4. The special case of dendrites. A *dendrite* may be defined as a Peano continuum which contains no simple closed curve. As has been noted elsewhere, a Peano continuum is arcwise connected and among the Peaño continua the property of being a dendrite is equivalent to being hereditarily únicoherent. It follows at once from Theorem 1 that a dendrite has the fixed point property for upper semi-continuous, continuum-valued mappings and, as remarked in the introduction, Wallace has previously obtained this result by other methods. In view of Plunkett's theorem we may assert the following at once.

THEOREM 3. *If X is a Peano continuum then the following statements are equivalent.*

(1) *X is a dendrite,*

(2) *X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings,*

(3) *X has the fixed point property for the class of continuous, closed set-valued mappings.*

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Axiomatizability of some many valued predicate calculi

by

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In a paper published in Volume 45 of the Fundamenta Mathematicae I proposed a generalization of the logical qunatifiers. Another generalization applicable in the two valued as well as in the many valued cases has been proposed and discussed by Rosser and Turquette [7]. According to their conception a quantifier is a function which correlates a truth value with a non-empty set of truth values (I disregard here a more general notion considered in [7] in which sets are replaced by relations). Rosser and Turquette ([7], Chapter V) discussed the problem of axiomatizability of the functional calculi with arbitrary quantifiers under the assumption that the set of truth values is finite and Rosser (in an address read at the 1959 meeting of the Association for Symbolic Logic and published in [6]) discussed a similar problem under the assumption that this set coincides with the interval $[0,1]$. In the present paper I take up the problem of axiomatizability under a more general assumption that the set of truth values is an ordered set which is bicompact in its order topology. The method of proof is illustrated in Section 3 where I discuss the case of a finite set of truth values and obtain a part of results of Rosser and Turquette. The chief feature of results set forth in the present paper is their non-effective character: I prove the existence of complete sets of axioms and rules of proof for the calculi in question without exhibiting them explicitly; the existence proofs are based on Tichonov's theorem.

1. Syntax. We consider a "language" S_0 whose expressions are built from the following symbols: $x_0, x_1, \ldots$ (individual variables), $F_0^j, F_1^j, \ldots$ (predicate variables with j arguments, $j = 0, 1, 2, \ldots$), $\mathfrak{F}_0, \mathfrak{F}_1, \ldots, \mathfrak{F}_a$ (propositional connectives), $\mathfrak{Q}_0, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_b$ (quantifiers). We denote by p_s the number of arguments of $\mathfrak{F}_s$ ($s = 0, 1, \ldots, a$). Formulas are expressions which belong to the smallest class K such that: (i) atomic expressions $F_n^j x_{i_1} \ldots x_{i_j}$ belong to K ($n, j = 0, 1, \ldots,$ $i_s = 0, 1, \ldots$ for $s = 1, 2, \ldots, j$); (ii) if $0 \leqslant s \leqslant a$ and $\Phi_1, \ldots, \Phi_{p_s}$ belong to K, then so does $\mathfrak{F}_s \Phi_1 \ldots \Phi_{p_s}$; (iii) if $0 \leqslant s \leqslant b$ and Φ belongs to K, then so does $\mathfrak{Q}_s x_q \Phi$, $q = 0, 1, \ldots$

The distinction between free and bound variables of a formula is assumed to be known. A formula without free variables is called closed. The result of the substitution of c for x_q in Φ is denoted by $\mathrm{Sb}(x_q/c)\Phi$.

Besides S_0 we shall also consider systems obtained from S_0 by adjunction of constants $c_0, c_1, \ldots$ whose number may be finite or infinite of any power. The "rules of formation" (i)-(iii) remain the same with the amendment that each x_{i_s} in (i) can be replaced by a constant.

We choose a Gödel numbering of expressions of S_0 and denote by $\ulcorner\Phi\urcorner$ the Gödel number of Φ; the expression with the Gödel number n is denoted by $\breve{n}$. We assume that the functions $\ulcorner\breve{n}\breve{m}\urcorner$ and $\ulcorner x_n\urcorner$ are recursive and increasing. From this assumption it easily follows:

1.1. *The following functions are recursive:*

(a) $f_1^0(n) = 0, 1, 2, 3$ *according as* $\breve{n}$ *is an atomic formula, a formula which begins with a connective, a formula which begins with a quantifier or* $\breve{n}$ *is not a formula or is undefined.*

(b) $f_2^0(n) = j, k, 0$ *according as* $f_1^0(n) = 1$ *and* $\breve{n}$ *begins with* $\mathfrak{F}_j$, $f_1^0(n) = 2$ *and* $\breve{n}$ *begins with* $\mathfrak{Q}_k$, *or* $f_1^0(n) \neq 1, 2$.

(c) $f_3^0(j, n) = 0$ *if* $f_1^0(n) \neq 1$ *or* $j = 0$ *or* $j > p_{f_2^0(n)}$; $f_2^0(j, n) = q_j$ *if* $f_1^0(n) = 1$ *and* $\breve{n}$ *has the form* $\mathfrak{F}_s q_1 q_2 \ldots q_{p_s}$.

(d) $f_4^0(n) = q$, $f_5^0(n) = r$ *if* $f_2^0(n) = 2$ *and* $\breve{n}$ *has the form* $\mathfrak{Q}_k x_q \breve{r}$; $f_4^0(n) = 0 = f_5^0(n)$ *in the remaining cases.*

2. Semantics. Let Z be a set, φ_s a mapping of $Z \times \ldots \times Z = Z^{p_s}$ into Z, Q_t a mapping of 2^Z (1) into Z $(0 \leqslant s \leqslant a, 0 \leqslant t \leqslant b)$. Let D be a subset of Z. We call elements of Z *truth values*, those of D *distinguished truth values*; φ_s are interpretations of connectives and Q_t interpretations of quantifiers.

A *model* of S_0 (or of a system resulting from S_0 by the adjunction of constants) in a set X is a mapping μ satisfying the following conditions. μx is defined if x is an individual constant or a predicate variable; in the former case $\mu x \in X$, in the latter $\mu x \in Z^{X^j} = Z^{X \times X \times \ldots \times X}$ where j is the number of arguments of x. A *valuation* of μ is an extension $\bar{\mu}$ of μ such that the domain of $\bar{\mu}$ consists of all individual constants, predicate variables and individual variables; if x is an individual variable, then $\bar{\mu}x \in Z$. Whenever $\bar{\mu}$ is a valuation of μ we denote by $W_{q,\bar{\mu}}$ the set of all valuations ν of μ which are identical with $\bar{\mu}$ except possibly for the argument x_q.

Let ν be a valuation of μ and let Φ be a formula. We define by induction the value of Φ at ν (denoted by $\mathrm{Val}_\nu\Phi$):

(1) 2^Z denotes the set of all non-void subsets of Z.

if Φ is $F_i^j t_1 \ldots t_j$ (where each t_k is either an individual variable or an individual constant), then $\mathrm{Val}_\nu\Phi = \nu(F_i^j)\big(\nu(t_1), \ldots, \nu(t_j)\big)$;

if Φ is $\mathfrak{F}_s\Phi_1 \ldots \Phi_{p_s}$, then $\mathrm{Val}_\nu\Phi = \varphi_s(\mathrm{Val}_\nu\Phi_1, \ldots, \mathrm{Val}_\nu\Phi_{p_s})$;

if Φ is $\mathfrak{Q}_t x_q \Psi$, then $\mathrm{Val}_\nu\Phi = Q_t(\{\mathrm{Val}_\varrho\Psi : \varrho \in W_{q,\nu}\})$.

The following lemmas are easily proved:

2.1. *If* μ *is a model and* ν *a valuation of* μ, *then* $\mathrm{Val}_\nu\Phi \in Z$ *for every formula* Φ.

2.2. *If* ν', ν'' *are valuations of a model* μ *and if* $\nu' x_q = \nu'' x_q$ *for all* q *such that* x_q *is free in* Φ, *then* $\mathrm{Val}_{\nu'}\Phi = \mathrm{Val}_{\nu''}\Phi$.

2.3. *If* Φ *is closed, then* $\mathrm{Val}_\nu\Phi$ *depends only on the model* μ *of which* ν *is a valuation.*

$\mathrm{Val}_\nu\Phi$ is denoted in this case by $\mathrm{Val}_\mu\Phi$.

2.4. *If* c *is an individual constant,* ν *a valuation of a model* μ, $\nu' \in W_{q,\nu}$, $\nu' x_q = \nu c$, *then* $\mathrm{Val}_\nu\mathrm{Sb}(x_q/c)\Phi = \mathrm{Val}_{\nu'}\Phi$.

A formula Φ is called *satisfiable* if there are a set X, a model μ in X, and a valuation ν of μ such that $\mathrm{Val}_\nu\Phi \in D$; Φ is *valid* if $\mathrm{Val}_\nu\Phi \in D$ for every set X, every model μ in X and every valuation ν of μ.

3. N-valued logics. In this section we assume that $Z = \{0, 1, \ldots, N-1\}$ where N is a positive integer and that $D = \{0, 1, \ldots, M-1\}$ where M is an integer $\leqslant N$. We define a sequence of systems S_n; S_0 is the system described in Section 1, S_{n+1} results from S_n by adjunction of constants $A_{h,q,\Phi}$ where $h = 0, 1, \ldots, N-1$, $q = 0, 1, 2, \ldots$ and Φ is a formula of S_n which is not a formula of S_{n-1} and has at most one free variable x_q. Let S_∞ be the union of all systems S_n. It is not difficult to see that a Gödel numbering of expressions of S_∞ can be chosen so that $\ulcorner A_{h,q,\Phi}\urcorner$ is a recursive function of $h, q, \ulcorner\Phi\urcorner$. It follows that there exists a recursive function g which enumerates the Gödel numbers of all individual constants of S_∞ and recursive functions $f_1^1, f_2^1, f_3^1, f_4^1$ satisfying conditions analogous to 1.1 (a)-(d) but with "formula" replaced by "closed formula of S_∞".

Let μ, μ' be models in X of systems S_n, S_m, $m > n$, $m = 1, 2, \ldots, \infty$. If $\mu' c = \mu c$ for every individual constant of S_n and $\mu F_i^j = \mu' F_i^j$ for $i, j = 0, 1, \ldots$, then we say that μ' is an *extension* of μ. The following lemmas are obvious:

3.1. *If* μ *is a model of* S_n, μ' *its extension, and* ν, ν' *are valuations of* μ, μ' *such that* $\nu x_q = \nu' x_q$ *for every* q, *then* $\mathrm{Val}_\nu\Phi = \mathrm{Val}_{\nu'}\Phi$ *for every formula* Φ *of* S_n.

3.2. *If* μ_n *is a model of* S_n *and* μ_{n+1} *is an extension of* μ_n $(n = 0, 1, 2, \ldots)$, *then there is a model* μ_∞ *of* S_∞ *which is a joint extension of all the* μ_n*'s.*

3.3. *Every model μ of S_0 can be extended to a model μ_∞ of S_∞ in such a way that for every formula Φ of S_∞ with at most one free variable x_q the following equation holds* (²)

$$(1)\quad \{\mathrm{Val}_\nu\Phi\colon\ \nu \textit{ is a valuation of } \mu_\infty\} = \{\mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_q/A_{0,q,\Phi})\Phi, \ldots, \mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_q/A_{N-1,q,\Phi})\Phi\}.$$

Proof. Put $\mu_0 = \mu$ and assume that an extension μ_n of μ has been constructed such that μ_n is a model of S_n and that

$$(2)\quad \{\mathrm{Val}_\varrho\Phi\colon\ \varrho \textit{ is a valuation of } \mu_n\} = \{\mathrm{Val}_{\mu_n}\mathrm{Sb}(x_q/A_{0,q,\Phi})\Phi, \ldots, \mathrm{Val}_{\mu_n}\mathrm{Sb}(x_q/A_{N-1,q,\Phi})\Phi\}$$

for every formula Φ of S_{n-1}, with at most one free variable x_q. This assumption is clearly satisfied for $n = 0$ for S_{n-1} in this case is empty. We shall extend μ_n to a valuation μ_{n+1} of S_{n+1} and we therefore have to define $\mu_{n+1}A_{j,q,\Phi}$ for $j = 0, 1, \ldots, N-1$, $q = 0, 1, \ldots$ and such Φ which are formulas of S_n but not of S_{n-1} and which have at most one free variable x_q. Let Φ be such a formula. Since the set $\{\mathrm{Val}_\varrho\Phi\colon\ \varrho \textit{ is a valuation of } \mu_n\}$ is contained in Z, we may assume that it consists of integers $s_1, \ldots, s_m < N$ where $1 \leqslant m < N$. Choose valuations ϱ_i of μ_n such that $\mathrm{Val}_{\varrho_i}\Phi = s_i$, and put $\mu_{n+1}A_{i,q,\Phi} = \varrho_i x_q$ for $i = 1, 2, \ldots, m$, $\mu_{n+1}A_{j,q,\Phi} = \varrho_m x_q$ for $j = m+1, \ldots, N-1$. The mapping μ_{n+1} thus defined is an extension of μ_n and hence of μ. If Φ is a formula of S_{n-1} with at most one free variable x_q, and ϱ is a valuation of μ_{n+1}, then ϱ restricted to symbols of S_n is a valuation of μ_n and hence we have equation (2) from which, in view of 3.1, we obtain

$$\{\mathrm{Val}_\varrho\Phi\colon\ \varrho \textit{ is a valuation of } \mu_{n+1}\} = \{\mathrm{Val}_{\mu_{n+1}}\mathrm{Sb}(x_q/A_{0,q,\Phi})\Phi, \ldots, \mathrm{Val}_{\mu_{n+1}}\mathrm{Sb}(x_q/A_{N-1,q,\Phi})\Phi\}.$$

If Φ is a formula of S_n but not of S_{n-1}, then the same equation holds true in view of the construction of μ_{n+1}. Thus we obtain a sequence of successive extensions μ_n of μ satisfying (2) for each n. If μ_∞ is a joint extension of the models μ_n, then clearly equation (1) holds for every formula Φ of S_∞ with at most one free variable x_q.

We shall now express arithmetically the notions of satisfiability and of validity. We put $f_6^1(j, n) = \ulcorner\mathrm{Sb}(x_{f_4^1(n)}/A_{j,f_4^1(n),\breve{f}_5^1(n)})\breve{f}_5^1(n)\urcorner$ if $f_1^1(n) = 2$ and $0 \leqslant j < N$ and $f_6^1(j, n) = 0$ otherwise. Furthermore we put $f_7^1(k, n) = \ulcorner\mathrm{Sb}(x_{f_4^1(n)}/\breve{g}(k))\breve{f}_5^1(n)\urcorner$ if $f_1^1(n) = 2$ and $f_7^1(k, n) = 0$ otherwise. Functions f_6^1, f_7^1 are recursive.

(²) $\{f(t)\colon \ldots t \ldots\}$ denotes the set of all $f(t)$ where t satisfies the condition $\ldots t \ldots$; $\{a, b, \ldots, m\}$ denotes the set consisting exclusively of $a, b, \ldots, m$.

Let α be a function from integers to integers. We call α an *A-model* if the following conditions are satisfied:

(3) if $0 \leqslant f_1^1(n) < 3$, then $0 \leqslant \alpha(n) < N$; $\alpha(n) = 0$ for $f_1^1(n) \geqslant 3$;

(4) if $f_1^1(n) = 1$, then $\alpha(n) = \varphi_{f_2^1(n)}\big(\alpha(f_3^1(1, n)), \ldots, \alpha(f_3^1(p_{f_2^1(n)}, n))\big)$;

(5) if $f_1^1(n) = 2$, then (³) $(q_0, \ldots, q_{N-1})_N\big\{(j)_N[\alpha(f_6^1(j, n)) = q_j] \supset [\alpha(n) = Q_{f_2^1(n)}(\{q_0, \ldots, q_{N-1}\})]\big\}$;

(6) if $f_1^1(n) = 2$, then $(k)(\mathrm{E}j)_N\big[\alpha(f_7^1(k, n)) = \alpha(f_6^1(j, n))\big]$.

3.4. *A closed formula Φ of S_0 is satisfiable (valid) if and only if $\alpha(\ulcorner\Phi\urcorner) < M$ for an (every) A-model α.*

Proof. Let μ_∞ be a model of S_∞ and put $\alpha(n) = \mathrm{Val}_{\mu_\infty}(\breve{n})$ if $0 \leqslant f_1^1(n) < 3$, $\alpha(n) = 0$ otherwise. First we show that if μ_∞ satisfies (1), then α is an A-model.

Condition (3) is obviously satisfied.

If $f_1^1(n) = 1$, then $\breve{n} = \mathfrak{F}_{f_2^1(n)}\breve{f}_3^1(1, n) \ldots \breve{f}_3^1(p_{f_2^1(n)}, n)$ and hence

$$\alpha(n) = \mathrm{Val}_{\mu_\infty}(\breve{n}) = \varphi_{f_2^1(n)}\big(\mathrm{Val}_{\mu_\infty}\breve{f}_3^1(1, n), \ldots, \mathrm{Val}_{\mu_\infty}\breve{f}_3^1(p_{f_2^1(n)}, n)\big) = \varphi_{f_2^1(n)}\big(\alpha(f_3^1(1, n)), \ldots, \alpha(f_3^1(p_{f_2^1(n)}, n))\big).$$

This proves (4).

If $f_1^1(n) = 2$ then $\breve{n} = \mathfrak{Q}_{f_2^1(n)}x_{f_4^1(n)}\breve{f}_5^1(n)$ and hence

$$\alpha(n) = \mathrm{Val}_{\mu_\infty}(\breve{n}) = Q_{f_2^1(n)}\big(\{\mathrm{Val}_\nu\breve{f}_5^1(n)\colon\ \nu \textit{ is a valuation of } \mu_\infty\}\big),$$

whence by (1)

$$\alpha(n) = Q_{f_2^1(n)}\big(\{\mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_{f_4^1(n)}/A_{0,f_4^1(n),\breve{f}_5^1(n)})\breve{f}_5^1(n), \ldots, \mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_{f_4^1(n)}/A_{N-1,f_4^1(n),\breve{f}_5^1(n)})\breve{f}_5^1(n)\}\big) = Q_{f_2^1(n)}\big(\{\mathrm{Val}_{\mu_\infty}f_6^1(0, n), \ldots, \mathrm{Val}_{\mu_\infty}f_6^1(N-1, n)\}\big).$$

This proves (5).

If $f_1^1(n) = 2$ then $\breve{f}_7^1(k, n) = \mathrm{Sb}(x_{f_4^1(n)}/\breve{g}(k))\breve{f}_5^1(n)$ and hence by 2.4 $\alpha(f_7^1(k, n)) = \mathrm{Val}_{\mu_\infty}\breve{f}_7^1(k, n)$ is equal to $\mathrm{Val}_\varrho\breve{f}_5^1(n)$ where $\varrho \in W_{f_4^1(n),\mu_\infty}$ and $\varrho x_{f_4^1(n)} = \mu_\infty\breve{g}(k)$. By (1) there is a $j < N$ such that $\mathrm{Val}_\varrho\breve{f}_5^1(n) = \mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_{f_4^1(n)}/A_{j,f_4^1(n),\breve{f}_5^1(n)})\breve{f}_5^1(n) = \mathrm{Val}_{\mu_\infty}f_6^1(j, n) = \alpha(f_6^1(j, n))$.

This proves (6).

(³) $(i, j, \ldots, p)_N$ means: for arbitrary integers $i, j, \ldots, p < N$; similarly $(\mathrm{E}i, j, \ldots, p)_N$ means: there are integers $i, j, \ldots, p < N$.

Now let a be an A-model. We define a model μ_∞ of S_∞ in the set X_∞ of all constants of S_∞ as follows: for c in X_∞ we put $\mu_\infty c = c$ and we let $\mu_\infty F_i^j$ to be a function ψ such that $\psi(u_1, \ldots, u_j) = a(\ulcorner F_i^j u_1 \ldots u_j \urcorner)$ for $u_1, \ldots, u_j$ in X_∞.

For any formula Φ of S_∞ we denote by $\hat{\Phi}$ the set of closed formulas of S_∞ which can be obtained from Φ by substitutions of individual constants for free variables; and by $\hat{\hat{\Phi}}$ the set of formulas of S_∞ which have at most one free variable and which result from Φ by substitutions. We shall show that if Φ is a formula of S_∞ and Ψ is in $\hat{\Phi}$ then

$$\mathrm{Val}_{\mu_\infty}\Psi = a(\ulcorner\Psi\urcorner)\,.$$

Case 1: *Φ is an atomic formula.* In this case any Ψ in $\hat{\Phi}$ has the form $F_i^j u_1 \ldots u_j$ where $u_1, \ldots, u_j \in X_\infty$ and hence by the definition of μ_∞

$$\mathrm{Val}_{\mu_\infty}\Psi = a(\ulcorner F_i^j u_1 \ldots u_j\urcorner) = a(\ulcorner\Psi\urcorner)\,.$$

Case 2. *Φ has the form $\mathfrak{F}_j\Phi_1 \ldots \Phi_{p_j}$.* In this case any Ψ in $\hat{\Phi}$ has the form $\mathfrak{F}_j\Psi_1 \ldots \Psi_{p_j}$ where Ψ_i is in $\hat{\Phi}_i$ for $i = 1, 2, \ldots, p_j$ and hence

$$\mathrm{Val}_{\mu_\infty}\Psi = \varphi_j(\mathrm{Val}_{\mu_\infty}\Psi_1, \ldots, \mathrm{Val}_{\mu_\infty}\Psi_{p_j})\,.$$

Using inductive assumption and (4) we obtain

$$\mathrm{Val}_{\mu_\infty}\Psi = \varphi_j\big(a(\ulcorner\Psi_1\urcorner), \ldots, a(\ulcorner\Psi_{p_j}\urcorner)\big) = a(\ulcorner\Psi\urcorner)$$

because $f_1^1(\ulcorner\Psi\urcorner) = 1$, $f_2^1(\ulcorner\Psi\urcorner) = j$, $f_3^1(i, \ulcorner\Psi\urcorner) = \ulcorner\Psi_i\urcorner$ for $i = 1, 2, \ldots, p_j$.

Case 3. *Φ has the form $\mathfrak{Q}_j x_q \Xi$.* In this case any Ψ in $\hat{\Phi}$ has the form $\mathfrak{Q}_j x_q \Pi$ where Π is in $\hat{\hat{\Xi}}$ or in $\hat{\Xi}$ according as x_q is or is not free in Ξ.

Subcase 3[a]. *x_q is not free in Ξ.* In this case Π is closed and

$$\mathrm{Val}_{\mu_\infty}\Psi = Q_j(\{\mathrm{Val}_{\mu_\infty}\Pi\})\,, \text{ (4)}$$

and hence

$$\ulcorner\Pi\urcorner = f_6^1(i, \ulcorner\Psi\urcorner) \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

$$\mathrm{Val}_{\mu_\infty}\Psi = Q_j\big(\{\mathrm{Val}_{\mu_\infty}\breve{f}_6^1(0, \ulcorner\Psi\urcorner), \ldots, \mathrm{Val}_{\mu_\infty}\breve{f}_6^1(N-1, \ulcorner\Psi\urcorner)\}\big)\,,$$

whence by the inductive assumption and by (5)

$$\mathrm{Val}_{\mu_\infty}\Psi = Q_j\Big(\big\{a\big(f_6^1(0, \ulcorner\Psi\urcorner)\big), \ldots, a\big(f_6^1(N-1, \ulcorner\Psi\urcorner)\big)\big\}\Big) = a(\ulcorner\Psi\urcorner)\,.$$

Subcase 3[b]. *x_q is free in Ξ.* In this case Π has just one free variable x_q and

$$\mathrm{Val}_{\mu_\infty}\Psi = Q_j(\{\mathrm{Val}_\varrho\Pi\colon\ \varrho \textit{ is a valuation of } \mu_\infty\})\,.$$

If ϱ is a valuation of μ_∞ then $\varrho x_q = c$ is in X_∞ and hence, by 2.4, $\mathrm{Val}_\varrho\Pi = \mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_q/c)\Pi$. If $\ulcorner c\urcorner = g(k)$, then by the inductive assumption

(4) $\{a\}$ is the unit set with the sole element a.

and the remark that $\mathrm{Sb}(x_q/c)\Pi$ is in $\hat{\Xi}$ we infer that the right hand side is equal to $\mathrm{Val}_{\mu_\infty}\breve{f}_7^1(k, \ulcorner\Psi\urcorner) = a\big(f_7^1(k, \ulcorner\Psi\urcorner)\big)$ and hence, by (6), to $a\big(f_6^1(j, \ulcorner\Psi\urcorner)\big)$ where j is an integer $< N$. Conversely $a\big(f_6^1(j, \ulcorner\Psi\urcorner)\big)$ is an element of the set $\{\mathrm{Val}_\varrho\Pi\colon$ *ϱ is a valuation of μ_∞*$\}$ for $a\big(f_6^1(j, \ulcorner\Psi\urcorner)\big) = \mathrm{Val}_\varrho\Pi$ where $\varrho \in W_{a,\mu_\infty}$ and $\varrho x_q = \mu_\infty A_{j,q,\Pi} = A_{j,q,\Pi}$. This proves that

$$\mathrm{Val}_{\mu_\infty}\Psi = Q_j\Big(\big\{a\big(f_6^1(0, \ulcorner\Psi\urcorner)\big), \ldots, a\big(f_6^1(N-1, \ulcorner\Psi\urcorner)\big)\big\}\Big)$$

and hence by (5) that $\mathrm{Val}_{\mu_\infty}\Psi = a(\ulcorner\Psi\urcorner)$.

Now let Φ be a closed formula of S_0. If Φ is satisfiable then there is a model μ of S_0 such that $\mathrm{Val}_\mu\Phi < M$. We extend μ to a model μ_∞ of S_∞ satisfying (1) according to 3.3 and obtain thus an A-model a such that $a(n) = \mathrm{Val}_{\mu_\infty}\breve{n}$ whenever $0 \leqslant f_1^1(n) < 3$. In particular, $a(\ulcorner\Phi\urcorner) = \mathrm{Val}_{\mu_\infty}\Phi = \mathrm{Val}_\mu\Phi < M$. Conversely if there is an A-model a such that $a(\ulcorner\Phi\urcorner) < M$, then there is a model μ_∞ of S_∞ in the set X_∞ such that $\mathrm{Val}_{\mu_\infty}\Phi = a(\ulcorner\Phi\urcorner) < M$. Restricting μ_∞ to symbols of S_0 we obtain a model μ of S_0 in X_∞ such that $\mathrm{Val}_\mu\Phi < M$ and thus Φ is satisfiable.

If Φ is valid and a is an A-model, then (as shown above) there is a model μ of S_0 in X_∞ such that $\mathrm{Val}_\mu\Phi = a(\ulcorner\Phi\urcorner)$ and hence $a(\ulcorner\Phi\urcorner) < M$. Conversely, if this inequality holds for every A-model a and μ is a model of S_0 in a set X then there is an extension of μ to a model μ_∞ of S_∞ in X satisfying (1). We proved above that there is an A-model a such that $\mathrm{Val}_{\mu_\infty}\Phi = a(\ulcorner\Phi\urcorner)$ and hence $\mathrm{Val}_\mu\Phi = \mathrm{Val}_{\mu_\infty}\Phi < M$ which shows that Φ is valid. Theorem 3.4 is thus proved.

3.5. *The predicate "Φ is a closed satisfiable formula of S_0" is expressible in the form $(\mathsf{E}a)_H(x)R\big(\bar{a}(x), \ulcorner\Phi\urcorner\big)$ where R is a recursive binary relation, $H = \big\{a\colon (x)\big(a(x) < N\big)\big\}$ and $(\mathsf{E}a)_H$ means: there is an a in H. The predicate "Φ is a closed valid formula of S_0" is expressible in the dual form $(a)_H(\mathsf{E}x)S\big(\bar{a}(x), \ulcorner\Phi\urcorner\big)$ with recursive S.*

Proof. There is obviously a recursive predicate C such that: *Φ is a closed formula of S_0* $\equiv C(\ulcorner\Phi\urcorner)$.

By 3.4 we have the equivalence:

$$\{\Phi \textit{ is a closed and satisfiable formula of } S_0\}$$
$$\equiv C(\ulcorner\Phi\urcorner) \;\&\; (\mathsf{E}a)(x)(y)\Big(\big[\big(0 \leqslant f_1^1(x) < 3\big) \supset \big(0 \leqslant a(x) < N\big)\big]$$
$$\&\; \big[\big(f_1^1(x) \geqslant 3\big) \supset \big(a(x) = 0\big)\big] \;\&\; \Big\{\big(f_1^1(x) = 1\big) \supset (\mathsf{E}r)_{a+1}\Big[\big(r = f_2^1(x)\big)$$
$$\&\; \Big(a(x) = \varphi_r\Big(a\big(f_3^1(1, x)\big), \ldots, a\big(f_3^1(p_r, x)\big)\Big)\Big)\Big]\Big\}$$
$$\&\; \Big\{\big(f_1^1(x) = 2\big) \supset (q_0, \ldots, q_{N-1})_N(r)_{b+1}\big[(j)_N\big(r = f_2^1(x)\big)$$
$$\&\; \Big(a\big(f_6^1(j, x)\big) = q_j\Big) \supset \big(a(x) = Q_r(\{q_0, \ldots, q_{N-1}\})\big)\big]\Big\}$$
$$\&\; \Big\{\big(f_1^1(x) = 2\big) \supset (\mathsf{E}j)_N\big[a\big(f_7^1(y, x)\big) = a\big(f_6^1(j, x)\big)\big]\Big\} \;\&\; \big(a(\ulcorner\Phi\urcorner) < M\big)\Big)\,.$$

The above predicate can be written in the form [5]

$$(7)\qquad C(\ulcorner\Phi\urcorner)\ \&\ (\mathsf{E}\alpha)(x, y)P\Big(x, \alpha(x), \alpha\big(f_3^1(1, x)\big), \ldots, \alpha\big(f_3^1(p, x)\big), \alpha\big(f_6^1(0, x)\big), \ldots, \alpha\big(f_6^1(N-1, x)\big), \alpha\big(f_7^1(y, x)\big), \alpha(\ulcorner\Phi\urcorner)\Big)$$

where $p = \max(p_0, \ldots, p_a)$ and P is a recursive predicate with $N+p+4$ arguments.

Let $K(n), L(n)$ be the usual pairing functions, $f(n) = \max\Big(K(n), f_3^1\big(1, K(n)\big), \ldots, f_3^1\big(p, K(n)\big), f_6^1\big(0, K(n)\big), \ldots, f_6^1\big(N-1, K(n)\big), f_7^1\big(L(n), K(n)\big)\Big)$ and denote by $(s)_j$ the exponent of the j-th prime in the prime power expansion of s (cf. Kleene [3], p. 230). Put $c(j, s) = (s)_j \dot{-} 1$. If $R(s, t)$ is the predicate [6]

$$C(t)\ \&\ Seq(s)\ \&\ (x)_{lh(s)+1}\Big[\big(f(x) \leqslant lh(s)\big)\ \&\ \big(t \leqslant lh(s)\big) \supset P\Big(K(x), c\big(K(x), s\big), c\big(f_3^1(1, K(x)), s\big), \ldots, c\big(f_3^1(p, K(x)), s\big), c\big(f_6^1(0, K(x)), s\big), \ldots, c\big(f_6^1(N-1, K(x)), s\big), c\big(f_7^1(L(x), K(x)), s\big), c(t, s)\Big)\Big]$$

then (7) is equivalent to $(\mathsf{E}\alpha)_H(z)R\big(\overline{\alpha}(z), \ulcorner\Phi\urcorner\big)$. This accomplishes the proof of the first part of 3.5. Proof of the second part can be obtained by taking dual formulas.

3.6. *The set of closed valid formulas of S_0 is recursively enumerable and the set of closed satisfiable formulas of S_0 is a complement of a recursively enumerable set.*

Proof. By König's "Unendlichkeitslemma" (cf. [5], p. 126) the set $\{n:\ (\mathsf{E}\alpha)_H(x)R\big(\overline{\alpha}(x), n\big)\}$ is a complement of a recursively enumerable set and the set $\{n:\ (\alpha)_H(\mathsf{E}x)S\big(\overline{\alpha}(x), n\big)\}$ is recursively enumerable.

Theorem 3.6 can obviously be inferred from results in [7], Chapter V.

4. The case of a continuous set of truth values. In sections 4, 5, and 6 we assume that $b = 1$, i.e. that we are dealing with just two quantifiers $\mathfrak{Q}_0, \mathfrak{Q}_1$ which we denote by symbols $\bigvee$ and $\bigwedge$. We shall assume that Z is a linearly ordered complete [7] set with a denumerable dense subset Z' and that the interpretations Q_0, Q_1 of the quantifiers are defined as

$$Q_0(Y) = \text{l.u.b.}\ Y, \quad Q_1(Y) = \text{g.l.b.}\ Y \quad \text{for} \quad 0 \neq Y \subseteq Z.$$

Similarly, as in Section 3, we define a sequence of auxiliary systems S_n. The system S_{n+1} is obtained from S_n by adjoining constants $B_{z',q,\Phi}$, $C_{z',q,\Phi}$

[5] Note that in view of the finiteness of Z there is a recursive relation T such that $T(r, s, q_0, \ldots, q_{N-1}) \equiv Q_r(\{q_0, \ldots, q_{N-1}\}) = s$ for arbitrary $r \leqslant a$ and $s, q_0, \ldots, q_{N-1} < N$.

[6] $Seq(s)$ is the predicate "s is a sequence number"; cf. [4], p. 230.

[7] "Complete" means that every non void subset of Z has an l.u.b. and a g.l.b.

for $q = 0, 1, 2, \ldots$, z' in Z' and every formula Φ of S_n which is not a formula of S_{n-1} and which has at most one free variable x_q. The notion of an extension of a model is the same as in Section 3 and Lemmas 3.1 and 3.2 remain valid. We denote by S_∞ the union of systems S_n.

4.1. *Every model μ of S_0 can be extended to a model μ_∞ of S_∞ in such a way that for every z' in Z' and every formula Φ of S_∞ with at most one free variable x_q the following conditions hold:*

$$(8)\qquad \begin{array}{lll} z' \geqslant \mathrm{Val}_{\mu_\infty}\bigvee x_q\Phi & \textit{or} & z'\,\mathrm{non} \geqslant \mathrm{Val}_{\mu_\infty}\mathrm{Sb}\,(x_q/B_{z',q,\Phi})\Phi\,, \\ z' \leqslant \mathrm{Val}_{\mu_\infty}\bigwedge x_q\Phi & \textit{or} & z'\,\mathrm{non} \leqslant \mathrm{Val}_{\mu_\infty}\mathrm{Sb}\,(x_q/C_{z',q,\Phi})\Phi\,. \end{array}$$

Proof. Let $\mu_0 = \mu$ and assume that an extension μ_n of μ which is a model of S_n has been defined in such a way that

$$(9)\qquad \begin{array}{lll} z' \geqslant \mathrm{Val}_{\mu_n}\bigvee x_q\Phi & \text{or} & z'\,\mathrm{non} \geqslant \mathrm{Val}_{\mu_n}\mathrm{Sb}\,(x_q/B_{z',q,\Phi})\Phi\,, \\ z' \leqslant \mathrm{Val}_{\mu_n}\bigwedge x_q\Phi & \text{or} & z'\,\mathrm{non} \leqslant \mathrm{Val}_{\mu_n}\mathrm{Sb}\,(x_q/C_{z',q,\Phi})\Phi \end{array}$$

for every z' in Z', $q = 0, 1, 2, \ldots$ and every formula Φ of S_{n-1} with at most one free variable x_q. This assumption is satisfied if $n = 0$ since in this case S_{n-1} is empty. We shall now extend μ_n to a model of S_{n+1} and have therefore to define $\mu_{n+1}B_{z',q,\Phi}$ and $\mu_{n+1}C_{z',q,\Phi}$ for every z' in Z', $q = 0, 1, 2, \ldots$ and every formula Φ of S_n which is not a formula of S_{n-1} and which has at most one free variable x_q. Let Φ be such a formula and put $Y = \{\mathrm{Val}_\varrho\Phi:\ \varrho \in W_{q,\mu_n}\}$. If l.u.b. Y non $\leqslant z'$, then there is a ϱ in W_{q,μ_n} such that $\mathrm{Val}_\varrho\Phi$ non $\leqslant z'$; we choose a ϱ of this sort and put $\mu_{n+1}B_{z',q,\Phi} = \varrho x_q$. If l.u.b. $Y \geqslant z'$, then we choose $\mu_{n+1}B_{z',q,\Phi}$ arbitrarily. If g.l.b. Y non $\geqslant z'$, then there is a σ in W_{q,μ_n} such that $\mathrm{Val}_\sigma\Phi$ non $\geqslant z'$. We choose again a σ of this sort and put $\mu_{n+1}C_{z',q,\Phi} = \sigma x_q$. If g.l.b. $Y \geqslant z'$, then we choose $\mu_{n+1}C_{z',q,\Phi}$ arbitrarily. The mapping μ_{n+1} thus defined is an extension of μ_n and hence of μ. If Φ is a formula of S_{n-1} with at most one free variable x_q, then $\mathrm{Val}_{\mu_{n+1}}\bigvee x_q\Phi = \mathrm{Val}_{\mu_n}\bigvee x_q\Phi$, and $\mathrm{Val}_{\mu_{n+1}}\mathrm{Sb}\,(x_q/B_{z',q,\Phi})\Phi = \mathrm{Val}_{\mu_n}\mathrm{Sb}\,(x_q/B_{z',q,\Phi})\Phi$, whence in view of (9)

$$z' \geqslant \mathrm{Val}_{\mu_{n+1}}\bigvee x_q\Phi \quad \text{or} \quad z'\,\mathrm{non} \geqslant \mathrm{Val}_{\mu_{n+1}}\mathrm{Sb}\,(x_q/B_{z',q,\Phi})\Phi\,.$$

The same formula holds true if Φ is a formula of S_n which is not a formula of S_{n-1} as we immediately see from the definition of μ_{n+1} and 2.4. A similar relation is also provable for the formula $\bigwedge x_q\Phi$. We thus see that the sequence μ_n of models satisfies (9) for every n and every formula Φ of S_{n-1} with at most one free variable x_q. It is now obvious that (8) is true if we choose as μ_∞ the joint extension of models μ_n.

Remark 1. Theorem 4.1 holds under the assumption that Z is a complete lattice and Z' an arbitrary subset of Z.

We denote by ζ a fixed function which enumerates the elements of Z'. It is easy to see that a Gödel numbering of expressions of S_1 can be so chosen that $\ulcorner B_{\zeta(r),q,\Phi}\urcorner$, $\ulcorner C_{\zeta(r),q,\Phi}\urcorner$ be recursive functions of $r, q, \ulcorner\Phi\urcorner$; indeed we can choose as these Gödel numbers any integers uniquely determined by $q, \ulcorner\Phi\urcorner$ and r. It follows that there exists a Gödel numbering of formulas of S_1 such that the Gödel numbers of closed formulas and the Gödel numbers of formulas with at most one free variable form recursive sets. From this it follows again that it is possible to enumerate the expressions of S_2 in such a way that the Gödel numbers of the constants $B_{\zeta(r),q,\Phi}$, and $C_{\zeta(r),q,\Phi}$ of S_2 are recursive functions of $r, q, \ulcorner\Phi\urcorner$. Continuing in this way we infer that there is a Gödel numbering of S_∞ such that the Gödel numbers of the constants $B_{\zeta(r),q,\Phi}$, $C_{\zeta(r),q,\Phi}$ are recursive functions of $r, q, \ulcorner\Phi\urcorner$. Hence there is a recursive function g which enumerates the Gödel numbers of the constants of S_∞. We continue to denote by $\ulcorner\Phi\urcorner$ the Gödel number of Φ and by $\breve{n}$ the expression with the Gödel number of n.

A further easy consequence of the construction of the Gödel numbering outlined above is that there exist recursive functions $f_1^2, \ldots, f_5^2$ satisfying conditions similar to conditions 1.1 (a)-(d) but with "formula" replaced by "closed formula of S_∞".

We put (8) $f_7^2(k, n) = \ulcorner \mathrm{Sb}\left(x_{f_4^2(n)}/\breve{g}(k)\right)\breve{f}_5^2(n)\urcorner$ if $f_1^2(n) = 2$ and $f_7^2(k, n) = 0$ otherwise. This function is obviously recursive. We also put $f_8^2(r, n) = \ulcorner \mathrm{Sb}\left(x_{f_4^2(n)}/B_{\zeta(r),f_4^2(n),\breve{f}_5^2(n)}\right)\breve{f}_5^2(n)\urcorner$ if $f_1^2(n) = 2$, $f_2^2(n) = 0$ and $f_8^2(r, n) = 0$ otherwise. Similarly we put $f_9^2(r, n) = \ulcorner \mathrm{Sb}\left(x_{f_4^2(n)}/C_{\zeta(r),f_4^2(n),\breve{f}_5^2(n)}\right)\breve{f}_5^2(n)\urcorner$ if $f_1^2(n) = 2$, $f_2^2(n) = 1$ and $f_9^2(r, n) = 0$ otherwise. Both functions f_8^2 and f_9^2 are recursive.

Using these notations we shall express arithmetically the notions of satisfiability and of validity.

A mapping χ of the integers into Z is called a *B-model* if it satisfies the following conditions:

(10) *if* $f_1^2(n) = 1$, *then* $\chi(n) = \varphi_{f_2^2(n)}\left(\chi\left(f_3^2(1, n)\right), \ldots, \chi\left(f_3^2(p_{f_2^2(n)}, n)\right)\right)$,

(11) *if* $f_1^2(n) = 2$ *and* $f_2^2(n) = 0$, *then* $(k)\left(\chi(n) \geqslant \chi\left(f_7^2(k, n)\right)\right)$,

(12) *if* $f_1^2(n) = 2$ *and* $f_2^2(n) = 1$, *then* $(k)\left(\chi(n) \leqslant \chi\left(f_7^2(k, n)\right)\right)$,

(13) *if* $f_1^2(n) = 2$ *and* $f_2^2(n) = 0$, *then* $(r)\left[\zeta(r) \geqslant \chi(n) \vee \zeta(r)\,\mathrm{non} > \chi\left(f_8^2(r, n)\right)\right]$,

(14) *if* $f_1^2(n) = 2$ *and* $f_2^2(n) = 1$, *then* $(r)\left[\zeta(r) \leqslant \chi(n) \vee \zeta(r)\,\mathrm{non} < \chi\left(f_9^2(r, n)\right)\right]$.

(8) f_6^2 is omitted to preserve analogy with Section 3; f_7^2 will play a role analogous to f_7^1 whereas in place of the former f_6^1 we shall have two functions f_8^2 and f_9^2.

4.2. *A closed formula Φ of S_0 is satisfiable (valid) if and only if $\chi(\ulcorner\Phi\urcorner) \in D$ for a (every) B-model χ.*

Proof. Let μ be a model of S_0; construct an extension μ_∞ of μ satisfying (8). We shall show that any function χ such that $\chi(n) = \mathrm{Val}_{\mu_\infty}\breve{n}$ if $0 \leqslant f_1^2(n) < 3$ is a *B-model*.

Condition (10) is obvious. Condition (11) follows from the fact that if $f_1^2(n) = 2$ and $f_2^2(n) = 0$ then $\mathrm{Val}_{\mu_\infty}\breve{n}$ is the l.u.b. of a set Y of which $\mathrm{Val}_{\mu_\infty}\breve{f}_7^2(k, n)$ is an element. Proof of (12) is similar. To prove (13) let us assume that $f_1^2(n) = 2$ and $f_2^2(n) = 0$ and $\zeta(r)\,\mathrm{non} \geqslant \chi(n) = \mathrm{Val}_{\mu_\infty}\breve{n} = \mathrm{Val}_{\mu_\infty}\bigvee x_{f_4^2(n)}\breve{f}_5^2(n)$. It follows from (8) that $\zeta(r)\,\mathrm{non} \geqslant \mathrm{Val}_{\mu_\infty}\mathrm{Sb}\left(x_{f_4^2(n)}/B_{\zeta(r),f_4^2(n),\breve{f}_5^2(n)}\right)\breve{f}_5^2(n) = \mathrm{Val}_{\mu_\infty}\breve{f}_8^2(r, n) = \chi\left(f_8^2(r, n)\right)$. Hence $\zeta(r)\,\mathrm{non} > \chi\left(f_8^2(r, n)\right)$. Proof of (14) is similar.

Now let χ be a B-model and let X_∞ be the set of all constants of S_∞. Define a model μ_∞ of S_∞ in X_∞ by taking $\mu_\infty c = c$ for c in X_∞ and by letting $\mu_\infty F_i^j$ to be a function ψ such that $\psi(c_1, \ldots, c_j) = \chi(\ulcorner F_i^j c_1 \ldots c_j\urcorner)$ for arbitrary $c_1, \ldots, c_j$ in X_∞. We shall prove that if Φ is a formula of S_∞ then

$$\mathrm{Val}_{\mu_\infty}\Psi = \chi(\ulcorner\Psi\urcorner) \quad \textit{for every} \quad \Psi \textit{ in } \hat{\Phi}. \tag{15}$$

Case 1. *Φ is an atomic formula.* In this case Ψ has the form $F_i^j c_1 \ldots c_j$ with $c_1, \ldots, c_j$ in X_∞ and hence (15) follows from the definition of μ_∞.

Case 2. *Φ has the form $\mathfrak{F}_j\Phi_1 \ldots \Phi_{p_j}$.* In this case (15) follows from (10) and the inductive assumption.

Case 3. *Φ has the form $\bigvee x_q \Xi$.* In this case Ψ has the form $\bigvee x_q \Pi$ where $\Pi \in \hat{\Xi}$ or $\Pi \in \hat{\hat{\Xi}}$ according to whether or not x_q is free in Ξ.

Subcase 3a. *x_q is not free in Ξ.* In this case $\mathrm{Val}_{\mu_\infty}\Psi = \mathrm{l.u.b.}\{\mathrm{Val}_{\mu_\infty}\Pi\} = \mathrm{Val}_{\mu_\infty}\Pi$ and hence $\mathrm{Val}_{\mu_\infty}\Psi = \chi(\ulcorner\Pi\urcorner)$ by the inductive assumption. From (11) it follows that $\chi(\ulcorner\Psi\urcorner) \geqslant \chi\left(\ulcorner\mathrm{Sb}\left(x_q/\breve{g}(k)\right)\Pi\urcorner\right) = \chi(\ulcorner\Pi\urcorner)$. If $\chi(\ulcorner\Psi\urcorner)$ were $\neq \chi(\ulcorner\Pi\urcorner)$, then by the density of Z' there would be an r such that $\chi(\ulcorner\Pi\urcorner) < \zeta(r)$ and $\zeta(r)\,\mathrm{non} \geqslant \chi(\ulcorner\Psi\urcorner)$ which contradicts (13) since $f_8^2(r, \ulcorner\Psi\urcorner) = \ulcorner\Pi\urcorner$.

Subcase 3b. *x_q free in Ξ.* In this case $\mathrm{Val}_{\mu_\infty}\Psi = \mathrm{l.u.b.}\{\mathrm{Val}_\varrho\Pi$: *$\varrho$ is a valuation of μ_∞*$\}$; hence, by 2.4 and the inductive assumption,

$$\mathrm{Val}_{\mu_\infty}\Psi = \underset{c \in X_\infty}{\mathrm{l.u.b.}}\,\mathrm{Val}_{\mu_\infty}\mathrm{Sb}\,(x_q/c)\,\Pi = \underset{k=0,1,2,\ldots}{\mathrm{l.u.b.}}\,\chi\left(\ulcorner\mathrm{Sb}\left(x_q/\breve{g}(k)\right)\Pi\urcorner\right) = \underset{k=0,1,2,\ldots}{\mathrm{l.u.b.}}\,\chi\left(f_7^2(k, \ulcorner\Psi\urcorner)\right).$$

From (11) we obtain therefore $\mathrm{Val}_{\mu_\infty}\Psi \leqslant \chi(\ulcorner\Psi\urcorner)$. If $\chi(\ulcorner\Psi\urcorner)$ were different from $\mathrm{Val}_{\mu_\infty}\Psi$ there would be an r such that $\mathrm{Val}_{\mu_\infty}\Psi < \zeta(r)$ and

$\zeta(r)\,\mathrm{non} \geqslant \chi(\ulcorner\Psi\urcorner)$ whence $\mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_q/B_{\zeta(r),q,\Pi})\Pi < \zeta(r)$ and $\zeta(r)\,\mathrm{non} \geqslant \chi(\ulcorner\Psi\urcorner)$. This contradicts (13) since, by the inductive assumption, $\mathrm{Val}_{\mu_\infty}\mathrm{Sb}(x_q/B_{\zeta(r),q,\Pi})\Pi = \chi\big(f_8^2(r, \ulcorner\Psi\urcorner)\big)$.

Case 4. *Φ has the form $\bigwedge x_q \Xi$.* The proof of (15) is similar as in Case 3.

The rest of the proof of 4.2 does not differ from the corresponding part of the proof of 3.4.

Remark 2. Theorem 3.2 holds under the assumption made in Remark 1 and the additional assumption that Z' is a denumerable subset of Z such that $x < y$ implies $(\mathrm{E}z')_{z' \,\mathrm{in}\, Z'}[x \leqslant z' \;\&\; z'\,\mathrm{non} \geqslant y]$.

We consider Z as a topological space in the interval topology (see [1], Chapter IV, § 8). Thus Z is a bicompact space (see l.c., Theorem 14). We assume that the functions φ_j $(j = 0, 1, \ldots, a)$ are continuous.

We put $p = \max(p_0, p_1, \ldots, p_a)$ and denote by lower case German letters (other than $\mathfrak{m}, \mathfrak{n}$) strings consisting of $p+5$ elements of Z. The elements which occur in such a string will be denoted by the corresponding Roman letters with indices: $\mathfrak{w} = (w, w_1, \ldots, w_p, w', w'', w''', \overline{w})$. Consider the following condition on $\mathfrak{w}$ (depending on numeric parameters n, r)

$$\Big\{\big(f_1^2(n) = 1\big) \supset (\mathrm{E}j)_{a+1}\Big[\big(f_2^2(n) = j\big) \;\&\; \big(w = \varphi_j(w_1, \ldots, w_{p_j})\big)\Big]\Big\} \;\&$$
$$\Big\{\big(f_1^2(n) = 2\big) \supset \Big[\big(f_2^2(n) = 0\big) \supset (w \geqslant w')\Big] \;\&\; \Big[\big(f_2^2(n) = 1\big) \supset (w \leqslant w')\Big]\Big\} \;\&$$
$$\Big\{\big(f_1^2(n) = 2\big) \;\&\; \big(f_2^2(n) = 0\big) \supset \Big[\big(\zeta(r) \geqslant w\big) \vee \big(\zeta(r)\,\mathrm{non} > w''\big)\Big]\Big\} \;\&$$
$$\Big\{\big(f_1^2(n) = 2\big) \;\&\; \big(f_2^2(n) = 1\big) \supset \Big[\big(\zeta(r) \leqslant w\big) \vee \big(\zeta(r)\,\mathrm{non} < w'''\big)\Big]\Big\}\,.$$

We call this condition briefly $C_{n,r}(\mathfrak{w})$.

By $\mathfrak{m}, \mathfrak{n}$ (with or without indices) we denote triples of integers. For every triple $\mathfrak{m} = (m, k, r)$, every closed formula Φ of S_0 and every $\mathfrak{w} \in Z^{p+5}$ we denote by $T_{\mathfrak{m},\mathfrak{w},\Phi}$ the "schema"

$$\begin{pmatrix} m & f_3^2(1, m) \ldots f_3^2(p, m) & f_7^2(k, m) & f_8^2(r, m) & f_9^2(r, m)\,, & \ulcorner\Phi\urcorner \\ w & w_1 \quad \ldots \quad w_p & w' & w'' & w''' & \overline{w} \end{pmatrix}.$$

We shall write $E(\Phi, \mathfrak{m}, \mathfrak{n}, \mathfrak{w}, \mathfrak{v})$ if the schemas $T_{\mathfrak{m},\mathfrak{w},\Phi}$ and $T_{\mathfrak{n},\mathfrak{v},\Phi}$ are consistent in the sense that any equation between the elements of the upper rows of the schemas implies the identity of the corresponding elements of the lower rows. In particular $E(\Phi, \mathfrak{m}, \mathfrak{n}, \mathfrak{w}, \mathfrak{v})$ implies that schemas $T_{\mathfrak{m},\mathfrak{w},\Phi}$ and $T_{\mathfrak{n},\mathfrak{v},\Phi}$ define mappings of the elements of their upper rows onto the elements of the lower rows.

4.5. *The set $\{\mathfrak{w}\colon C_{n,r}(\mathfrak{w})\}$ is closed in Z^{p+5} for any n, r.*

Proof. By the continuity of the functions φ_j and the remark that the set $\{w\colon \zeta(r) \geqslant w\}$ is closed and the set $\{w\colon \zeta(r) > w\}$ is open.

4.6. *If D is a closed subset of Z, then a closed formula Φ of S_0 is satisfiable if and only if for every integer s*

$$(15)\qquad (\mathfrak{n}_0, \ldots, \mathfrak{n}_s)(\mathrm{E}\mathfrak{w}_0, \ldots, \mathfrak{w}_s)(i, j)_{s+1}[E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j) \;\&\; C_{n_i,r_i}(\mathfrak{w}_i) \;\&\; (\overline{w}_i \in D)]$$

(we assume that $\mathfrak{n}_i = (n_i, k_i, r_i)$).

If D is an open subset of Z, then a closed formula Φ of S_0 is valid if and only if there is an integer s such that

$$(16)\qquad (\mathrm{E}\mathfrak{n}_0, \ldots, \mathfrak{n}_s)(\mathfrak{w}_0, \ldots, \mathfrak{w}_s)(\mathrm{E}i, j)_{s+1}[E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j) \;\&\; C_{n_i,r_i}(\mathfrak{w}_i) \supset \overline{w}_i \in D]\,.$$

Proof. Let Φ be satisfiable and let χ be a B-model such that $\chi(\ulcorner\Phi\urcorner) \in D$. Choose an integer s and triples $\mathfrak{n}_j = (n_j, k_j, r_j)$, $j = 0, 1, \ldots, s$, and take

$$(17)\qquad \mathfrak{w}_j = \Big(\chi(n_j), \chi\big(f_3^2(1, n_j)\big), \ldots, \chi\big(f_3^2(p, n_j)\big), \chi\big(f_7^2(k_j, n_j)\big), \chi\big(f_8^2(r_j, n_j)\big), \chi\big(f_9^2(r_j, n_j)\big), \chi(\ulcorner\Phi\urcorner)\Big)\,.$$

It is obvious that for arbitrary $i, j \leqslant s$ the consistency condition $E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j)$ is satisfied. Since χ is a B-model, (10)-(14) hold true for $j = 0, 1, \ldots, s$ which means that the conditions $C_{n_i,r_i}(\mathfrak{w}_i)$ are satisfied. Finally from $\chi(\ulcorner\Phi\urcorner) \in D$ we obtain $\overline{w}_j \in D$.

Assume now that (15) holds for arbitrary s. For given $s, \mathfrak{n}_0, \ldots, \mathfrak{n}_s$ denote by $\mathcal{F}_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$ the family of functions χ which map integers into Z and are such that the strings (17) satisfy $C_{n_j,r_j}(\mathfrak{w}_j)$ and $\overline{w} \in D$. The family $\mathcal{F}_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$ is non-void. Indeed, choose any strings $\mathfrak{w}_0, \ldots, \mathfrak{w}_j$ satisfying (15) and define χ on the elements

$$(18)\qquad n_j,\; f_3^2(1, n_j)\,,\; \ldots,\; f_3^2(p, n_j)\,,\; f_7^2(k_j, n_j)\,,\; f_8^2(r_j, n_j)\,,\; f_9^2(r_j, n_j)\,,\; \ulcorner\Phi\urcorner\,, \qquad j = 0, 1, \ldots, s\,,$$

by identifying χ restricted to these elements with the mapping $T_{\mathfrak{n}_j,\mathfrak{w}_j,\Phi}$. Completing χ by choosing its value arbitrarily on elements different from (18) we obtain (in view of the consistency conditions $E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j)$, $i, j \leqslant s$) a function which obviously belongs to $\mathcal{F}_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$. Since $\mathcal{F}_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$ is closed in the Tichonov topology of Z^ω (9) (here we use the assumption that D is closed), we infer that there is a function χ which belongs to all $\mathcal{F}_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$. If $\mathfrak{w}_j$ is defined by (17), then we have $C_{n_j,r_j}(\mathfrak{w}_j)$ for an arbitrary $\mathfrak{n}_j = (n_j, k_j, r_j)$, whence we infer that χ is a B-model. Since $\chi(\ulcorner\Phi\urcorner) = \chi(\overline{w}_j) \in D$, we obtain that Φ is satisfiable.

Now let D be open and put $D' = Z - D$. According to 4.2 Φ is non valid if and only if there is a B-model χ such that $\chi(\ulcorner\Phi\urcorner) \in D'$. According

(9) Z^ω is the space of all infinite sequences of the elements of Z.

to the part of the theorem which is already proved the condition for the existence of such a χ is expressible in the form (15) with D replaced by D'. Hence the validity of Φ is equivalent to (16).

Remark 3. Theorem 4.5 and 4.6 are valid under assumptions made in Remark 2 and the following additional assumptions: Z is a bicompact space, functions φ_j are continuous and the sets $\{(x, y)\colon x \leqslant y\}$, $\{(x, y)\colon x \text{non} > y\}$ are closed in $Z \times Z$.

5. Applications. Consider arbitrary relations $R_1, R_2, \ldots$ with the field Z and let $\mathcal{T}$ be the elementary theory of these relations, i.e. the applied 1st order functional calculus in which the predicate variables are interpreted as $R_1, R_2, \ldots$

5.1. *Let the following assumptions be satisfied: the functions $\varphi_0, \ldots, \varphi_a$ and the relation $\leqslant$ are definable in $\mathcal{T}$; there is a recursive sequence F_r of formulas of $\mathcal{T}$ such that F_r defines the set $\{z\colon \zeta(r) \geqslant z\}$; there is a formula of $\mathcal{T}$ which defines the set D; the theory $\mathcal{T}$ is decidable; then* (a) *if D is a closed set, the set of satisfiable formulas is a complement of a recursively enumerable set;* (b) *if D is an open set, then the set of valid formulas is recursively enumerable.*

Proof. For given $\mathfrak{m}, \mathfrak{n}, \Phi$ the relation $E(\Phi, \mathfrak{m}, \mathfrak{n}, \mathfrak{w}, \mathfrak{v})$ between $\mathfrak{w}$ and $\mathfrak{v}$ is definable by means of a formula of $\mathcal{T}$ depending recursively on $\ulcorner\Phi\urcorner, \mathfrak{m}, \mathfrak{n}$. The same is true of the relations $C_{n,r}(\mathfrak{w})$ and $w \in D$ and hence of the relation

$$(i, j)_{s+1}[E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j) \;\&\; C_{n_i,r_i}(\mathfrak{w}_i) \;\&\; (\overline{w}_i \in D)]\,.$$

If the formula $G_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s,\Phi}$ of $\mathcal{T}$ (with $k = (s+1)(p+5)$ free variables $y_0, \ldots, y_{k-1}$) defines this relation, then the formula $H_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s,\Phi} = (\mathrm{E}y_0, \ldots, y_{k-1})\, G_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s,\Phi}$ defines the relation

$$(\mathrm{E}\mathfrak{w}_0, \ldots, \mathfrak{w}_s)(i, j)_{s+1}[E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j) \;\&\; C_{n_i,r_i}(\mathfrak{w}_i) \;\&\; \overline{w}_i \in D]\,.$$

Hence by 4.6 the condition that Φ be satisfiable is expressible (under the assumption that D be closed) in the form: $(s, \mathfrak{n}_0, \ldots, \mathfrak{n}_s)[H_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s,\Phi}$ *is a theorem of* $\mathcal{T}$]. The theory $\mathcal{T}$ being decidable and $H_{\mathfrak{n}_0,\ldots,\mathfrak{n}_s,\Phi}$ depending recursively on $\mathfrak{n}_0, \ldots, \mathfrak{n}_s, \ulcorner\Phi\urcorner$, it follows that the set of $\ulcorner\Phi\urcorner$ for which Φ is satisfiable forms a complement of a recursively enumerable set. This proves part (a) of 5.1. Proof of (b) is similar.

As a particular case of 5.1 we note

5.2. *If Z is the closed interval $\langle 0, 1\rangle$ ordered in the usual way, D is either an open interval $(m/n, p/q)$ where m, n, p, q are integers and $0 \leqslant m/n < p/q \leqslant 1$ or one of the intervals $\langle 0, m/n)$, $(p/q, 1\rangle$, and if functions $\varphi_0, \ldots, \varphi_a$ are continuous and definable in the elementary theory of real closed fields, then the set of valid formulas is recursively enumerable.*

Indeed, the assumptions of 5.1 are satisfied since we can take as Z' the set of rationals contained in Z and as $\zeta(r)$ the function $K(r)/[K(r)+L(r)+1]$. Obviously there is a recursive sequence of formulas F_r of the elementary theory of real closed fields such that F_r defines the relation $(0 \leqslant z \leqslant 1) \;\&\; z[K(r)+L(r)+1] \leqslant K(r)$. Decidability of the elementary theory of real closed fields is a well-known result of Tarski.

5.3. *If Z is the set $\{0, 1\}^{\omega}$ of all zero-one sequences ordered lexicographically, D is the set of sequences a in Z such that $a(0) = 0$ and φ_j is a recursive mapping of Z^{p_j} into Z, $j = 0, 1, \ldots, a$, then the set of satisfiable formulas is the complement of a recursively enumerable set and the set of valid formulas is recursively enumerable.*

Proof. A mapping φ_j of Z^{p_j} into Z is recursive if the value which the function $\varphi_j(a_1, \ldots, a_{p_j}) = \gamma$ takes for the argument n is a general recursive functional $F(a_1, \ldots, a_{p_j}, n)$ in the sense of [4], p. 275 or, in other words, if the following condition holds: there is an integer e_j such that for arbitrary $a_1, \ldots, a_{p_j}, n$

$$(\mathrm{E}k)\, T_1^{p_j}(\overline{a}_1(k), \ldots, \overline{a}_{p_j}(k), e_j, n, k)\,,$$

$$\gamma(n) = U\big(\mu k T_1^{p_j}(\overline{a}_1(k), \ldots, \overline{a}_{p_j}(k), e_j, n, k)\big)\,.$$

Z is obviously a linearly ordered complete set. If we choose as Z' the set of ultimately vanishing functions, then all assumptions of Theorem 4.6 are satisfied since D is closed and open in Z and the mappings φ_j are continuous according to [2], p. 180, [4], p. 277. We choose the enumerating function ζ so that if $r = 2^{r_0}+2^{r_1}+\ldots+2^{r_k}-1$, $(r_0 < r_1 < \ldots < r_k)$, then $\zeta(r)$ vanishes everywhere except at points $r_0, r_1, \ldots, r_k$.

Let us fix an integer s, $s+1$ triples $\mathfrak{n}_0, \ldots, \mathfrak{n}_s$ and a closed formula Φ of S_0. The relation (between two elements $\mathfrak{w}_i, \mathfrak{w}_j$ of Z^{p+5}) $E(\Phi, \mathfrak{n}_i, \mathfrak{n}_j, \mathfrak{w}_i, \mathfrak{w}_j)$ is expressible as a conjunction of equations between the members of $\mathfrak{w}_i$ and $\mathfrak{w}_j$; this conjunction depends recursively on $\mathfrak{n}_0, \ldots, \mathfrak{n}_s$ and Φ. For given $\mathfrak{n}_i = (n_i, k_i, r_i)$ the relation $C_{n_i,r_i}(\mathfrak{w}_i)$ is equivalent to one of the relations

$$w = \varphi_j(w_1, \ldots, w_{p_j})\,, \quad w \geqslant w'\,, \quad w \leqslant w'\,, \quad \zeta(r_i) \geqslant w \vee \zeta(r_i) \leqslant w''\,,$$
$$\zeta(r_i) \leqslant w \vee \zeta(r_i) \geqslant w'''$$

and again it can be decided recursively to which of the above relations $C_{n_i,r_i}(\mathfrak{w}_i)$ is reducible. It follows that (15) is expressible in the form

$$(s)(\mathfrak{n}_0, \ldots, \mathfrak{n}_s)(\mathrm{E}a_1, \ldots, a_t)\, M_{\Phi,\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$$

where the a_i's run over Z and $M_{\Phi,\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$ is a disjunction (depending recursively on $\ulcorner\Phi\urcorner, \mathfrak{n}_0, \ldots, \mathfrak{n}_s$) of conjunctions of the following relations

$$a_u = \varphi_j(a_{v_1}, \ldots, a_{v_{p_j}})\,, \quad a_i \leqslant a_j\,, \quad \zeta(r_j) \geqslant a_i\,,$$
$$\zeta(r_j) \leqslant a_i\,, \quad a_i \in D\,.$$

We now notice that $a_i \leqslant a_j \equiv (n)[a_i(n) \leqslant a_j(n)]$, $\zeta(r_j) \leqslant a_i \equiv (n)[c(r_j, n) \leqslant a_i(n)]$, $\zeta(r_j) \geqslant a_i \equiv (n)[c(r_j, n) \geqslant a_i(n)]$, $a_i \in D \equiv a_i(0) = 0$ where $c(r, n)$ is a recursive function which gives the value of $\zeta(r)$ at point n. Finally

$$a_u = \varphi_j(a_{v_1}, \ldots, a_{v_{p_j}}) \equiv (n, k)\big[T_1^{p_j}(\bar{a}_{v_1}(k), \ldots, \bar{a}_{v_{p_j}}(k), e_j, n, k) \supset \big(U(k) = a_u(n)\big)\big].$$

Introducing the right-hand sides of these equivalences for the left-hand ones in $M_{\Phi,\mathfrak{n}_0,\ldots,\mathfrak{n}_s}$ and reducing, we infer that (15) is equivalent to a relation of the form

$$(s)(\mathfrak{n}_0, \ldots, \mathfrak{n}_s)(\mathrm{E}a_1, \ldots, a_t)(n)P_{\Phi,\mathfrak{n}_0,\ldots,\mathfrak{n}_s}(\bar{a}_1(n), \ldots, \bar{a}_t(n), n)$$

where P is a recursive relation between $\ulcorner\Phi\urcorner, \mathfrak{n}_0, \ldots, \mathfrak{n}_s, \bar{a}_1(n), \ldots, \bar{a}_t(n), n$. König's lemma enables us to replace

$$(\mathrm{E}a_1, \ldots, a_t)(n)P_{\Phi,\mathfrak{n}_0,\ldots,\mathfrak{n}_s}(\bar{a}_1(n), \ldots, \bar{a}_t(n), n)$$

by $(m)P'(\ulcorner\Phi\urcorner, \mathfrak{n}_0, \ldots, \mathfrak{n}_s, m)$ where P' is recursive. Thus, finally, (15) is equivalent to $(s, \mathfrak{n}_0, \ldots, \mathfrak{n}_s, m)P'(\ulcorner\Phi\urcorner, \mathfrak{n}_0, \ldots, \mathfrak{n}_s, m)$, whence we see that the set of satisfiable formulas is the complement of a recursively enumerable set.

Proof of recursive enumerability of the set of valid formulas follows from the remark that Φ is not valid if and only if it becomes satisfiable after D is replaced by $Z-D$.

6. The case of a well-ordered set of truth values. In the present section we shall deal with the case when Z is the set of ordinals $\leqslant v$ where v is a preassigned ordinal. Z is obviously a bicompact space but does not possess a denumerable dense subset. For this reason the theory set forth in Section 4 is not applicable and we shall have to use a slightly different technique. The chief obstacle to be overcome is the lack of the Gödel numbering (in the usual sense) of the formulas of the auxiliary system S_∞.

We start with the same system S_0 as in Section 4. Formulas of S_0 are said to have the rank 0; we agree that the set of constants of S_0 is empty. S_{n+1} is a system obtained from S_n by adjoining constants $D_{\xi,q,\Phi}$, $E_{q,\Phi}$, $F_{q,\Psi}$ where $\xi \leqslant v$, $q = 0, 1, 2, \ldots$, Φ is a closed formula exactly of rank n having the form $\bigvee x_q H$ and Ψ is a closed formula exactly of rank n having the form $\bigwedge x_q H$. The constants $D_{\xi,q,\Phi}$ whose cardinal number is (for $v \geqslant \omega_1$) uncountable can be thought of as triples $\langle\xi, q, \Phi\rangle$ and formulas of S_{n+1} as finite sequences of symbols. Formulas of S_{n+1} are said to have a rank $\leqslant n+1$. We take as S_∞ the union of all S_n.

It is easy to see that every formula Φ of S_∞ arises from a well determined formula Φ_0 of S_0 by a (uniquely determined) substitution of constants. We now define a "Gödel numbering" of formulas and constants S_∞. For formulas of rank 0 we take as $\llcorner\Phi\lrcorner$ the pair $\langle 0, \ulcorner\Phi\urcorner\rangle$ consisting of the void set and of the usual Gödel number of Φ. If $\llcorner c\lrcorner$ and $\llcorner\Phi\lrcorner$ are already defined for constants and formulas of S_n then we take $\langle\xi, q, \llcorner\Phi\lrcorner\rangle$, $\langle q, \llcorner\Phi\lrcorner\rangle$, $\langle q, \llcorner\Psi\lrcorner\rangle$ as the "Gödel numbers" of the constants $D_{\xi,q,\Phi}$, $E_{q,\Phi}$, $F_{q,\Psi}$ of S_{n+1}. If Φ is a formula of S_{n+1} which arises from a formula Φ_0 of S_0 by the substitution $\mathrm{Sb}(x_{q_1}/c_1 \ldots x_{q_r}/c_r)$, then we take as $\llcorner\Phi\lrcorner$ the ordered pair

$$\left\langle \begin{pmatrix} q_1 & \ldots & q_r \\ \llcorner c_1\lrcorner & \ldots & \llcorner c_r\lrcorner \end{pmatrix}, \ulcorner\Phi_0\urcorner \right\rangle$$

(the symbol in paranthesis denotes a mapping which carries q_j into $\llcorner c_j\lrcorner$; to simplify writing we shall sometimes denote this mapping by a single letter T).

Let N be the set of "Gödel numbers" $\llcorner\Phi\lrcorner$ of formulas of S_∞. We put for $\llcorner\Phi\lrcorner = \langle T, \ulcorner\Phi_0\urcorner\rangle$ in N

$$f_1(\llcorner\Phi\lrcorner) = \begin{cases} f_1^0(\ulcorner\Phi_0\urcorner) & \text{if } \Phi \text{ is closed (i.e. the indices of all free variables of } \Phi_0 \text{ occur in the upper row of } T), \\ 3 & \text{otherwise;} \end{cases}$$

$$f_2(\llcorner\Phi\lrcorner) = f_2^0(\ulcorner\Phi_0\urcorner);$$

$$f_3(j, \llcorner\Phi\lrcorner) = \langle T', f_3^0(j, \ulcorner\Phi_0\urcorner)\rangle \quad \text{where } T' \text{ arises from } T \text{ by restricting it to such } q \text{ for which } x_q \text{ is free in } \breve{f}(j, \ulcorner\Phi_0\urcorner),$$

$$f_4(\llcorner\Phi\lrcorner) = f_4^0(\ulcorner\Phi_0\urcorner);$$

$$f_5(\llcorner\Phi\lrcorner) = \langle T, f_5^0(\ulcorner\Phi_0\urcorner)\rangle.$$

Let C be the set of "Gödel numbers" of constants of S_∞. We define f_6, f_7, f_8, f_9 as follows:

$$f_6(k, \llcorner\Phi\lrcorner) = \begin{cases} \llcorner\Phi\lrcorner & \text{if } f_1(\llcorner\Phi\lrcorner) \neq 2 \text{ or } x_{f_4(\llcorner\Phi\lrcorner)} \text{ is not free in } \breve{f}_5(\llcorner\Phi\lrcorner), \\ \left\langle T \cup \begin{pmatrix} f_4(\llcorner\Phi\lrcorner) \\ k \end{pmatrix}, f_5^0(\ulcorner\Phi_0\urcorner)\right\rangle & \text{otherwise (10);} \end{cases}$$

$$f_7(\xi, \llcorner\Phi\lrcorner) = \begin{cases} f_6(\llcorner D_{\xi, f_4(\llcorner\Phi\lrcorner), \Phi}\lrcorner, \llcorner\Phi\lrcorner) & \text{if } f_1(\llcorner\Phi\lrcorner) = 2,\ f_2(\llcorner\Phi\lrcorner) = 0, \\ \llcorner\Phi\lrcorner & \text{otherwise;} \end{cases}$$

$$f_8(\llcorner\Phi\lrcorner) = \begin{cases} f_6(\llcorner E_{f_4(\llcorner\Phi\lrcorner), \Phi}\lrcorner, \llcorner\Phi\lrcorner) & \text{if } f_1(\llcorner\Phi\lrcorner) = 2,\ f_2(\llcorner\Phi\lrcorner) = 0, \\ \llcorner\Phi\lrcorner & \text{otherwise;} \end{cases}$$

$$f_9(\llcorner\Phi\lrcorner) = \begin{cases} f_6(\llcorner F_{f_4(\llcorner\Phi\lrcorner), \Phi}\lrcorner, \llcorner\Phi\lrcorner) & \text{if } f_1(\llcorner\Phi\lrcorner) = 2,\ f_2(\llcorner\Phi\lrcorner) = 1, \\ \llcorner\Phi\lrcorner & \text{otherwise;} \end{cases}$$

(10) It is assumed that k runs over C.

Functions f_1-f_5 perform with respect to the present system S_∞ the same role that functions f_1^0-f_5^0 do with respect to S_0. $f_6(k, \llcorner\Phi\lrcorner)$ is the "Gödel number" of a formula which we obtain from Φ by dropping the initial quantifier (if any) and substituting $\breve{k}$ for the variable bound by this quantifier; f_7, f_8, f_9 are special cases of f_6 obtained for special values of k.

The notion of extension of a model is the same as in Section 3; Lemmas 3.1 and 3.2 remain valid.

6.1. *Every model μ of S_0 can be extended to a model μ_∞ of S_∞ in such a way that the following conditions be satisfied for every formula Φ of S_∞ with at most one free variable x_q:*

(19) *If* $\mathrm{Val}_{\mu_\infty} \bigvee x_q \Phi$ *is a limit number, then for every* ξ

$$\xi \geqslant \mathrm{Val}_{\mu_\infty} \bigvee x_q \Phi \vee \xi+1 \leqslant \mathrm{Val}_{\mu_\infty} \mathrm{Sb}\,(x_q/D_{\xi,q,\bigvee x_q\Phi})\,\Phi\,,$$

(20) *If* $\mathrm{Val}_{\mu_\infty} \bigvee x_q \Phi$ *is not a limit number, then*

$$\mathrm{Val}_{\mu_\infty} \bigvee x_q \Phi = \mathrm{Val}_{\mu_\infty} \mathrm{Sb}\,(x_q/E_{q,\bigvee x_q\Phi})\,\Phi;$$

(21) $$\mathrm{Val}_{\mu_\infty} \bigwedge x_q \Phi = \mathrm{Val}_{\mu_\infty} \mathrm{Sb}\,(x_q/F_{q,\bigwedge x_q\Phi})\,\Phi\,.$$

Proof of this theorem is similar to that of 4.1 and can be omitted.

A function χ which maps N into Z is called a *C-model* if it satisfies the following conditions for arbitrary n in N, k in C and $\xi \leqslant v$:

(22) *If* $f_1(n) = 1$, *then* $\chi(n) = \varphi_{f_2(n)}\big(\chi(f_3(1, n)), \ldots, \chi(f_3(p_{f_2(n)}, n))\big)$.

(23) *If* $f_1(n) = 2$ *and* $f_2(n) = 0$, *then* $\chi(n) \geqslant \chi(f_6(k, n))$.

(24) *If* $f_1(n) = 2$ *and* $f_2(n) = 1$, *then* $\chi(n) \leqslant \chi(f_6(k, n))$.

(25) *If* $f_1(n) = 2$ *and* $f_2(n) = 0$, *then* $\chi(n) = \chi(f_8(n))$ $\vee\,(\chi(n)$ *is a limit number*$)\;\&\;[\xi \geqslant \chi(n) \vee \xi+1 \leqslant \chi(f_7(\xi, n))]$.

(26) *If* $f_1(n) = 2$ *and* $f_2(n) = 1$, *then* $\chi(n) = \chi(f_9(n))$.

6.2. *A closed formula Φ of S_0 is satisfiable (valid) if and only if $\chi(\llcorner\Phi\lrcorner) \in D$ for at least one (for each) C-model χ.*

Proof. It is sufficient to show that for every model μ_∞ of S_∞ satisfying (19)-(21) there is a C-model χ such that

(27) $\chi(\llcorner\Phi\lrcorner) = \mathrm{Val}_{\mu_\infty}\Phi$ *for every closed formula Φ of S_∞*

and that every C-model χ determines a model μ_∞ of S_∞ such that (27) holds.

The first statement is proved by verifying that if μ_∞ satisfies (19)-(21), then the function χ defined by (27) is a C-model. The second statement

is proved as follows. Let χ be a C-model. Define a model μ_∞ of S_∞ in C (set of all constants) by putting $\mu_\infty c = c$ for c in C and letting $\mu_\infty F_k^j$ be a function ψ such that $\psi(c_1, \ldots, c_j) = \chi(\llcorner F_k^j c_1, \ldots, c_j\lrcorner)$ for $c_1, \ldots, c_j$ in C. It is sufficient to prove (27) for every formula Φ in $\hat{\Psi}$ where Ψ is an arbitrary formula of S_∞.

The cases when Ψ is atomic or begins with a propositional connective are dealt with exactly as in 4.3.

Case 3. *Ψ has the form* $\bigvee x_q \Xi$. Hence Φ has the form $\bigvee x_q H$ where $H \in \hat{\hat{\Xi}}$ or $H \in \hat{\Xi}$ according as x_q is or is not free in Ξ and $\mathrm{Val}_{\mu_\infty}\Phi = \underset{c \in C}{\mathrm{l.u.b.}}\,\chi\big(f_6(\llcorner c\lrcorner, \llcorner\Phi\lrcorner)\big)$. In both cases we obtain from (23) $\chi(\llcorner\Phi\lrcorner) \geqslant \mathrm{Val}_{\mu_\infty}\Phi$. If $\chi(\llcorner\Phi\lrcorner)$ is not a limit number, then, by (25),

$$\chi(\llcorner\Phi\lrcorner) = \chi\big(f_8(\llcorner\Phi\lrcorner)\big) = \chi\big(\llcorner \mathrm{Sb}\,(x_q/E_{q,\Phi})\,H\lrcorner\big)$$
$$= \chi\big(f_6(\llcorner E_{q,\Phi}\lrcorner, \llcorner\Phi\lrcorner)\big) \leqslant \underset{c \in C}{\mathrm{l.u.b.}}\,\chi\big(f_6(\llcorner c\lrcorner, \llcorner\Phi\lrcorner)\big)$$

which proves (27). If $\chi(\llcorner\Phi\lrcorner)$ is a limit number, then again by (25)

$$(\xi)\,[\xi < \chi(\llcorner\Phi\lrcorner) \supset \xi+1 \leqslant \chi\big(f_6(\llcorner D_{\xi,q,\Phi}\lrcorner, \llcorner\Phi\lrcorner)\big)]\,.$$

If we had $\chi(\llcorner\Phi\lrcorner) > \mathrm{Val}_{\mu_\infty}\Phi$, then we would obtain

$$\chi\big(f_6(\llcorner D_{\xi,q,\Phi}\lrcorner, \llcorner\Phi\lrcorner)\big) < \mathrm{Val}_{\mu_\infty}\Phi + 1 \leqslant \chi\big(f_6(\llcorner D_{\xi,q,\Phi}\lrcorner, \llcorner\Phi\lrcorner)\big)$$

which is a contradiction. (27) is thus proved in Case 3.

Case 4. *Ψ has the form* $\bigwedge x_q \Xi$. Hence Φ has the form $\bigwedge x_q H$ where $H \in \hat{\Xi}$ or $H \in \hat{\hat{\Xi}}$ and $\mathrm{Val}_{\mu_\infty}\Phi = \underset{c \in C}{\mathrm{g.l.b.}}\,\big(f_6(\llcorner c\lrcorner, \llcorner\Phi\lrcorner)\big)$. From (24) we obtain

$$\chi(\llcorner\Phi\lrcorner) \leqslant \mathrm{Val}_{\mu_\infty}\Phi$$

and from (26)

$$\chi(\llcorner\Phi\lrcorner) = \chi\big(f_9(\llcorner\Phi\lrcorner)\big) = \chi\big(f_6(\llcorner F_{q,\Phi}\lrcorner, \llcorner\Phi\lrcorner)\big)$$
$$\geqslant \underset{c \in C}{\mathrm{g.l.b.}}\,\chi\big(f_6(\llcorner c\lrcorner, \llcorner\Phi\lrcorner)\big) = \mathrm{Val}_{\mu_\infty}\Phi\,,$$

whence we obtain (27). 6.2 is thus proved.

Our next task will be to express the conditions for validity and satisfiability so as to make evident their recursive character. To this end we shall consider an arbitrary but fixed finite set of relations $R_1, \ldots, R_k$ defined in the set Z and denote by $\mathcal{T}$ the elementary theory of relations $R_1, \ldots, R_k, \leqslant$. The variables of $\mathcal{T}$ will be denoted by Greek letters $\alpha, \beta, \ldots$ with or without indices. We extend $\mathcal{T}$ to a theory $\mathcal{T}^*$ by adjoining variables $\mathrm{m}, \mathrm{n}, \ldots$ (with or without indices) ranging over N, variables $\mathrm{k}, \mathrm{l}, \ldots$ (with or without indices) ranging over C, variables $\varkappa, \lambda, \ldots$ (with or without indices) ranging over integers, constants for individual integers and symbols $\bar{f}_1, \ldots, \bar{f}_9$ for functions f_1-f_9. Variables $\mathrm{m}, \mathrm{n}, \ldots$ are N-terms, $\mathrm{k}, \mathrm{l}, \ldots$ are C-terms, $\varkappa, \lambda, \ldots$ and numerals (constants for integers) are ω-terms and $\alpha, \beta, \ldots$ are Z-terms. If ν is an N-term, $\bar{k}$ a C-term, $\bar{\xi}$ a Z-variable and $\bar{\varkappa}$ an ω-variable or a numeral, then $\bar{f}_1(\nu), \bar{f}_2(\nu), \bar{f}_4(\iota)$ are ω-terms, $\bar{f}_3(\bar{\varkappa}, \nu), \bar{f}_5(\nu), \bar{f}_6(\bar{k}, \nu), \bar{f}_7(\bar{\xi}, \nu), \bar{f}_8(\nu), \bar{f}_9(\nu)$ are N-terms. This concludes the description of terms of $\mathcal{T}^*$. Atomic formulas of $\mathcal{T}^*$ are those of $\mathcal{T}$ and equations between terms of the same kind. Other formulas of $\mathcal{T}^*$ are constructed from the atomic ones in the usual way. It is clear how to define the notion of satisfaction for formulas of $\mathcal{T}^*$ or $\mathcal{T}$. We shall write $\models_{\mathcal{T}} M[\xi_1, \ldots, \xi_r]$ instead: "$\xi_1, \ldots, \xi_r$ *satisfy formula* M *of* $\mathcal{T}$" and similarly for the theory $\mathcal{T}^*$. We shall need two simple lemmas:

6.3. *Let M be a formula of $\mathcal{T}^*$ whose bound variables are exclusively the Z-variables and whose free variables are* $\mathrm{n}_1, \ldots, \mathrm{n}_p, \mathrm{k}_1, \ldots, \mathrm{k}_q, \varkappa_1, \ldots, \varkappa_r$. *Let $a_1, \ldots, a_r$ be integers, $c_1, \ldots, c_q$ elements of C and*

$$\llcorner \Phi_j \lrcorner = \left\langle \begin{pmatrix} s_{j1}, \ldots, s_{ju_j} \\ c_{j1}, \ldots, c_{ju_j} \end{pmatrix}, \ulcorner \Phi_{j0} \urcorner \right\rangle, \qquad j = 1, 2, \ldots, p,$$

elements of N. Under these assumptions there is a formula M' of $\mathcal{T}^*$ depending recursively on $u_1, \ldots, u_p, s_{11}, \ldots, s_{pu_p}, \ulcorner \Phi_{10} \urcorner, \ldots, \ulcorner \Phi_{p0} \urcorner, a_1, \ldots, a_r, M$ *and such that*

$$(28) \qquad \models_{\mathcal{T}^*} M[\llcorner \Phi_1 \lrcorner, \ldots, \llcorner \Phi_p \lrcorner, c_1, \ldots, c_q, a_1, \ldots, a_r] \equiv \models_{\mathcal{T}^*} M'[c_{11}, \ldots, c_{1u_1}, \ldots, c_{p1}, \ldots, c_{pu_p}, c_1, \ldots, c_q].$$

Proof. Values of terms depending on the variables $\mathrm{n}_1, \ldots, \mathrm{n}_p$ can be evaluated, i.e. represented in the form of numerals or in the form

$$\left\langle \begin{pmatrix} v_1, \ldots, v_t \\ d_1, \ldots, d_t \end{pmatrix}, \ulcorner \Psi \urcorner \right\rangle$$

where the integers $v_1, \ldots, v_t, \ulcorner \Psi \urcorner$ are effectively calculable from the $s_{ji}, \ulcorner \Phi_{j,0} \urcorner, a_j$ and each d_j is either one of the c_{ji} or one of the c_m. Every equation between these values can be expressed as equations between integers or between the constants d_j. The former equations are then replaced by their truth values. In this way the left hand side of (28) is transformed into a condition representable by the right hand side of (28) and the formula M' can be constructed effectively (i.e. recursively) from M and integers $u_1, \ldots, u_p, s_{11}, \ldots, s_{pu_p}, \ulcorner \Phi_{10} \urcorner, \ldots, \ulcorner \Phi_{p0} \urcorner, a_1, \ldots, a_r$.

6.4. *Let M be a formula of $\mathcal{T}^*$ whose bound variables are exclusively the Z-variables and whose free variables are* $\mathrm{m}_1, \ldots, \mathrm{m}_f, \mathrm{k}_1, \ldots, \mathrm{k}_e, \varkappa_1, \ldots, \varkappa_r$. *Let $m_1, \ldots, m_f$ be elements of N, let*

$$k_j = \llcorner D_{\xi, a_j, \breve{n}_j} \lrcorner \quad \textit{for} \quad j = 1, 2, \ldots, g,$$
$$k_j = \llcorner E_{a_j, \breve{n}_j} \lrcorner \quad \textit{for} \quad j = g+1, \ldots, h,$$
$$k_j = \llcorner F_{a_j, \breve{n}_j} \lrcorner \quad \textit{for} \quad j = h+1, \ldots, e,$$

and let $a_1, \ldots, a_r$ be integers. Then there is a formula M' of $\mathcal{T}^*$ depending recursively on $g, h, q_1, \ldots, q_e, a_1, \ldots, a_r$ and M such that

$$\models_{\mathcal{T}^*} M[m_1, \ldots, m_f, k_1, \ldots, k_e, a_1, \ldots, a_r] \equiv \models_{\mathcal{T}^*} M'[\xi_1, \ldots, \xi_g, m_1, \ldots, m_f, n_1, \ldots, n_e].$$

Proof. Similarly as in the proof of 6.3 we evaluate the values of terms depending on $m_1, \ldots, m_f, k_1, \ldots, k_e, a_1, \ldots, a_r$ and replace equations between these values by equations between ordinals or between integers or between elements of N. Equations between integers are then replaced by their truth values.

Let now M_u be a recursive sequence of formulas of $\mathcal{T}^*$ whose free variables are $\lambda, \varkappa_1, \ldots, \varkappa_{p_u}, \mathrm{n}_1, \ldots, \mathrm{n}_{q_u}, \mathrm{k}_1, \ldots, \mathrm{k}_{r_u}$ and whose bound variables are exclusively the Z-variables.

6.5. *There is a recursive sequence $G_{p,e}$ of closed formulas of $\mathcal{T}$ such that*

$$(29) \qquad (u) \models_{\mathcal{T}^*} \big((\varkappa_1, \ldots, \varkappa_{p_u}, \mathrm{n}_1, \ldots, \mathrm{n}_{q_u}, \mathrm{k}_1, \ldots, \mathrm{k}_{r_u}) M_u\big)[e] \equiv (p) \models_{\mathcal{T}} G_{p,e}.$$

Proof. The left hand side of (29) is equivalent to [11]

$$(h, u)_\omega (a_1, \ldots, a_{p_u})_\omega (n_1, \ldots, n_{q_u})_{N_h} (c_1, \ldots, c_{r_u})_{C_h} \models_{\mathcal{T}^*} M_u[e, a_1, \ldots, a_{p_u}, n_1, \ldots, n_{q_u}, c_1, \ldots, c_{r_u}]$$

where N_h is the set of "Gödel numbers" of formulas of S_h and C_h is the set of "Gödel numbers" of constants of S_h. It will be sufficient to show (by induction on h) that there exist closed formulas $H_{h,p,e}$ of $\mathcal{T}$ which depend recursively on h, p, e such that

$$(30) \qquad (u)_\omega (a_1, \ldots, a_{p_u})_\omega (n_1, \ldots, n_{q_u})_{N_h} (c_1, \ldots, c_{r_u})_{C_h} \models_{\mathcal{T}^*} M_u[e, a_1, \ldots, a_{p_u}, n_1, \ldots, n_{q_u}, c_1, \ldots, c_{r_u}] \equiv p \models_{\mathcal{T}} H_{h,p,e}.$$

Consider first the case $h > 0$. We shall show how to reduce the left hand side of (30) to a similar condition with h replaced by $h-1$.

The left hand side of (30) is equivalent to

$$\begin{aligned} (31) \qquad & (u)_\omega (a_1, \ldots, a_{p_u})_\omega (n_1, \ldots, n_{q_u})_{N_{h-1}} (c_1, \ldots, c_{r_u})_{C_h} \\ & \models_{\mathcal{T}^*} M_u[e, a_1, \ldots, a_{p_u}, n_1, \ldots, n_{q_u}, c_1, \ldots, c_{r_u}] \\ & \;\&\; (u)_\omega (a_1, \ldots, a_{p_u})_\omega (n_1, \ldots, n_{q_u})_{N_h - N_{h-1}} (c_1, \ldots, c_{r_u})_{C_h} \\ & \models_{\mathcal{T}^*} M_u[e, a_1, \ldots, a_{p_u}, n_1, \ldots, n_{q_u}, c_1, \ldots, c_{r_u}]. \end{aligned}$$

If we replace in the second conjunct every n_j by

$$\left\langle \begin{pmatrix} s_{j1}, \ldots, s_{jv_j} \\ c_{j1}, \ldots, c_{jv_j} \end{pmatrix}, n_j' \right\rangle, \qquad j = 1, \ldots, q_u,$$

[11] $(x)_X$ means: for every x in X; we denote by ω the set of integers.

where the s_{ij} run over integers, the c_{ji} run over C_h and the n'_j over the set $X_{s_{j1}\dots s_{ju_j}}$ of Gödel numbers of formulas of S_0 with the free variables $s_{j1}, \dots, s_{ju_j}$, we obtain a condition equivalent to (31). Using Lemma 6.3 we can therefore replace (31) by an equivalent condition

$$\begin{aligned}
&(u)_\omega(a_1, \dots, a_{p_u})_\omega(n_1, \dots, n_{q_u})_{N_{h-1}}(c_1, \dots, c_{r_u})_{C_h}\\
&\models_{\mathcal{T}^*} M_u[e, a_1, \dots, a_{p_u}, n_1, \dots, n_{q_u}, c_1, \dots, c_{q_u}]\\
&\& (u, v_1, \dots, v_{q_u}, a_1, \dots, a_{p_u}, s_{11}, \dots, s_{1v_1} \dots s_{q_u 1}, \dots, s_{q_u v_{q_u}})_\omega\\
&(n'_1)_{X_{s_{11}\dots s_{1v_1}}} \dots (n'_{q_u})_{X_{s_{q_u 1},\dots,s_{q_u v_{q_u}}}}(c_1, \dots, c_{r_u}, c_{11}, \dots, c_{1v_1}, \dots, c_{q_u v_{q_u}})_{C_h}\\
&\models_{\mathcal{T}^*} M'_{u,v_1,\dots,v_{q_u},a_1,\dots,a_{p_u},s_{11},\dots,s_{q_u v_{q_u}},n'_1,\dots,n'_{q_u}}[c_1, \dots, c_{r_u}, c_{11}, \dots, c_{q_u v_{q_u}}]
\end{aligned}$$

where the formula M' depends recursively on the indices. Contracting we transform this condition to a condition

$$(32)\qquad (u)_\omega(b_1, \dots, b_{t_u})_\omega(n_1, \dots, n_{w_u})_{N_{h-1}}(d_1, \dots, d_{y_u})_{C_h} \models_{\mathcal{T}^*} M^*_u[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, d_1, \dots, d_{y_u}]\,.$$

We divide this formula into two parts letting the d_j in the first part to run over C_{h-1} and in the second over $C_h - C_{h-1}$. In the second part each d_j can be replaced either by $\llcorner D_{\xi_j, q_j, \breve{m}_j}\lrcorner$ or by $\llcorner E_{q_j, \breve{m}_j}\lrcorner$ or by $\llcorner F_{q_j, \breve{m}_j}\lrcorner$, i.e. by $\langle \xi_j, q_j, m_j\rangle$ or $\langle q_j, m_j\rangle$ where m_j is assumed to run over a subset of N_{h-1}. In this way (32) is replaced by

$$\begin{aligned}
(33)\qquad &(u)_\omega(b_1, \dots, b_{t_u})_\omega(n_1, \dots, n_{w_u})_{N_{h-1}}(d_1, \dots, d_{y_u})_{C_{h-1}}\\
&\models_{\mathcal{T}^*} M^*_u[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, d_1, \dots, d_{y_u}]\ \&\\
&(u)_\omega(b_1, \dots, b_{t_u})_\omega(n_1, \dots, n_{w_u})_{N_{h-1}}(i)_{i \leqslant y_u}(\xi_1, \dots, \xi_i)_Z\\
&(m_1, \dots, m_i)_{N_{h-1}}(q_1, \dots, q_i)_\omega(j)_{i<j\leqslant y_u}\\
&(q_{i+1}, \dots, q_{j_u})_\omega(m_{i+1}, \dots, m_{y_u})_{N_{h-1}}\{(f_1(m_1) = \dots = f_1(m_{y_u}) = 2)\\
&\& (f_2(m_1) = \dots = f_2(m_j) = 0)\ \&\ (f_2(m_{j+1}) = \dots = f_2(m_{y_u}) = 1)\\
&\supset \models_{\mathcal{T}^*} M^*_u[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, \langle \xi_1, q_1, m_1\rangle, \dots,\\
&\langle \xi_i, q_i, m_i\rangle, \langle q_{i+1}, m_{i+1}\rangle, \dots, \langle q_{y_u}, m_{y_u}\rangle]\}\,.
\end{aligned}$$

We now use Lemma 6.4 and replace

$$\models_{\mathcal{T}^*} M^*_u[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, \langle \xi_1, q_1, m_1\rangle, \dots, \langle \xi_i, q_i, m_i\rangle, \langle q_{i+1}, m_{i+1}\rangle, \dots, \langle q_{y_u}, m_{y_u}\rangle]$$

by an equivalent condition

$$\models_{\mathcal{T}^*} M^{**}_{u,i,j,q_1,\dots,q_{y_u}}[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, m_1, \dots, m_{y_u}, \xi_1, \dots, \xi_i]$$

with M^{**} depending recursively on the indices. Further we denote the formula

$$\begin{aligned}
(\alpha_1, \dots, \alpha_i)\,[(\bar{\mathrm{f}}_1(\mathrm{m}_1) = 2)\ \&\ \dots\ \&\ (\bar{\mathrm{f}}_1(\mathrm{m}_i) = 2)\ \&\ (\bar{\mathrm{f}}_2(\mathrm{m}_1) = 0)\ \&\ \dots\ \&\\
(\bar{\mathrm{f}}_2(\mathrm{m}_j) = 0)\ \&\ (\bar{\mathrm{f}}_2(\mathrm{m}_{j+1}) = 1)\ \&\ \dots\ \&\ (\bar{\mathrm{f}}_2(\mathrm{m}_{y_u}) = 1) \supset M^{**}_{u,q_1,\dots,q_{y_u}}]
\end{aligned}$$

by $M^{***}_{u,i,j,q_1,\dots,q_{y_u}}$.

(33) is thus equivalent to

$$\begin{aligned}
&(u)_\omega(b_1, \dots, b_{t_u})_\omega(n_1, \dots, n_{w_u})_{N_{h-1}}(d_1, \dots, d_{y_u})_{C_{h-1}}\\
&\models_{\mathcal{T}^*} M^*_u[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, d_1, \dots, d_{y_u}]\\
&\& (u)_\omega(b_1, \dots, b_{t_u})_\omega(n_1, \dots, n_{w_u}, m_1, \dots, m_{y_u})_{N_{h-1}}(i, j)_{i<j\leqslant y_u}\\
&(q_1, \dots, q_{y_u})_\omega \models_{\mathcal{T}^*} M^{***}_{u,i,j,q_1,\dots,q_{y_u}}[e, b_1, \dots, b_{t_u}, n_1, \dots, n_{w_u}, m_1, \dots, m_{y_u}]
\end{aligned}$$

which, after contradiction can be brought to the same form as the left hand side of (30) but with h replaced by $h-1$.

In order to prove (30) it remains to prove it for $h = 0$. The left hand side of (30) has in this case the form

$$(u)_\omega(a_1, \dots, a_{p_u})_\omega(n_1, \dots, n_{q_u})_{N_0} \models_{\mathcal{T}^*} M_u[e, a_1, \dots, a_{p_u}, n_1, \dots, n_{q_u}]\,.$$

Performing the same operations as in the first step of the reduction of h to $h-1$ we are left with a condition of the form

$$(u)_\omega(a_1, \dots, a_{p_u})_\omega(n'_1, \dots, n'_{q_u})_{X_0} \models_{\mathcal{T}} M'_{u,n'_1,\dots,n'_{q_u},a_1,\dots,a_{p_u},e}$$

where M' is a formula of $\mathcal{T}$ and X_0 is the set of Gödel numbers of closed formulas of S_0. Contracting, we finally obtain a condition of the form $(p) \models_{\mathcal{T}} H_{0,p,e}$ where $H_{0,p,e}$ is a closed formula of $\mathcal{T}$ depending recursively on p and e. Lemma 6.5 is thus proved.

We now repeat with minor changes the construction carried out on pp. 176-178. Let sequences of $p+6$ ordinals be denoted by German letters and their terms denoted by corresponding Roman letters:

$$\mathfrak{w} = (w, w_1, \dots, w_p, w', w'', w''', w^{\mathrm{iv}}, \overline{w})$$

where

$$p = \max(p_0, \dots, p_a)\,.$$

Assume that the relation $w = \varphi_j(w_1, \dots, w_{p_j})$ is definable in $\mathcal{T}$ and let $F_j(\alpha, \alpha_1, \dots, \alpha_{p_j})$ be a formula of $\mathcal{T}$ which defines this relation, $j = 0, 1, \dots, a$. Notice that relations $\xi \leqslant \eta$, $\xi + 1 \leqslant \eta$ and "*ξ is a limit number*" are definable in $\mathcal{T}$ (which was supposed to be an extension of the elementary theory of the $\leqslant$ relation). Let formulas which define these relations be $\alpha \leqslant \beta$ [12], $\alpha + 1 \leqslant \beta$, $\Lambda(\alpha)$. Consider the relation $C(n, \xi, \mathfrak{w})$ where $n \in N$, $\xi \in Z$, $\mathfrak{w} \in Z^{p+6}$:

[12] Whenever convenient we write $\beta \geqslant \alpha$ instead of $\alpha \leqslant \beta$.

$$\{(f_1(n)=1)\supset(\mathrm{E}j)_{a+1}[(f_2(n)=j)\ \&\ (w=\varphi_j(w_1,\ldots,w_{p_j}))]\}$$
$$\&\ \{(f_1(n)=2)\supset[(f_2(n)=0)\supset(w\geqslant w')]\ \&\ [(f_2(n)=1)\supset(w\leqslant w')]\}$$
$$\&\ \big((f_1(n)=2)\ \&\ (f_2(n)=0)\supset\{(w=w'')\vee(w\text{ is a limit number})$$
$$\&\ [(\xi\geqslant w)\vee(\xi+1\leqslant w''')]\}\big)\ \&\ \{(f_1(n)=2)\ \&\ (f_2(n)=1)\supset(w=w^{\mathrm{iv}})\}\,.$$

This relation is obviously definable in $\mathcal{T}^*$ by the formula $\Gamma(\mathrm{n},\alpha,\alpha_1,\ldots,\alpha_p,\alpha',\alpha'',\alpha''',\alpha^{\mathrm{iv}},\overline{\alpha},\beta)$:

$$\{(\bar{\mathrm{f}}_1(\mathrm{n})=1)\supset(\mathrm{E}j)_{a+1}[(\bar{\mathrm{f}}_2(\mathrm{n})=j)\ \&\ F_j(\alpha,\alpha_1,\ldots,\alpha_{p_j})]\}$$
$$\&\ \{(\bar{\mathrm{f}}_1(\mathrm{n})=2)\supset[(\bar{\mathrm{f}}_2(\mathrm{n})=0)\supset(\alpha\geqslant\alpha')]\ \&\ [(\bar{\mathrm{f}}_2(\mathrm{n})=1)\supset(\alpha\leqslant\alpha')]\}$$
$$\&\ \big((\bar{\mathrm{f}}_1(\mathrm{n})=2)\ \&\ (\bar{\mathrm{f}}_2(\mathrm{n})=0)\supset\{(\alpha=\alpha'')\vee\Lambda(\alpha)\ \&\ [(\beta\geqslant\alpha)\vee$$
$$\vee(\beta+1\leqslant\alpha''')]\}\big)\ \&\ \{(\bar{\mathrm{f}}_1(\mathrm{n})=2)\ \&\ (\bar{\mathrm{f}}_2(\mathrm{n})=1)\supset(\alpha=\alpha^{\mathrm{iv}})\}\,.$$

We abbreviate this formula as $\Gamma(\mathrm{n},\boldsymbol{\alpha})$. The symbol $(\mathrm{E}j)_{a+1}$ is of course an abbreviation for a logical sum of $a+1$ terms. Note that only Z-variables are bound in Γ.

Next we introduce the "consistency relation" $E(\ulcorner\Phi\urcorner,n_1,k_1,\xi_1,n_2,k_2,\xi_2,\mathfrak{w}_1,\mathfrak{w}_2)$ where $n_i\in N$, $k_i\in C$, $\xi_i\in Z$, $i=1,2$ and Φ is a closed formula of S_0. To define this relation we consider "schemas" $T_{n,k,\xi,\mathfrak{w},\Phi}$

$$\begin{pmatrix} n, & f_3(1,n), & \ldots, & f_3(p,n), & f_6(k,n), & f_7(\xi,n), & f_8(n), & f_9(n), & \ulcorner\Phi\urcorner \\ w, & w_1, & \ldots, & w_p, & w', & w'', & w''', & w^{\mathrm{iv}}, & \overline{w} \end{pmatrix}$$

and agree that $E(\ulcorner\Phi\urcorner,n_1,k_1,\xi_1,n_2,k_2,\xi_2,\mathfrak{w}_1,\mathfrak{w}_2)$ holds if and only if the identity of any two elements in the upper rows of schemas $T_{n_1,k_1,\xi_1,\mathfrak{w}_1,\Phi}$, $T_{n_2,k_2,\xi_2,\mathfrak{w}_2,\Phi}$ implies the identity of the corresponding elements in the lower rows. The consistency relation is obviously definable in $\mathcal{T}^*$ by means of an open formula involving only the identity predicate. We note this formula as $\mathrm{E}(\varkappa,\mathrm{n}_1,\mathrm{k}_1,\beta_1,\mathrm{n}_2,\mathrm{k}_2,\beta_2,\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$.

6.6. *If the functions $\varphi_0,\ldots,\varphi_a$ are continuous, then the set $\{\mathfrak{w}: C(n,\xi,\mathfrak{w})\}$ is closed in Z^{p+6} for arbitrary n in N and ξ in Z.*

The proof is obvious.

6.7 *If the set D is closed in Z, then a closed formula Φ of S_0 is satisfiable if and only if for every integer s*

$$(34)\quad \begin{gathered}(n_0,\ldots,n_s)_N(k_0,\ldots,k_s)_C(\xi_0,\ldots,\xi_s)_Z(\mathrm{E}\mathfrak{w}_0,\ldots,\mathfrak{w}_s)(i,j)_{s+1}\\ [E(\ulcorner\Phi\urcorner,n_i,k_i,\xi_i,n_j,k_j,\xi_j,\mathfrak{w}_i,\mathfrak{w}_j)\ \&\ C(n_i,\xi_i,\mathfrak{w}_i)\ \&\ (\overline{w}_i\in D)]\,.\end{gathered}$$

If D is open in Z, then a closed formula Φ of S_0 is valid if and only if there is an integer s such that

$$(35)\quad \begin{gathered}(\mathrm{E}n_0,\ldots,n_s)_N(\mathrm{E}k_0,\ldots,k_s)_C(\mathrm{E}\xi_0,\ldots,\xi_s)_Z(\mathfrak{w}_0,\ldots,\mathfrak{w}_s)\\ (i,j)_{s+1}[E(\ulcorner\Phi\urcorner,n_i,k_i,\xi_i,n_j,k_j,\xi_j,\mathfrak{w}_i,\mathfrak{w}_j)\ \&\ C(n_i,\xi_i,\mathfrak{w}_i)\supset(\overline{w}_i\in D)]\,.\end{gathered}$$

The proof does not differ from that of 4.5.

6.8. *If $\mathcal{T}$ is a decidable extension of the elementary theory of the $\leqslant$ relation, if the functions φ_j are definable in $\mathcal{T}$ and if the set D is definable in $\mathcal{T}$ and open and closed in Z, then the set of (Gödel numbers of) valid formulas of S_0 is recursively enumerable and the set of (Gödel numbers of) satisfiable formulas of S_0 is a complement of such a set.*

Proof. Φ is satisfiable if and only if (34) holds. Now (34) is equivalent to the condition

$$(36)\qquad (s)\models_{\mathcal{T}^*}(\mathrm{n}_0,\ldots,\mathrm{n}_s)(\mathrm{k}_0,\ldots,\mathrm{k}_s)(\beta_0,\ldots,\beta_s)(\mathrm{E}\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_s)M_s[\ulcorner\Phi\urcorner]$$

where M_s is the following formula of $\mathcal{T}^*$

$$(i,j)_{s+1}[\mathrm{E}(\varkappa,\mathrm{n}_i,\mathrm{k}_i,\beta_i,\mathrm{n}_j,\mathrm{k}_j,\beta_j,\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_j)\ \&\ \Gamma(\mathrm{n}_i,\beta_i,\boldsymbol{\alpha}_i)\ \&\ \Delta(\overline{\alpha}_i)];$$

here the quantifier $(i,j)_{s+1}$ is an abbreviation for a conjunction with $(s+1)^2$ factors, Δ is a formula of $\mathcal{T}$ defining D, and $\overline{\alpha}_i$ is the last variable of the string $\boldsymbol{\alpha}_i$.

By Lemma 6.5, condition (36) is equivalent to $(p)\models_{\mathcal{T}}G_{p,\ulcorner\Phi\urcorner}$ where $G_{p,q}$ is a recursive sequence of closed formulas of $\mathcal{T}$. Since $\mathcal{T}$ is decidable, it follows that the set $\{\ulcorner\Phi\urcorner: (p)\models_{\mathcal{T}}G_{p,\ulcorner\Phi\urcorner}\}$ is a complement of a recursively enumerable set.

The recursive enumerability of the set of valid sentences is proved by passing to dual formulas.

As an example to Theorem 6.8 we can take $\mathcal{T}$ to be the elementary theory of addition of ordinals $\leqslant\omega_a$ modifying the addition in such a way that $\omega_a+\xi=\omega_a$ for every ξ. Decidability of $\mathcal{T}$ was proved by Ehrenfeucht (in a paper not yet published). As D we can take for example the unit set $\{0\}$ and as φ_j any continuous functions definable in $\mathcal{T}$. We obtain in this way numerous examples of functional calculi with recursively enumerable sets of valid sentences.

7. We conclude with some unsolved problems:

A. Let v_1,v_2 be two ordinals and S_{01},S_{02} functional calculi defined in the last paragraph of Section 6 by taking as Z either the set $\{\xi:\ \xi\leqslant\omega_{v_1}\}$ or the set $\{\xi:\ \xi\leqslant\omega_{v_2}\}$. Do the sets of valid formulas of S_{01}, S_{02} coincide?

B. Let Z be the set of all subsets of an infinite set X and let φ_j be functions definable in a decidable fragment of an extension of the elementary theory of the inclusion relation. If S_0 is the functional calculus with two quantifiers $\bigwedge$ and $\bigvee$ corresponding to this choice of Z and the φ_j, is the set of valid formulas of S_0 recursively enumerable?

C. Same question as in B but with Z replaced by the family of closed subsets of a topological space.

D. Same question but with Z replaced by the complete lattice of closed domains of a topological space X.

The lattice Z in problem B is not separable but methods used in Section 6 should be sufficient to overcome this difficulty. However when one tries to adapt methods of Section 4 to problem B (and to problems C and D as well) one is faced with the difficulty that not only the set $\{(x, y): x \leqslant y\}$ but also the set $\{(x, y): x \text{non} > y\}$ should be closed in $Z \times Z$. No reasonable topology seems to satisfy this condition and this is the chief reason why it is an open question as to whether or not methods similar to those of Section 4 are applicable to our problems.

We limited ourselves chiefly to the study of quantifiers whose interpretations were the l.u.b. and the g.l.b. operations. It is easy to construct examples showing that for an infinite Z, e.g. for $Z = \{\xi: \xi \leqslant \omega\}$, another choice of quantifiers may lead to a "functional calculus", in which the set of valid formulas is not recursively enumerable. It would be interesting to solve the following problem:

E. What is the general characterization of quantifiers which lead to functional calculi with recursively enumerable sets of valid formulas?

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The family of dendrites $\mathfrak{R}$-ordered similarly to the segment

by

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1. Introduction. The continuous mapping f of the topological space X onto the space Y is called the $\mathfrak{R}$-*mapping* if there exists a continuous mapping $g: Y \to X$ such that $fg = \text{identity}$ ([1] and [2]). It is easy to show ([1]) that the $\mathfrak{R}$-mappings are the same as the mappings of the form hr, where r is a retraction and h a homeomorphism.

If there exists an $\mathfrak{R}$-mapping $f: X \to Y$ then we shall write $Y \underset{\mathfrak{R}}{\leqslant} X$. If $Y \underset{\mathfrak{R}}{\leqslant} X$ and $X \underset{\mathfrak{R}}{\leqslant} Y$ then we shall write $X \underset{\mathfrak{R}}{=} Y$. If $X \underset{\mathfrak{R}}{\leqslant} Y$ but $X \underset{\mathfrak{R}}{\neq} Y$ then we shall write $X \underset{\mathfrak{R}}{<} Y$. The relation $\underset{\mathfrak{R}}{<}$ establishes the partial order of every class of spaces.

2. The family of dendrites (1) **ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the segment.** At the end of the paper [2] K. Borsuk raised the following questions:

(i) Does there exist an uncountable family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$?

(ii) Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ in a dense manner?

(iii) Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the set of all real numbers?

In the present paper we shall construct the family of dendrites ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the segment. It solves the three mentioned problems even in the stronger formulation concerning compact 1-dimensional AR-sets.

(1) A dendrite is a locally connected continuum containing no simple closed curve. Dendrites are the same as compact 1-dimensional AR-sets. See for example [3], p. 224 and p. 290.