Remarks on the homotopic join of maps

by

K. Borsuk (Warszawa)

1. Introduction.. Let Y^X denote the set of all continuous maps of a space X into another space Y. A map $f \in Y^X$ is said to be a *homotopic join* of maps $f_1, f_2 \in Y^X$ (see [1] and [2]) provided there exist two disjoint open subsets G_1, G_2 of X, a point $y_0 \in Y$ and three maps $f', f_1', f_2' \in Y^X$ homotopic to f, f_1, f_2 respectively, such that

$$f_1'(X-G_1)\,,\ f_2'(X-G_2) \subset (y_0)\,,$$

and

$$f'(x) = \begin{cases} y_0 & \text{for} \quad x \in X - G_1 - G_2\,, \\ f_1'(x) & \text{for} \quad x \in G_1\,, \\ f_2'(x) & \text{for} \quad x \in G_2\,. \end{cases}$$

In the case when X is the n-sphere ($n > 0$), the operation of the homotopic joining is always performable and it leads (for a fixed point y_0) to the notion of the n-dimensional homotopy group $\pi_n(Y)$ of the space Y ([6]). In the case when Y is the n-sphere, the operation of the homotopic join is performable under the hypothesis $\dim X < 2n$. Then the homology types $\{f\}$ of maps $f \in Y^X$ (i.e. the subsets of Y^X consisting of all maps homologous one to another) constitute by the operation of the homotopic join, an abelian group. If Y is the n-sphere and $\dim X < 2n-1$ then the operation of the homotopic join leads to the n-dimensional cohomotopy group $\pi^n(X)$ of the space X (see [1] and [7]). However in the case of arbitrary spaces X and Y the operation of the homotopic join is in general not performable (see [3]), and consequently it does not allow to introduce in Y^X the structure of a group.

In the present note I prove that in the case when X is an n-dimensional (finite) polytope, Y is an ANR-set and k is an integer $> \frac{n}{2} - 1$, the operation of the homotopic join is performable in the set $Y^X(k/y_0)$ consisting of all such maps $f \in Y^X$ which are homotopic to a constant $y_0 \in Y$ on every, at most k-dimensional, closed subset of X. It follows, in particular, that in this case the homology types $\{f\}$ of the maps $f \in Y^X(k/y_0)$ constitute, by the operation of the homotopic join, an abelian group.

2. Condensators. A set A is said to be a *condensator for a map* $f \in Y^X$ rel. $y_0 \in Y$ provided:

1. A is a closed subset of X,

2. For every neighbourhood U of A in X there exists a map $f_U \in Y^X$ homotopic to f and such that $f_U(x) = y_0$ every for point $x \in X - U$.

Evidently if A is a condensator for $f \in Y^X$ rel. $y_0 \in Y$, and if B is a closed subset of X containing A, then B is also a condensator for f rel. y_0.

If M is a subset of Y^X, that is a class of maps of X into Y, and if A is a condensator rel. $y_0 \in Y$ for every map $f \in M$, then A is said to be a *condensator* rel. y_0 *for the class* M.

Examples. 1. If a map $f \in Y^X$ is homotopic to the constant y_0 then every closed subset of X is a condensator for f rel. y_0.

2. If X is the n-sphere S_n and Y is a connected ANR-set then every closed and not empty subset of X is a condensator for Y^X relatively to every point y_0 of Y.

3. If Y is the n-sphere then for every map $f \in Y^X$ and every point $y_0 \in Y$ the set $f^{-1}(y_0)$ is a condensator for f rel. y_0.

4. If Y is a connected ANR-set, y_0 a point of Y, and A a closed subset of X such that $X - A$ is contractible over X to a point, then A is a condensator rel. y_0 for Y^X.

5. Let A be a condensator rel. y_0 for the class Y^X of all continuous maps of X into an ANR-set Y and let h be a homeomorphism mapping X onto itself. Then the set $A' = h(A)$ is also a condensator rel. y_0 for Y^X.

In order to prove it, consider a neighbourhood V of A' in X. Then $U = h^{-1}(V)$ is a neighbourhood of A in X. Now, if f is an arbitrary map of X into Y then there exists a map $g \in Y^X$ homotopic to fh and satisfying the condition

$$g(x) = y_0 \quad \text{for} \quad x \in X - U .$$

It is clear that the map $gh^{-1} \in Y^X$ is homotopic to $fhh^{-1} = f$ and we have

$$gh^{-1}(X - V) = gh^{-1}\big(X - h(U)\big) = g(X - U) \subset (y_0) .$$

Consequently A' is a condensator for f rel. y_0.

6. Let A be a closed subset of X and B be a subset of X which we obtain from $X - A$ by a continuous deformation over X. Let y_0 be a point of an ANR-set Y. If a map $f \in Y^X$ is homotopic to y_0 on B, i.e. the partial map $f/B \in Y^B$ is homotopic to the constant y_0, then A is a condensator for f rel. y_0.

In order to prove it, let us consider a continuous deformation φ of $X - A$ into B over X. Hence $\varphi(x, t)$ is defined and continuous for $(x, t) \in (X - A) \times \langle 0, 1\rangle$ and

$$\varphi(x, t) \in X \quad \text{for every} \quad (x, t) \in (X - A) \times \langle 0, 1\rangle ,$$
$$\varphi(x, 0) = x \quad \text{and} \quad \varphi(x, 1) \in B \quad \text{for every} \quad x \in X - A .$$

By our hypothesis, the partial map $f_B = f/B$ is homotopic to y_0. We infer (since Y is an ANR-set) that there exists a map $f' \in Y^X$ homotopic to f and such that

$$f'(x) = y_0 \quad \text{for every} \quad x \in B .$$

Now let us consider an arbitrary open neighbourhood U of A in X. The map $f'\varphi$, considered only on the closed subset $(X - U) \times \langle 0, 1\rangle$ of $X \times \langle 0, 1\rangle$, is a homotopy, joining the partial map $f'/X - U$ with the constant y_0. It follows that the map f', hence also the map f, is homotopic in Y^X to a map $f'' \in Y^X$ satisfying the condition

$$f''(X - U) = f'(X - U) \subset (y_0) .$$

Thus we have shown that A is a condensator for f rel. y_0.

3. Maps homotopic to y_0 in dimension k. A map $f \in Y^X$ is said to be *homotopic to* $y_0 \in Y$ *in dimension* k, provided for every closed set $X_0 \subset X$ satisfying the condition

$$\dim X_0 \leqslant k$$

the partial map $f/X_0 \subset Y^{X_0}$ is homotopic to y_0. The set of all maps $f \in Y^X$ homotopic to y_0 in dimension k will be denoted by $Y^X(k/y_0)$.

In [4] I have introduced the notion of the homotopic k-skeleton of space X, defined as a closed subset A of X such that $\dim A \leqslant k$ and that every, at most k-dimensional closed subset X_0 of X can be transformed into A by a continuous deformation over the space X. In particular, if X is a polytope and T is one of its triangulations, then the union of all at most k-dimensional simplexes of T is a homotopic k-skeleton of X. Let us prove the following

Theorem. *If A is a homotopic k-skeleton of space X and y_0 is a point of an ANR-set Y then the set $Y^X(k/y_0)$ coincides with the set Z of all maps $f \in Y^X$ homotopic to y_0 on A.*

Proof. Since $\dim A \leqslant k$, we have

$$Y^X(k/y_0) \subset Z .$$

On the other hand, if B is a subset of X and $\dim B \leqslant k$, then there exists a family of maps $\{\varphi_t\} \subset X^B$ dependent continuously on the parameter

$0 \leqslant t \leqslant 1$ and joining the inclusion $\varphi_0 = i_B$ of B into X with the map φ_1 of B into A.

Now let us consider a map $f \in Z$ and let f_A and f_B denote the partial maps f/A and f/B respectively. Then the maps $\psi_t = f\varphi_t \in Y^B$ constitute a family dependent continuously on the parameter $0 \leqslant t \leqslant 1$ and satisfying the conditions:

$$\psi_0 = f\varphi_0 = f_B; \quad \psi_1 = f\varphi_1 = f_A\varphi_1 .$$

But $f \in Z$, hence f_A is homotopic in Y^A to y_0 and we infer that ψ_1, and consequently also $f_B = \psi_0$, is homotopic in Y^B to y_0. Thus we see that $f \in Y^X(k/y_0)$, i.e. $Z \subset Y^X(k/y_0)$ and the proof is concluded.

4. Homotopic k-coskeletons. A closed subset C of space X will be said to be a *homotopic k-coskeleton of* X provided:

1. $\dim C \leqslant \operatorname{Max}(-1, \dim X - k - 1)$,

2. Every closed subset of X, disjoint to C, can be transformed by a continuous deformation over X into a set of dimension $\leqslant k$.

Evidently the condition 2. is a consequence of the following one:

2′. The set $X - C$ is deformable over itself to a closed subset of X with dimension $\leqslant k$.

Manifestly, for $k \geqslant \dim X$, the empty subset of X is the unic homotopic k-coskeleton of X. For $k < \dim X$, the problem of the existence of homotopic k-coskeletons remains open, even for ANR-sets satisfying the condition (Δ), for which the existence of homotopic k-skeletons is proved (see [4]). Only for polytopes we can solve it. This will be done in the next two Nr's.

Now let us prove the following

THEOREM. *Let X be a space and y_0 a point of an ANR-set Y. Then every homotopic k-coskeleton C of X is a condensator for $Y^X(k/y_0)$ rel. y_0.*

Proof. Let U be an open neighbourhood of C in X. Then $B = X - U$ is a closed subset of X disjoint to C. Consequently, there exists a family of maps $\{\varphi_t\} \subset X^B$, dependent continuously on the parameter $0 \leqslant t \leqslant 1$ and such that φ_0 coincides with the inclusion i_B of B into X and that the dimension of the set $D = \varphi_1(B)$ is $\leqslant k$.

Consider now a map $f \in Y^X(k/y_0)$ and let f_B and f_D denote the partial maps f/B and f/D respectively. Then $\{f\varphi_t\}$ is a family of maps joining in Y^B the map $f\varphi_0 = f_B$ with the map $f\varphi_1 = f_D\varphi_1$. But $f \in Y^X(k/y_0)$ implies that f_D is homotopic in Y^D to y_0. Hence $f_D\varphi_1$, and consequently also f_B, is homotopic in Y^B to y_0. Regarding that B is a closed subset of X and Y is an ANR-set, we infer that the homotopy of f_B to y_0 in Y^B implies the homotopy of f in Y^X to a map f' satisfying the condition $f'(B) = f(X - U) \subset (y_0)$. Thus it is shown that C is a condensator for f, and the proof is concluded.

5. Homotopic k-coskeletons for polytopes. The question of the existence of homotopic k-coskeletons of polytopes is answered by the following

THEOREM. *Let T be a triangulation of a polytope X and k a non-negative integer. Let X^k denote the union of all at most k-dimensional simplexes of T. Then there exists in $X - X^k$ a polytope being a homotopic k-coskeleton of X. Moreover, if $k > \frac{1}{2}\dim X - 1$, then for every natural N there exists a system consisting of N disjoint k-coskeletons of X which are polytopes of dimension $\leqslant n - k - 1$.*

First let us reduce the proof of this theorem to the proof of a lemma with a little more complicated formulation.

Let T be a triangulation of an n-dimensional polytope X and let

$$\Delta_1^j, \Delta_2^j, \ldots, \Delta_{\nu_j}^j, \quad \text{where} \quad j = 0, 1, \ldots, n,$$

be all distinct closed j-dimensional simplexes of T. Let us set

$$X^m = \bigcup_{j=0}^{m} \bigcup_{i=0}^{\nu_j} \Delta_i^j \quad \text{for every} \quad m = 0, 1, \ldots, n .$$

By a set marking the triangulation T, we understand a set P consisting of points p_i^j, where p_i^j is a point chosen arbitrarily in the interior of the simplex Δ_i^j, $i = 0, 1, \ldots, \nu_j$; $j = 0, 1, \ldots, n$.

Using these notations, we formulate our lemma as follows:

LEMMA. *To every set P marking a triangulation T of an n-dimensional polytope X corresponds a system of polytopes*

$$Q_0(P), Q_1(P), \ldots, Q_{n-k-1}(P)$$

satisfying the following conditions:

1_m. *$Q_m(P)$ is an m-dimensional subpolytope of the polytope X^{k+m+1}, for every $m = 0, 1, \ldots, n-k-1$.*

2_m. *X^k is a deformation retract of $X^{k+m+1} - Q_m(P)$, for every $m = 0, 1, \ldots, n-k-1$.*

3_m. *If $m < k+1$ and if N is a natural number then there exists a system of N sets $P_1, P_1, \ldots, P_N$ marking the triangulation T, such that*

$$Q_m(P_\mu) \cap Q_m(P_{\mu'}) = 0 \quad \text{for} \quad \mu, \mu' = 1, 2, \ldots, N;\ \mu \neq \mu' .$$

The proof of this lemma is given in Nr. 6. Here let us observe that our theorem is included in this lemma. Since for $k \geqslant n$ the empty set is a homotopic k-coskeleton of X, we can restrict ourselves to the case $k < n$. Then setting $m = n-k-1$, we infer by condition 2_m, that the set $X^{k+m+1} - Q_m(P) = X - Q_m(P)$ is deformable over itself to the k-dimensional polytope X^k. Hence the $(n-k-1)$-dimensional polytope $Q_m(P)$ satisfies the condition 2′ of Nr. 4 and consequently it is a homotopic k-coskeleton of X.

Moreover, if $k > \frac{1}{2}\dim X - 1 = \frac{1}{2}n - 1$ then $k+1 > n-k-1 = m$ and we infer, by condition 3_m, that for every natural N there exists a system of sets $P_1, P_2, \ldots, P_N$ marking the triangulation T, such that the homotopic k-coskeletons $Q_m(P_\mu)$, where $\mu = 1, 2, \ldots, N$, are polytopes disjoint one to other.

Thus we see that our theorem will be proved while we prove the lemma.

6. Proof of the lemma. We define the polytopes $Q_0(P), Q_1(P), \ldots, Q_{n-k-1}(P)$ by the recurrence:

$Q_0(P)$ is defined as the set consisting of all points $p_1^{k+1}, \ldots, p_{\nu_{k+1}}^{k+1}$. It is clear that it fulfills the conditions 1_0-3_0. Now let us assume that for an integer m such that $0 \leqslant m < n-k-1$ the polytope $Q_m(P)$, satisfying the conditions 1_m-3_m, is defined. We define the polytope $Q_{m+1}(P)$ as the union of the polytope $Q_m(P)$ and of all segments joining one of the points p_i^{k+m+2}, where $i = 1, 2, \ldots, \nu_{k+m+2}$, with a point belonging to the common part of $Q_m(P)$ and of the boundary of the simplex Δ_i^{k+m+2}.

It is clear that the set $Q_{m+1}(P)$ defined in this way is a polytope of dimension $m+1$ included in the set X^{k+m+2}, i.e. the condition 1_{m+1} is satisfied. Moreover we have

$$Q_{m+1}(P) \cap X^{k+m+1} = Q_m(P) .$$

In order to show that the condition 2_{m+1} holds, let us denote by $r(x)$, for every point $x \in \Delta_i^{k+m+2} - (p_i^{k+m+2})$, the projection of x from the centre p_i^{k+m+2} onto the boundary of the simplex Δ_i^{k+m+2}. Let $r(x, t)$ denote, for every $0 \leqslant t \leqslant 1$, the point dividing the segment $\overline{xr(x)}$ at the ratio $t:1-t$. If we set

$$r(x, t) = x \quad \text{for every} \quad x \in X^{k+m+1} - Q_m(P) \quad \text{and} \quad 0 \leqslant t \leqslant 1 ,$$

then we get a retraction by deformation of the set $X^{k+m+2} - Q_{m+1}(P)$ to the set $X^{k+m+1} - Q_m(P)$. But, by the hypothesis of the recurrence, the set X^k is a deformation retract of the set $X^{k+m+1} - Q_m(P)$ and consequently the set X^k is also a deformation retract of the set $X^{k+m+2} - Q_{m+1}(P)$. Thus the condition 2_{m+1} is satisfied.

Now let us pass to the condition 3_{m+1}. Let $m+1 < k+1$. Then $m < k$ and we infer, by the hypothesis of recurrence, that for the triangulation T there exists a system of N marking sets $P_1, P_2, \ldots, P_N$, where $P_\mu = \{p_{i\mu}^j\}$ with $p_{i\mu}^j$ belonging to the interior of Δ_i^j, such that

$$Q_m(P_\mu) \cap Q_m(P_{\mu'}) = 0 \quad \text{for} \quad \mu \neq \mu' .$$

Now let us replace the sets $P_1, P_2, \ldots, P_N$ by some other sets $\overset{*}{P}_1, \overset{*}{P}_2, \ldots, \overset{*}{P}_N$ where $\overset{*}{P}_\mu = \{\overset{*}{p}{}_{i\mu}^j\}$ with $\overset{*}{p}{}_{i\mu}^j$ belonging to the interior of Δ_i^j.

Let us show that, setting

$$\overset{*}{p}{}_{i\mu}^j = p_{i\mu}^j \quad \text{for} \quad j \neq k+m+2 ,$$

we can choose the points $\overset{*}{p}{}_{i\mu}^{k+m+2}$ so that

$$Q_{m+1}(\overset{*}{P}_\mu) \cap Q_{m+1}(\overset{*}{P}_{\mu'}) = 0 \quad \text{for} \quad \mu \neq \mu' .$$

Consider a simplex $\Delta = \Delta_i^{k+m+2}$ of the triangulation T. For every $\mu = 1, 2, \ldots, N$ let us set

$$A_\mu = Q_m(P) \cap \Delta ,$$

and let B_μ denote the union of all segments joining the point $p_\mu = \overset{*}{p}{}_{i\mu}^{k+m+2}$ with all points of the polytope A_μ. Recalling the construction of the set $Q_{m+1}(P)$, one sees at once that the condition 3_{m+1} will be proved, while we shall prove that

(1) *points $p_1, p_2, \ldots, p_N$ can be chosen so that the sets $B_1, B_2, \ldots, B_N$ are disjoint.*

As point p_1 we take an arbitrary point in the interior of Δ. Let us assume that, for an index $l < N$, the points $p_1, p_2, \ldots, p_l$ are already chosen so that

$$B_\mu \cap B_{\mu'} = 0 \quad \text{for} \quad \mu, \mu' \leqslant l;\ \mu \neq \mu' .$$

Now let us observe that

$$\dim B_\mu = \dim A_\mu + 1 \leqslant m+1 < k+1 .$$

It follows that the dimension of the set C_μ consisting of all points of Δ colinear with a point $x \in A_{l+1}$ and a point $y \in B_\mu$ is $\leqslant 2m+2$. Since $k+1 > m+1$, the dimension of the simplex Δ, equal to $k+m+2$, is $> 2m+2$. We infer that there exists in the interior of Δ a point p_{l+1} which does not belong to any of the sets $C_1, C_2, \ldots, C_l$. It follows that

$$B_\mu \cap B_{\mu'} = 0 \quad \text{for every} \quad \mu, \mu' \leqslant l+1;\ \mu \neq \mu' ,$$

and we infer that (1) holds. But this concludes the proof of the condition 3_{m+1}, and also the proof of the lemma.

7. Some algebraic notions. Let Z be an abstract set and J an operation assigning to each ordered pair x, y of elements of Z, a subset $J(x, y)$ of Z. If A and B are two subsets of Z then we set

$$J(A, B) = \bigcup_{x \in A} \bigcup_{y \in B} J(x, y) .$$

The operation J is said to be *performable in* Z if $J(x, y) \neq 0$ for every $x, y \in Z$. It is said to be *associative*, provided

$$J(J(x, y), z) \frown J(x, J(y, z)) \neq 0 \quad \text{for every} \quad x, y \in Z .$$

It is said to be *commutative*, provided

$$J(x, y) \frown J(y, x) \neq 0 \quad \text{for every} \quad x, y \in Z .$$

It is said to be *univalent* provided, for every $x, y \in Z$, the set $J(x, y)$ contains only one element z of Z. In this case we write simply

$$J(x, y) = z .$$

An element z_0 of Z is said to be *neutral* (for the operation J) if

$$x \in J(x, z_0) \frown J(z_0, x) \quad \text{for every} \quad x \in Z .$$

An element x of Z is said to be a *negative* of an element y of Z, provided each of the sets $J(x, y)$ and $J(y, x)$ contains at least one neutral element. Let us observe that if y is a negative of x then x is a negative of y.

It is clear that if the operation J is performable, associative, commutative and univalent, and if there exists for J a neutral element and also a negative of every x then the set Z with the operation J is an abelian group.

8. Operations of the join in $Y^X[k/y_0]$ and in $Y^X\{k/y_0\}$. Consider now a polytope X, an ANR-set Y, an integer $k \geqslant 0$ and a point $y_0 \in Y$. Evidently the set $Y^X(k/y_0)$ of all maps belonging to Y^X and homotopic to the constant y_0 on every at most k-dimensional closed subset of X is the union of some components of Y^X. The set of all these components will be denoted by $Y^X[k/y_0]$. Hence $Y^X[k/y_0]$ is a subset of the set $[Y^X]$ of all components of Y^X.

For every map $f \in Y^X$ let us denote by $[f]$ the component of Y^X containing f and by $\{f\}$ the class of all maps $f' \in Y^X$ which are homologous to f, i.e. which satisfy the condition

$$f(\gamma) \sim f'(\gamma) \text{ in } Y \quad \text{for every true cycle } \gamma \text{ lying in } X .$$

Manifestly $\{f\}$ is the union of some components of Y^X, in particular $[f]$ is a subset of $\{f\}$. The set of all classes $\{f\}$ with $f \in Y^X$ will be denoted by $\{Y^X\}$, and the set of all classes $\{f\}$ with $f \in Y^X(k/y_0)$ will be denoted by $Y^X\{k/y_0\}$. Hence $Y^X\{k/y_0\}$ is a subset of $\{Y^X\}$.

Consider now two maps $f_1, f_2 \in Y^X(k/y_0)$. Manifestly the set Φ consisting of all maps $f \in Y^X(k/y_0)$ being the joins of f_1 and f_2 depends only on the classes $[f_1]$ and $[f_2]$. Moreover, if $f \in \Phi$ then $[f] \subset \Phi$. Hence Φ is the union of some classes belonging to $Y^X[k/y_0]$. Let us denote the set consisting of these classes by $J([f_1], [f_2])$. Thus we get an operation J called the *operation of the homotopic join*, assigning to every ordered pair of elements $[f_1], [f_2] \in Y^X[k/y_0]$ a subset of $Y^X[k/y_0]$.

Similarly, if we denote by $\tilde{J}(\{f_1\}, \{f_2\})$ the subset of $Y^X\{k/y_0\}$ consisting of all classes $\{f\} \in Y^X\{k/y_0\}$ where f is a join of f_1 and f_2, then we get an operation $\tilde{J}$, called the *operation of the homological join*, assigning to every ordered pair of elements $\{f_1\}, \{f_2\} \in Y^X\{k/y_0\}$ a subset of $Y^X\{k/y_0\}$.

Using these notations, we can formulate the following

THEOREM. *Let X be an n-dimensional polytope, y_0 be a point of an ANR-set Y and k be an integer $> \frac{1}{2}n-1$. Then the operation J of the homotopic join is performable, associative and commutative in $Y^X[k/y_0]$. Moreover, there exists for J a neutral element and also a negative of every element of $Y^X[k/y_0]$.*

The operation $\tilde{J}$ of the homological join is performable, associative, commutative and univalent in $Y^X\{k/y_0\}$. Moreover, there exists for $\tilde{J}$ a neutral element and also a negative of every element of $Y^X\{k/y_0\}$. Consequently the set $Y^X\{k/y_0\}$ with the operation of the homological join is an abelian group.

9. Proof of the theorem. Let T be a triangulation of X and let X^k denote the union of all at most k-dimensional simplexes of T. By theorem of Nr. 5, there exist in $X - X^k$ two disjoint k-coskeletons C_1 and C_2 of X. Consider two disjoint open neighbourhoods G_1 and G_2 of C_1 and C_2 respectively. By theorem of Nr. 4, for every two maps $f_1, f_2 \in Y^X(k/y_0)$ there exist two maps $f_1', f_2' \in Y^X$ homotopic to f_1 and f_2 respectively and such that

$$f_1'(X - G_1) \subset (y_0); \quad f_2'(X - G_2) \subset (y_0) .$$

It follows that setting

$$f(x) = \begin{cases} y_0 & \text{for every} \quad x \in X - G_1 - G_2 , \\ f_1'(x) & \text{for every} \quad x \in G_1 , \\ f_2'(x) & \text{for every} \quad x \in G_2 , \end{cases}$$

we get a join f of f_1 and f_2. Moreover, $X^k \subset X - G_1 - G_2$ and consequently $f(X^k) \subset (y_0)$. But X^k is a homotopic k-skeleton of X (see [4], p. 611, (10)) and we infer, by theorem of Nr. 3, that $f \in Y^X(k/y_0)$. Thus we have shown that the homotopy class $[f]$ of the join f of two maps f_1 and f_2 belongs to $Y^X[k/y_0]$ and consequently the operation of the homotopic join J is performable in $Y^X[k/y_0]$. The commutativity of J is evident, because $[f]$ belongs to both sets $J([f_1], [f_2])$ and $J([f_2], [f_1])$.

In order to show that the operation J is associative, consider three maps $f_1, f_2, f_3 \in Y^X(k/y_0)$. By theorem of Nr. 5, there exist in $X - X^k$

three disjoint homotopic k-coskeletons C_1, C_2 and C_3 of X. Consider three disjoint open neighbourhoods G_1, G_2 and G_3 of C_1, C_2 and C_3 respectively. By theorem of Nr. 4, there exist three maps $f_1', f_2', f_3' \in Y^X$ homotopic to f_1, f_2, f_3 respectively and such that

$$f_\nu'(X-G_\nu) \subset (y_0) \quad \text{for} \quad \nu = 1, 2, 3 .$$

Setting

$$f(x) = \begin{cases} y_0 & \text{for} \quad x \in X-G_1-G_2, \\ f_1(x) & \text{for} \quad x \in G_1, \\ f_2(x) & \text{for} \quad x \in G_2, \end{cases}$$

$$g(x) = \begin{cases} y_0 & \text{for} \quad x \in X-G_2-G_3, \\ f_2(x) & \text{for} \quad x \in G_2, \\ f_3(x) & \text{for} \quad x \in G_3, \end{cases}$$

and

$$h(x) = \begin{cases} y_0 & \text{for} \quad x \in X-G_1-G_2-G_3, \\ f_1(x) & \text{for} \quad x \in G_1, \\ f_2(x) & \text{for} \quad x \in G_2, \\ f_3(x) & \text{for} \quad x \in G_3, \end{cases}$$

we see at once that f is a homotopic join of f_1 and f_2, g is a homotopic join of f_2 and f_3, and h is both: the join of f and f_3 and of f_1 and g. Moreover the images of X^k by each of maps f, g and h are contained in (y_0). Consequently f, g and h belongs to $Y^X(k/y_0)$ and $[h]$ belongs to the set $J(J([f_1],[f_2],[f_3])) \frown J([f_1], J([f_2],[f_3]))$ and we infer that the operation J is associative.

Now let us observe that the homotopy class of the constant map y_0 is a neutral element for the operation J.

In order to prove the existence of a negative for $[f] \in Y^X[k/y_0]$, let us consider two disjoint homotopic k-coskeletons C_1 and C_2 and two open disjoint neighbourhoods G_1 and G_2 of them. We can assume that $f(X-G_1) \subset (y_0)$. Now let us observe that $X-G_2 \subset X-C_2$. It follows that the partial map $f|X-G_2$ is homotopic to y_0 and consequently there exists a map $g \in Y^X$ homotopic to y_0 which coincides on $X-G_2$ with f. If we set

$$h(x) = \begin{cases} g(x) & \text{for} \quad x \in G_2, \\ y_0 & \text{for} \quad x \in X-G_2, \end{cases}$$

then we get a map $h \in Y^X(k/y_0)$ and we see at once that g is a homotopic join of f and h. But g is homotopic to y_0, hence $[g]$ is a neutral element for the operation J. It follows that $J([f],[g])$ contains a neutral element, i.e. $[h]$ is a negative for $[f]$. Thus the first part of theorem is proved.

In order to prove the second part, let us recall that if $f \in Y^X$ is a homotopic join of two maps $f_1, f_2 \in Y^X$, then for every true cycle γ lying in X we have

$$f(\gamma) \sim f_1(\gamma) + f_2(\gamma) \quad \text{in} \quad Y .$$

It follows that the homology type $\{f\}$ of f is uniquely determined by the homology types $\{f_1\}$ and $\{f_2\}$ of f_1 and f_2. If we recall that the operation of the homotopic join is in our case performable, we infer that the operation of the homological join $\tilde{J}$ is performable and univalent in $Y^X\{k/y_0\}$. Moreover it is clear that $\{y_0\}$ is the neutral element for $\tilde{J}$ and if $[h]$ is a negative of an element $[f]$ by the operation J, then $\{h\}$ is the negative of the element $\{f\}$ by the operation $\tilde{J}$. Consequently the existence of negatives by operation J implies the existence of negatives by operation $\tilde{J}$.

Thus the proof of theorem is concluded.

10. Final remarks. If Y is the n-sphere S_n and X is a polytope of dimension $< 2n-1$ then the operation of the homotopic join J is univalent ([3] and [7]) and consequently the set $S_n^X[k/y_0]$ with the operation J is an abelián group. Evidently, for $k < n$ the set $S_n^X[k/y_0]$ coincides with S_n^X, and our group coincides with the n-th cohomotopy group $\pi^n(X)$ of X.

The same proposition holds for $k = n$, if we assume that X is acyclic in the dimension n. It holds also for $k > n$, if we assume that X is aspheric in all dimensions $l \leqslant k$, i.e. if every continuous map of an l-dimensional sphere into X is homotopic to a constant for every $l \leqslant k$. In fact, then every at most k-dimensional closed subset of X is contractible in X to a point and consequently $S_n^X[k/y_0] = S_n^X$. But without these additional hypotheses the relation between the group $S^X[k/y_0]$ and the cohomotopy groups of X remains obscure.

Let us observe that in the case of an arbitrary connected ANR-set Y the hypothesis of the asphericity of the polytope X in all dimensions $l \leqslant k$ implies that

$$Y^X(k/y_0) = Y^X .$$

Now if we assume that $k > n/2-1$, then we conclude that the operation of the homotopic join J is performable, associative and commutative in the whole set $[Y^X]$. Also the existence of neutral and of negative elements is assured. But the question under what hypothesis the operation of the homotopic join is univalent remains open.

It follows by theorem of Nr. 8, that for $k > \frac{1}{2}\dim X - 1$ the collection $Y^X\{k/y_0\}$ of all homology types $\{f\}$ of maps f belonging to $Y^X(k/y_0)$ with the operation $\tilde{J}$ of the homological join, is an abelian group. In particular, if Y is the n-th sphere S_n and the dimension of X is $\leqslant n$ and $k = n-1$, then the set $Y^X(k/y_0)$ coincides with Y^X, and the group $Y^X\{k/y_0\}$ coincides with the Hopf group of X in the sense of Freudenthal [5].

References

[1] K. Borsuk, *Sur les groupes des classes de transformations continues*, C.R. de l'Ac. des Sc. 202 (1936), pp. 1400-1403.

[2] — *On a concept of dependence for continuous mappings*, Fund. Math. 43 (1956), pp. 95-113.

[3] — *Sur l'addition homologique des types de transformations continues en surfaces sphériques*, Annals of Math. 38 (1937), pp. 733-738.

[4] — *Some remarks concerning the position of sets in a space*, Bull. de l'Ac. Pol. Sc. 9 (1960), pp. 609-613.

[5] H. Freudenthal, *Die Hopfsche Gruppe*, Compositio Math. 2 (1935), pp. 134-162.

[6] W. Hurewicz, *Beiträge zur Topologie der Deformationen* (*I Höherdimensionale Homotopiegruppen*), Proc. Akad. Amsterdam 38 (1935), pp. 112-119.

[7] E. Spanier, *Borsuk's cohomotopy groups*, Annals of Math. 50 (1949), pp. 203-245.

Reçu par la Rédaction le 12. 11. 1960

Structure maps in group theory

by

B. Eckmann (Zürich) and **P. J. Hilton** (Birmingham)

Introduction and notation. In [3] the authors elaborate a theory of structure maps in general categories. If the category $\mathcal{C}$ admits direct products the notion of group structure (1) and the group axioms may be formulated entirely in terms of the maps of the category; and this enables us to carry over to categories other than categories of sets and element-maps certain classical notions of group theory. Moreover the definitions and results may be dualized and applied to categories admitting free products. We also discuss in [3] various generalizations of the classical notions of group theory, for example, the notions of unions and intersections of subgroups.

Among the concrete categories to which the notions of [3] may be applied is the category $\mathcal{G}$ of groups and homomorphisms; indeed the notions and terminology of [3] were in part inspired by the category $\mathcal{G}$. The present paper consists of a fairly detailed discussion of the application of the notions of [3] to the category $\mathcal{G}$. In the course of this discussion we naturally find ourselves introducing ideas and adopting arguments peculiar to the category of groups. Thus, unlike [3], we claim here no generality for our results which are all group-theoretic, and the present paper is intended to be, more or less, readable independently of [3], owing to [3] merely its motivation.

If $\mathcal{C}$ is a category of the type considered in [3] and if $X_1, \ldots, X_n$ are n objects of $\mathcal{C}$ then there is a natural (self-dual) map

$$\varkappa\colon X_1 * \ldots * X_n \to X_1 \times \ldots \times X_n$$

in $\mathcal{C}$ from the free product of $X_1, \ldots, X_n$ to the direct product. An important general construction described in [3] consists of the two (dual) factorizations of the map $\varkappa$, namely,

$$\text{(F)}\quad X_1 * \ldots * X_n = X^n \xrightarrow{\varkappa^{n-1}} X^{n-1} \to \ldots \to X^{p+1} \xrightarrow{\varkappa^p} X^p \to \ldots \xrightarrow{\varkappa^1} X^1 = X_1 \times \ldots \times X_n\,,$$

$$\text{(F')}\quad X_1 * \ldots * X_n = {}^1X \xrightarrow{{}^1\varkappa} \ldots \to {}^qX \xrightarrow{{}^q\varkappa} {}^{q+1}X \to \ldots \to {}^{n-1}X \xrightarrow{{}^{n-1}\varkappa} {}^nX = X_1 \times \ldots \times X_n\,,$$

(1) And, of course, semigroup structure.