_l(K)] \subseteq N_l(I).$$

Proof. It is classic that if A, B, C are normal subgroups of a group G then

$$[[A, B]C] \subseteq [[B, C]A][[C, A], B]. \tag{2.4}$$

We now prove the proposition by induction on $|J|$. If $|J| = 1$, then every element of $N_l(K)$ is expressible as a product of left-normed K-commutators and the commutator identity

$$[x, yz] = [x, z][x, y]^z \tag{2.5}$$

shows that every commutator $[u, v]$, $u \in N_l(J)$, $v \in N_l(K)$, belongs to $N_l(I)$ so that the proposition is proved in this case.

Now suppose the proposition proved for partitions (J', K') of I with $|J'| < |J|$, where $|J| \geqslant 2$. In the light of (2.5) it is sufficient to prove that if u is a left-normed J-commutator and v is left-normed K-commutator then $[u, v] \in N_l(I)$. Since $|J| \geqslant 2$, $u = [s, t]$ where $s \in \bar{G}_j$ and t is a left-normed J_j-commutator for some $j \in J$. Thus

$$\begin{aligned} [u, v] &\in [[\bar{G}_j, N_l(J_j)], N_l(K)] \\ &\subseteq [\bar{G}_j, [N_l(J_j), N_l(K)]][N_l(J_j), [\bar{G}_j, N_l(K)]] \quad \text{by (2.4)} \\ &\subseteq [\bar{G}_j, N_l(I_j)][N_l(J_j), N_l(K^j)], \quad \text{by the inductive hypothesis} \\ &\subseteq N_l(I), \quad \text{again by the inductive hypothesis.} \end{aligned}$$

THEOREM 2.6. $N(I) = N_l(I)$.

Proof. This is trivial if $|I| = 1$; assume it true for strings I' with $|I'| < |I|$, and $|I| \geqslant 2$. Then it is sufficient to prove that if $y \in N(J)$, $z \in N(K)$ for some partition (J, K) of I then $[y, z] \in N_l(I)$. But

$$\begin{aligned} [y, z] &\in [N(J), N(K)] \\ &= [N_l(J), N_l(K)] \quad \text{by the inductive hypothesis} \\ &\subseteq N_l(I) \quad \text{by Propositions 2.3.} \end{aligned}$$

COROLLARY 2.7. *Every I-commutator is expressible as a product of left-normed I-commutators.*

It is evident that if π_i projects G^n off G_i and if $i \in I$ then $\pi_i(N(I)) = (e)$. Thus

$$N(I) \subseteq \ker \pi_i = \bar{G}_i,$$

so that

$$N(I) \subseteq \bigcap_{i \in I} \bar{G}_i. \tag{2.8}$$

The main theorem of this section establishes that the inclusion (2.8) is actually an equality:

THEOREM 2.9. $N(I) = \bigcap_{i \in I} \bar{G}_i$.

Proof. We have only to establish the inclusion $\bigcap_{i \in I} \bar{G}_i \subseteq N(I)$. This is trivial if $|I| = 1$ and we proceed to prove it if $|I| = 2$. It is then plainly sufficient to take $n = 3$; for this case we change notation, writing F, G, H for G_1, G_2, G_3 where $I = (1, 2)$. We write N for $N(I) = [\bar{F}, \bar{G}]$.

Let $x \in \bar{F} \cap \bar{G}$; then x may be written as

$$x = f_1 g_1 h_1 f_2 g_2 h_2 \ldots f_m g_m h_m, \; f_i \in F, \; g_i \in G, \; h_i \in H,$$

where $g_1 h_1 g_2 h_2 \ldots g_m h_m = f_1 h_1 f_2 h_2 \ldots f_m h_m = e$. Then

$$\begin{aligned} x &= f_1 g_1 h_1 \ldots f_{m-1} g_{m-1} h_{m-1} f_m h_{m-1}^{-1} g_{m-1}^{-1} \ldots h_1^{-1} g_1^{-1} \\ &\equiv f_1 g_1 h_1 \ldots f_{m-2} g_{m-2} h_{m-2} f_{m-1} h_{m-1} f_m h_{m-1}^{-1} h_{m-2}^{-1} g_{m-2}^{-1} \ldots h_1^{-1} g_1^{-1} \operatorname{mod} N, \\ &\quad \text{commuting } g_{m-1} \text{ with } h_{m-1} f_m h_{m-1}^{-1}, \\ &\equiv f_1 g_1 h_1 \ldots f_{m-3} g_{m-3} h_{m-3} f_{m-2} h_{m-2} f_{m-1} h_{m-1} f_m h_{m-1}^{-1} h_{m-2}^{-1} h_{m-3}^{-1} g_{m-3}^{-1} \ldots \\ &\qquad h_1^{-1} g_1^{-1} \operatorname{mod} N, \\ &\quad \text{commuting } g_{m-2} \text{ with } h_{m-2} f_{m-1} h_{m-1} f_m h_{m-1}^{-1} h_{m-2}^{-1}, \\ &\quad \ldots \\ &\equiv f_1 h_1 \ldots f_{m-1} h_{m-1} f_m h_{m-1}^{-1} h_{m-2}^{-1} \ldots h_1^{-1} \operatorname{mod} N, \\ &= f_1 h_1 \ldots f_{m-1} h_{m-1} f_m h_m \\ &= e. \end{aligned}$$

Thus the theorem is proved if $|I| = 2$.

We now proceed by induction on $|I|$, assuming the theorem true for strings I' with $|I'| < |I|$, where $|I| > 2$. There is clearly no real loss of generality in taking $I = (1, 2, \ldots, r)$ and this we will henceforth do. With a view to establishing the inductive step we prove a lemma and its corollary.

LEMMA 2.10. *Let (J, K) be a partition of $(1, 2, \ldots, s-1)$, let $a, a' \in N(J)$, $b, b' \in N(K)$, and let $a \equiv a' \operatorname{mod} N(J^s)$, $b \equiv b' \operatorname{mod} N(K^s)$. Then*

$$[a, b] \equiv [a', b'] \operatorname{mod} N(1, 2, \ldots, s).$$

Proof. We have $a = ua'$, $b = vb'$ where $u \in N(J^s)$, $v \in N(K^s)$. Thus

$$\begin{aligned} [a, b] &= [ua', b] \\ &= [u, b]^{a'} [a', b] \quad \text{by} \quad (2.5) \\ &\equiv [a', b] \operatorname{mod} N(1, 2, \ldots, s) \\ &= [a', vb'] \\ &= [a', b'][a', v]^{b'} \quad \text{by} \quad (2.5) \\ &\equiv [a', b'] \operatorname{mod} N(1, 2, \ldots, s). \end{aligned}$$

COROLLARY 2.11. *If $j_1, \ldots, j_{t-1}$ is a permutation of $(1, 2, \ldots, t-1)$, if $g_k, g'_k \in \bar{G}_{j_k}$ and $g'_k \equiv g_k \operatorname{mod} N(j_k, t)$, $k = 1, \ldots, t-1$, then $[g_1, \ldots, g_{t-1}] \equiv [g'_1, \ldots, g'_{t-1}] \operatorname{mod} N(1, 2, \ldots, t)$.*

Proof. We apply Lemma 2.10 to prove that $[g_m, \ldots, g_{t-1}] \equiv [g'_m, \ldots, g'_{t-1}] \operatorname{mod} N(j_m, \ldots, j_{t-1}, t)$ successively for $m = t-1, \ldots, 2, 1$.

We now return to the proof of the inductive step establishing Theorem 2.9. Let $x \in \bigcap_{i \in I} \bar{G}_i$; then certainly $x \in \bigcap_{i \in I_r} \bar{G}_i$ so that, by the inductive hypothesis,

$$x = c_1 c_2 \ldots c_q$$

where $c_j = [g_{j_1}, \ldots, g_{j_{r-1}}]$, $g_{j_k} \in \bar{G}_{j_k}$, $j = 1, \ldots, q$, $k = 1, \ldots, r-1$; $j_1, \ldots, j_{r-1}$ being a permutation P_j of $1, \ldots, r-1$. We now project c_j off G_r. We obtain an element

$$c'_j = [g'_{j_1}, \ldots, g'_{j_{r-1}}],$$

where g'_{j_k} is the projection of g_{j_k} off G_r. Then clearly $g_{j_k} \equiv g'_{j_k} \operatorname{mod} \bar{G}_{j_k} \cap \bar{G}_r$. But since our theorem is already proved for 2-strings we know that

$$\bar{G}_{j_k} \cap \bar{G}_r = N(j_k, r).$$

Thus

$$g_{j_k} \equiv g'_{j_k} \operatorname{mod} N(j_k, r), \qquad k = 1, 2, \ldots, r-1,$$

and we may apply Corrollary 2.11 to infer that

$$c_j \equiv c'_j \operatorname{mod} N(I).$$

Then

$$x \equiv c'_1 c'_2 \ldots c'_q \operatorname{mod} N(I);$$

but $c'_1 c'_2 \ldots c'_q$ is just the projection of x off G_r and so $c'_1 c'_2 \ldots c'_q = e$ since $x \in \bar{G}_r$. Thus $x \in N(I)$ and the theorem is proved.

In [5]—and, of course, in the study of l-length $l(G)$—we were particularly interested in the case in which I is the full set $(1, 2, \ldots, n)$. Then Theorem 2.9 implies (see also [1])

COROLLARY 2.12. *$[G^n]$ is generated by the set of all (left-normed) commutators*

$$[g_{j_1}, \ldots, g_{j_n}], \; g_{j_k} \in \bar{G}_{j_k}, \qquad k = 1, \ldots, n.$$

where $j_1, \ldots, j_n$ is a permutation of $1, \ldots, n$.

In [5] we introduced the homomorphism $\lambda^p \colon G^n \to G^p$, $1 < p < n$, defined in terms of the factorization (F)

$$G^n \xrightarrow{\varkappa^{n-1}} G^{n-1} \to \ldots \to G^2 \xrightarrow{\varkappa^1} G^1 \tag{2.13}$$

of the natural map $\varkappa$ by $\lambda^p = \varkappa^p \circ \ldots \circ \varkappa^{n-1}$. Then $\lambda^1 = \varkappa$, $\lambda^{n-1} = \varkappa^{n-1} = \sigma$ and $[G^n] = \ker \lambda^{n-1}$; the group $\ker \lambda^1$ is the Cartesian subgroup of G^n and the groups $\ker \lambda^p$ were called in [1] *generalized Cartesian subgroups* of G^n.

For any string I let us call the left-normed I-commutator $[g_{j_1}, \dots, g_{j_r}]$ *special* if g_{j_k} belongs to the normal closure of G_{j_k} in G_I, $k = 1, \dots, r = |I|$. Then the following assertion is plainly a generalization of Corollary 2.12.

THEOREM 2.14. *$\ker\lambda^p$ is generated by the set of all special left-normed I-commutators, where I ranges over all strings such that $|I| > p$.*

Proof. Define $\lambda^{p,q} = \varkappa^p \circ \dots \circ \varkappa^{q-1}$: $G^q \to G^p$ for $q > p$. Then $\lambda^{p,n} = \lambda^p$ and $\varkappa^{q-1}$ plainly maps $\ker\lambda^{p,q}$ into $\ker\lambda^{p,q-1}$. Let $\bar\varkappa^{q-1}$: $\ker\lambda^{p,q} \to \ker\lambda^{p,q-1}$ be the induced map, so that (2.13) induces the factorization

$$\ker\lambda^p \xrightarrow{\bar\varkappa^{n-1}} \ker\lambda^{p,n-1} \to \dots \xrightarrow{\bar\varkappa^{p+1}} \ker\lambda^{p,p+1}. \tag{2.15}$$

We now consider the transformation τ^q: $G^q \to G^n$ given by (2.12) of [5], and claim that it maps $\ker\lambda^{p,q}$ into $\ker\lambda^p$. For $\ker\lambda^{p,q}$ is just $F_p(G^q)$ and it is plain that, if $g \in F_p(G^q)$, then $\theta_0(g) = \dots = \theta_p(g) = g$, $\beta_0(g) = \dots = \beta_{p-1}(g) = e$. Since θ_s maps into $F_s(G^q)$ and β_s maps $F_s(G^q)$ into $F_s(G^n)$, and since the subgroups F_s form a decreasing filtration, the assertion follows from the definition of τ^q.

It is now immediate that the transformation η^q: $G^q \to G^{q+1}$ of Theorem 2.5 of [5] maps $\ker\lambda^{p,q}$ into $\ker\lambda^{p,q+1}$ and thus induces $\bar\eta^q$: $\ker\lambda^{p,q} \to \ker\lambda^{p,q+1}$ such that $\bar\varkappa^q\bar\eta^q = 1$. Since $\ker\varkappa^q \subseteq \ker\lambda^{p,q+1}$, it follows that the map σ^q of Proposition 2.19 of [5] maps $\bigtimes_{|I|=q+1} [G_I] \times \ker\lambda^{p,q}$ onto $\ker\lambda^{p,q+1}$. Thus, by iteration, the (1,1) correspondence

$$\tau\colon \bigtimes_I [G_I] \leftrightarrow G^n$$

of Theorems 2.20, 2.21 of [5] maps $\bigtimes_{|I|>p} [G_I]$ onto $\ker\lambda^p$. This proves the theorem.

COROLLARY 2.16. *$\ker\lambda^p$ is the normal closure of the group generated by all left-normed I-commutators with $|I| = p+1$.*

Remark 1. It follows readily from the proof of Proposition 2.1 that every special right-normed I-commutator is equal to a special left-normed I-commutator. Thus in Corollary 2.12, Theorem 2.13 and Corollary 2.16 the given generating sets generate the appropriate groups *qua semigroups*—inverses are not required.

Remark 2. $N(I)$ is, of course, generated by all left-normed I-commutators. However, it is clear that it is also generated by the smaller set consisting of left-normed I-commutators

$$[g_{j_1}^{\gamma_1}, \dots, g_{j_r}^{\gamma_r}],$$

where $g_{j_k} \in G_{j_k}$, $k = 1, \dots, r$; $\gamma_k \in G^n$, $k = 1, \dots, r$; and $j_1, \dots, j_r$ is a permutation of the elements of I. This follows by an evident induction, using (2.5). Here again $N(I)$ is generated qua semigroup by the given set of elements.

Let us now suppose that $G_1 = \dots = G_n = G$ and let d: $G^n \to G$ be the map which is the identity on each factor. From Corollary 2.16 we immediately infer (see [1])

THEOREM 2.17. *The homomorphism d: $G^n \to G$ maps $\ker\lambda^p$ onto the $(p+1)$st term $G^{(p+1)}$ of the lower central series of G. Thus $d[G^n] = d\ker\sigma = d\ker\lambda^{n-1} = G^{(n)}$; hence the length of G is precisely its nilpotency class.*

We close this section with a brief study of the algebraic structure of $N(I)$. We first observe

THEOREM 2.18. *If $|I| > 1$, $N(I)$ is free.*

Proof. It is clearly sufficient to consider the case $|I| = 2$, and we may then suppose that $n = 3$. It is clear however that $[\bar G_1, \bar G_2] = \bar G_1 \cap \bar G_2$ is contained in the Cartesian subgroup of $G_1 * G_2 * G_3$. Since the Cartesian subgroup is free (see [4]) so is $[\bar G_1, \bar G_2]$.

Let $G_i \supset H_i$, $i = 1, \dots, n$, and let $N(I; G)$, $N(I; H)$ refer to the groups $N(I)$ computed from the groups $G_1, \dots, G_n$; $H_1, \dots, H_n$ respectively. Then Higman's argument in [5] applies here to show that, if $|I| > 1$, $N(I; G)$ is a free factor in $N(I; H)$. We may also infer easily from the definition of $N(I)$ that epimorphisms Φ_i: $G_i \to H_i$ induce an epimorphism of $N(I; G)$ onto $N(I; H)$.

We would wish to show that if each G_i is a *group complex* and $|I| > 1$ then $N(I)$ is a free group complex (see [6]). The argument given by Cohen in [1] will apply to this situation provided we can establish the facts when $|I| = 2$. This we now proceed to do.

Let F, G, H be 3 groups. We will describe a set of free generators of the group $\bar F \cap \bar G \subseteq F * G * H$; this set will be canonical in a sense which will be made clear later. Let, then, S be the set of commutators

$$[f_1^{h_1} \dots f_m^{h_m}, g_1^{k_1} \dots g_n^{k_n}],$$

where $f_i \in F$, $g_i \in G$, $h_i, k_i \in H$, $f_i \neq e$, $g_i \neq e$, $h_i \neq h_{i+1}$, $k_i \neq k_{i+1}$.

We prove

THEOREM 2.19. *The set S is a free generating set for $\bar F \cap \bar G$.*

Proof. Let T be the set S enlarged by the removal of the restrictions $f_i \neq e$, $g_i \neq e$, $h_i \neq h_{i+1}$, $k_i \neq k_{i+1}$. Then the group N generated by T coincides with the group generated by S and we prove that S is a generating set for $\bar F \cap \bar G$ by showing that $N = \bar F \cap \bar G$.

We first observe that N is normal in $F * G * H$; this follows from the identities

$$[\gamma, \delta]^f = [\gamma f, \delta][\delta, f],$$
$$[\gamma, \delta]^g = [g, \gamma][\gamma, \delta g],$$
$$[\gamma, \delta]^h = [\gamma^h, \delta^h],$$

where $f \in F$, $g \in G$, $h \in H$, $\gamma = f_1^{h_1} \dots f_m^{h_m}$, $\delta = g_1^{k_1} \dots g_n^{k_n}$.

Now, by Remark 2, $\bar{F} \cap \bar{G}$ is generated by commutators $[f^\alpha, g^\beta]$, $f \in F$, $g \in G$, $\alpha, \beta \in F*G*H$. Thus it remains to show that $[f^\alpha, g^\beta] \in N$ for all f, g, α, β; by the normality of N it is sufficient to take $\beta = e$. We will complete the argument with the help of the following reduction lemma.

LEMMA 2.20. *Let $f \in F$, $g \in G$, $u, v, w, \in F*G*H$ and let $[f^{uw}, g] \in N$. Then $[f^{uvw}, g] \in N$ if* (i) *$v \in F$ and $[(v^{-1})^w, g] \in N$ or* (ii) *$v \in G$ and $[f^u, v] \in N$.*

Proof of Lemma. Consider first the case (ii). We use the identity

$$[\varrho^\sigma, \tau] = [\sigma, \varrho][\varrho, \tau][\varrho, \sigma]^\tau. \tag{2.21}$$

Thus $[f^{uv}, g^{w^{-1}}][= [v, f^u][f^u, g^{w^{-1}}][f^u, v]^\tau$, where $\tau = g^{w^{-1}}$. By hypothesis $[f^u, v] \in N$ and $[f^{uw}, g] \in N$ and N is normal. Thus $[f^{uv}, g^{w^{-1}}] \in N$ so that $[f^{uvw}, g] \in N$ as asserted.

Case (i) is proved in the same way: we consider $[f, g^{(uvw)^{-1}}]$ and interchange the roles of F and G. We omit the details and return to the proof of the Theorem.

Any element $\alpha \in F*G*H$ is expressible as

$$\alpha = f_1 g_1 h_1 \dots f_n g_n h_n, \quad f_i \in F, \quad g_i \in G, \quad h_i \in H;$$

if α is precisely so expressible we write $l(\alpha) \leqslant n$, and say that α admits length n. Suppose that α admits length 1 so that $\alpha = f_1 g_1 h_1$. Now obviously $[f^{f_1 h_1}, g] = [(f^{f_1})^{h_1}, g] \in N$ and $[f^{f_1}, g_1] \in N$ so that, by 2.20 (ii), $[f^{f_1 g_1 h_1}, g] \in N$. Suppose now that $[f^\beta, g] \in N$ for all $f \in F$, $g \in G$, $\beta \in F*G*H$ with $l(\beta) \leqslant n-1$, where $n \geqslant 2$; and let $\alpha = \beta f_n g_n h_n$, with $l(\beta) \leqslant n-1$. By the inductive hypothesis $[f^{\beta h_n}, g] \in N$ and $[f^\beta, g_n] \in N$ so that, by 2.20 (ii), $[f^{\beta g_n h_n}, g] \in N$. But we have proved that $[(f_n^{-1})^{g_n h_n}, g] \in N$ so that, by 2.20 (i), $[f^{\beta f_n g_n h_n}, g] \in N$. This completed the induction and establishes that S is a generating set for $\bar{F} \cap \bar{G}$.

It remains to prove that S is a free generating set. The reader will be familiar with the argument when the group H is trivial [7]. The argument in the general case is essentially the same and we will omit the details. This completes the proof of Theorem 2.19.

The free generating set S is, in a slightly generalized sense, verbal. Certainly it has the property that, if F_0, G_0, H_0 are subgroups of F, G, H and if S_0 is the free generating set for $\bar{F}_0 \cap \bar{G}_0$ (normal closure in $F_0 * G_0 * H_0$), then

$$S_0 = S \cap (\bar{F}_0 \cap \bar{G}_0). \tag{2.22}$$

For plainly $S_0 \subset S$; and if $s \in S \cap (\bar{F}_0 \cap \bar{G}_0)$ then s is expressible in terms of the members of S_0 and this would contradict the freedom of S unless

[7] If $c_i = [f_i, g_i]$, $i = 1, 2, \dots$, then $c_1^{\varepsilon_1} c_2^{\varepsilon_2} \dots = e$ only if, for some i, $\varepsilon_i = -\varepsilon_{i+1}$ and $c_i = c_{i+1}$.

$s \in S_0$. Relation (2.22)—or indeed the nature of S itself—shows that if $G_1, \dots, G_n$ are complexes and $|I| = 2$, then $N(I)$ is a free group complex. We may then use the arguments of [1] to show that 'verbal' free generating sets may be obtained for $N(I)$ whenever $|I| \geqslant 2$ and thus, if $G_1, \dots, G_n$ are group complexes, $N(I)$ is a free group complex provided $|I| \geqslant 2$.

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