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Characterization of the fixed point property for a class of set-valued mappings *

by

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1. Introduction. Let X be a space and $\mathcal{C}$ a class of set-valued mappings of X into itself. We say that X has the *fixed point property for* $\mathcal{C}$ if, for each $f \in \mathcal{C}$, there exists $x \in X$ such that $x \in f(x)$. In this paper our primary interest is to prove that if X is an arcwise connected compact metric space, then X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings *if and only if* X is hereditarily unicoherent. Incidental to this result we note that such spaces also have the fixed point property for continuous mappings whose values are arbitrary closed sets, but we have not been able to characterize this fixed point property. It seems likely that, for arcwise connected compacta, this fixed point property is also characterized by hereditary unicoherence.

These results are all generalizations of what is usually called the Scherrer fixed point theorem, and they have a lengthy history. Four earlier papers are of especial interest in what follows, and we mention them here. In [4] A. D. Wallace proved that a dendrite has the fixed point property for upper semi-continuous, continuum-valued mappings, and his result was later generalized by the Eilenberg-Montgomery fixed point theorem ([1]). Our results include the Wallace theorem but they neither include nor are included by the Eilenberg-Montgomery theorem. R. L. Plunkett ([3]) proved that a dendrite has the fixed point property for continuous, closed set-valued mappings and, conversely, that if a Peano continuum has this fixed point property then it is a dendrite. As we shall observe, his proof of the converse proposition is equally valid for the mappings considered by Wallace. Finally, in [5] the author proved that a hereditarily unicoherent, arcwise connected continuum has the fixed

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point property for continuous, closed set-valued mappings, thus generalizing one half of the Plunkett theorem.

There appears to be a remarkable overlap between the class of spaces which have the continuous, closed set-valued fixed point property on the one hand, and the class possessing the upper semi-continuous, continuum-valued fixed point property on the other. As indicated above, we have not been able to determine whether these classes are identical among arcwise connected compacta. But the final theorem of this paper, which is a corollary of our results and Plunkett's, shows that the two classes are identical when restricted to Peano continua.

2. Preliminaries. A *compactum* is a compact metric space. A *continuum* is a connected compactum. A continuum X is *unicoherent* if, for each representation $X = A \cup B$ where A and B are subcontinua, it follows that $A \cap B$ is connected. A continuum is *hereditarily unicoherent* if each of its subcontinua is unicoherent.

The study of hereditarily unicoherent, arcwise connected compacta is facilitated by the inherent order structure of these spaces. This order structure has been developed in our earlier paper on the subject ([5]), and we shall briefly review and extend that study here.

If X is a space and $\leqslant$ is a partial order on X, we write $L(x) = \{a\colon a \leqslant x\}$ and $M(x) = \{a\colon x \leqslant a\}$ for each $x \in X$. It is natural and convenient to define $[x, y] = M(x) \cap L(y)$ and, if $A \subset X$, we write $M(A)$ for the union of all $M(x)$ for which $x \in A$. A subset C of X is a *chain* (relative to $\leqslant$) if, for each x and y in C, $x \leqslant y$ or $y \leqslant x$. An *antichain* of X is a subset in which no two distinct elements are comparable under the partial order. The partial order is *dense* if, for each two elements x and z of X such that $x < z$, there exists $y \in X$ such that $x < y < z$. A *zero* of the set $A \subset X$ is an element $a_0 \in A$ such that $A \subset M(a_0)$.

The fundamental theorem on partial order in hereditarily unicoherent, arcwise connected compacta, which was proved in a slightly more general form in [5], is stated below without proof as Theorem A. Theorem B was also proved in [5]. We take this occasion to note that the hypothesis of hereditary decomposability employed in [5] is superfluous, for any arcwise connected compactum is decomposable, and, moreover, if the compactum is hereditarily unicoherent then each of its subcontinua is arcwise connected (Lemma 1), hence decomposable.

THEOREM A. *Let X be a compact metric space. A necessary and sufficient condition that X be hereditarily unicoherent and arcwise connected is that X admit a partial order $\leqslant$ satisfying the following five conditions.*

(I) *There exists $e \in X$ such that $M(e) = X$,*

(II) *$\leqslant$ is dense,*

(III) *if $x \in X, y \in X$ such that $x < y$ then $[x, y]$ is a closed chain,*

(IV) *if $x \in Y, y \in Y$ where Y is a subcontinuum of X and $x < y$, then $[x, y] \subset Y$,*

(V) *if A is an antichain of X and P is a continuum contained in $M(A)$ then $P \subset M(x)$ for some $x \in A$.*

If X is a hereditarily unicoherent, arcwise connected compactum, then the partial order of Theorem A can be constructed in the following manner. For each x and y in X with $x \neq y$, let $A(x, y)$ denote the unique arc whose endpoints are x and y. Select $e \in X$ and let $x \leqslant y$ mean that either $x = e$ or $x \in A(e, y)$. Note that if $x < y$ then $[x, y] = A(x, y)$.

THEOREM B. *Each subcontinuum of a hereditarily unicoherent, arcwise connected compactum has a zero and each chain has a supremum.*

LEMMA 1. *Each subcontinuum of a hereditarily unicoherent, arcwise connected compactum is arcwise connected.*

Proof. If Y is such a subcontinuum, then by Theorem B, Y has a zero, y_0. If $y \in Y - y_0$ then by (IV) of Theorem A the arc $[y_0, y]$ is contained in Y so that Y is arcwise connected.

LEMMA 2. *Let X be a hereditarily unicoherent, arcwise connected compactum, $x \in X$, and A a continuum contained in $X - x$. If A meets $M(x)$ then $A \subset M(x) - x$.*

Proof. By Theorem B, A has a zero a_0. If $a_0 \leqslant x < a$ where $a \in A \cap M(x)$ then by (IV) of Theorem A, $x \in [a_0, a] \subset A$, a contradiction. Hence $A \subset M(a_0) \subset M(x) - x$.

If Y is a space, we denote by 2^Y the set of non-empty closed subsets of Y and by $\mathcal{K}(Y)$ the family of connected members of 2^Y. Let $f\colon X \to 2^Y$ be a mapping. Following the traditional usage we say that f is *upper semi-continuous* if, for each $x \in X$ and open set V of Y such that $f(x) \subset V$, there exists a neighborhood U of x such that $f(t) \subset V$ for each $t \in U$. The mapping f is *lower semi-continous* if, for each $x \in X$ and open set V of Y such that $f(x) \cap V \neq 0$, there exists a neighborhood U of x such that $f(t) \cap V \neq 0$ for each $t \in U$. If f is both upper and lower semi-continuous then f is said to be *continuous*. Mappings into 2^Y may be termed *closed set-valued*; if Y is compact and the range of f is contained in $\mathcal{K}(Y)$ then f is *continuum-valued.* It is known (see, for example, [2], 9.2) that $f\colon X \to 2^Y$ is upper (lower) semi-continuous if and only if $\{x\colon f(x) \cap A \neq 0\}$ is closed (open) whenever A is closed (open) in Y. Further ([2], 9.6), if f is upper semi-continuous and each $f(x)$ is a compact set, then $f(K) = \bigcup \{f(x)\colon x \in K\}$ is compact whenever K is compact in X.

LEMMA 3. *If X and Y are compact and $f\colon X \to \mathcal{K}(Y)$ is upper semi-continuous, then $f(K)$ is a continuum whenever $K \in \mathcal{K}(X)$.*

Proof. It suffices to prove that $f(K)$ is connected when $K \in \mathcal{K}(X)$. Suppose $K \in 2^X$ and $f(K) = P_1 \cup P_2$, where P_1 and P_2 are disjoint, non-empty closed sets. If

$$Q_i = \{x \in K\colon f(x) \subset P_i\}, \quad i = 1, 2,$$

then Q_1 and Q_2 are disjoint and, since f is continuum-valued, $Q_1 \cup Q_2 = K$, so that K is not connected. Therefore $K \in \mathcal{K}(X)$ implies that $f(K)$ is connected.

The following lemma is due to Plunkett, who stated it in [3] only for Peano continua and closed set-valued mappings. His proof, however, goes through without changes for compacta and continuum-valued mappings. We conclude this section by stating it without proof.

LEMMA 4. *If Y is a compactum and S is a simple closed curve contained in Y, then there exists a continuous mapping $r\colon Y \to \mathcal{K}(S)$ such that $r(t) = t$ for each $t \in S$.*

3. The characterization theorem. In this section we establish the characterization theorem described in the introduction. The first half of this result (Theorem 1 below) can actually be obtained in slightly greater generality with practically no change in the proof. Specifically, X need only be Hausdorff rather than metrizable, with arcwise connectivity replaced by the condition that each pair of points lies in some topological chain ([5]). We shall require metrizability in the converse, however, and the added generality is hardly justified in the present setting.

THEOREM 1. *If X is a hereditarily unicoherent, arcwise connected compactum, and if $f\colon X \to \mathcal{K}(X)$ is upper semi-continuous, then there exists $p_0 \in X$ such that $p_0 \in f(p_0)$.*

Proof. Fix $e \in X$ and give X the partial order of Theorem A with minimal element e. Let P denote the set of all $x \in X$ with the property that $M(x) \cap f(x) \neq 0$. Since $M(e) = X$, it is clear that P is non-empty. If C is a non-empty chain of P then, by Theorem B, C has a supremum, x_0, and we claim that $x_0 \in P$. To prove this, note that $[e, x_0]$ is an arc and hence there exists a non-decreasing sequence x_n in $[e, x_0] \cap C$ such that $\lim x_n = x_0$. We may assume that no $x_n \in f(x_n)$ and hence, by Lemma 2, $f(x_n) \subset M(x_n) - x_n$. In particular, the zero, z_n, of $f(x_n)$ lies in $M(x_n) - x_n$, and some limit point z_0 of the sequence z_n is in the intersection of all $M(x_n)$. That is to say, each x_n lies in the arc $[e, z_0]$ and thus $x_0 \in [e, z_0]$, i.e. $z_0 \in M(x_0)$. By upper semi continuity, z_0 is also a member of $f(x_0)$ so that $x_0 \in P$.

By Zorn's lemma, P has a maximal element p_0, and we shall prove that $p_0 \in f(p_0)$. For if not, it follows by Lemma 2 that $f(p_0) \subset M(p_0) - p_0$. Let q_0 be the zero of $f(p_0)$; then we may choose a decreasing sequence p_n in $[p_0, q_0] - p_0$ such that $\lim p_n = p_0$. Let V be an open set containing $f(p_0)$, so chosen that for some natural number N, $V \cap L(p_n) = 0$ whenever $n > N$. By upper semi-continuity it follows that $f([p_0, p_n]) \subset V$ for sufficiently large n, and by Lemma 3, $f([p_0, p_n])$ has a zero which is necessarily between p_n and q_0. Hence, for large n, $f(p_n) \subset M(p_n)$, contradicting the maximality of p_0 in P. Therefore $p_0 \in f(p_0)$.

THEOREM 2. *Let X be an arcwise connected compactum. If X is not hereditarily unicoherent then there exists an upper semi-continuous mapping $f\colon X \to \mathcal{K}(X)$ which is fixed point free.*

Proof. If X is not hereditarily unicoherent then X contains subcontinua A and B whose intersection is the union of disjoint closed sets P and Q. If $p_1 \in P$ and $q_1 \in Q$ then there is an arc $A(p_1, q_1)$, with p_1 and q_1 for endpoints, which cannot be contained in both A and B; let us assume that $A(p_1, q_1)$ fails to be contained in B. Let p and q be the endpoints of some component of $A(p_1, q_1) - B$, let $A(p, q)$ be the arc consisting of that component and its endpoints, and let B' be a subcontinuum of B which is irreducible between p and q. For each $x \in X$, we define $[x] = x$ if $x \in X - B'$ and $[x] = B'$ otherwise. Let $[X]$ be the space of all $[x]$, endowed with the quotient topology, that is, if $\sigma\colon X \to [X]$ is the natural mapping $\sigma(x) = [x]$, then V is an open subset of $[X]$ if and only if $\sigma^{-1}(V)$ is open in X. We note that σ is continuous and monotone and that $\sigma(A(p, q)) = S$ is a simple closed curve. The space $[X]$ satisfies the hypothesis of Lemma 4 and hence there exists a continuous mapping $r\colon [X] \to \mathcal{K}(S)$ such that $r([x]) = [x]$ for each $[x] \in S$. Let $h\colon S \to S$ be a fixed point free homeomorphism and define $f\colon X \to 2^X$ by $f(x) = \sigma^{-1} h r \sigma(x)$. By Lemma 3, each $r\sigma(x)$ is a continuum and, since h is a homeomorphism, so is $hr\sigma(x)$. Since σ is monotone, each $f(x) = \sigma^{-1} h r \sigma(x)$ is a continuum, and therefore $f\colon X \to \mathcal{K}(X)$. To see that f is upper semi-continuous we need only show that $\{x\colon f(x) \cap F \neq 0\}$ is closed whenever F is closed. Since X is compact and σ and r are continuous, it follows that

$$r^{-1}\sigma(F) = \{[x]\colon r([x]) \cap F \neq 0\}$$

is closed, and hence $\sigma^{-1}h^{-1}r^{-1}\sigma(F) = \{x\colon f(x) \cap F \neq 0\}$ is closed. Finally, suppose there exists $x \in f(x)$; then $\sigma(x) \in hr\sigma(x)$ and, since $\sigma(x) \in S$, we have $\sigma(x) \in h\sigma(x)$, whereas h was assumed to be fixed point free. Hence f is without fixed points.

Combining Theorems 1 and 2 we have

COROLLARY. *Let X be an arcwise connected compactum. A necessary and sufficient condition that X have the fixed point property for the class of upper semi-continuous, continuum-valued mappings is that X be hereditarily unicoherent.*

4. The special case of dendrites. A *dendrite* may be defined as a Peano continuum which contains no simple closed curve. As has been noted elsewhere, a Peano continuum is arcwise connected and among the Peano continua the property of being a dendrite is equivalent to being hereditarily unicoherent. It follows at once from Theorem 1 that a dendrite has the fixed point property for upper semi-continuous, continuum-valued mappings and, as remarked in the introduction, Wallace has previously obtained this result by other methods. In view of Plunkett's theorem we may assert the following at once.

THEOREM 3. *If X is a Peano continuum then the following statements are equivalent.*

(1) *X is a dendrite,*

(2) *X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings,*

(3) *X has the fixed point property for the class of continuous, closed set-valued mappings.*

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Axiomatizability of some many valued predicate calculi

by

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In a paper published in Volume 45 of the Fundamenta Mathematicae I proposed a generalization of the logical qunatifiers. Another generalization applicable in the two valued as well as in the many valued cases has been proposed and discussed by Rosser and Turquette [7]. According to their conception a quantifier is a function which correlates a truth value with a non-empty set of truth values (I disregard here a more general notion considered in [7] in which sets are replaced by relations). Rosser and Turquette ([7], Chapter V) discussed the problem of axiomatizability of the functional calculi with arbitrary quantifiers under the assumption that the set of truth values is finite and Rosser (in an address read at the 1959 meeting of the Association for Symbolic Logic and published in [6]) discussed a similar problem under the assumption that this set coincides with the interval $[0,1]$. In the present paper I take up the problem of axiomatizability under a more general assumption that the set of truth values is an ordered set which is bicompact in its order topology. The method of proof is illustrated in Section 3 where I discuss the case of a finite set of truth values and obtain a part of results of Rosser and Turquette. The chief feature of results set forth in the present paper is their non-effective character: I prove the existence of complete sets of axioms and rules of proof for the calculi in question without exhibiting them explicitly; the existence proofs are based on Tichonov's theorem.

1. Syntax. We consider a "language" S_0 whose expressions are built from the following symbols: $x_0, x_1, \ldots$ (individual variables), $F_0^j, F_1^j, \ldots$ (predicate variables with j arguments, $j = 0, 1, 2, \ldots$), $\mathfrak{F}_0, \mathfrak{F}_1, \ldots, \mathfrak{F}_a$ (propositional connectives), $\mathfrak{Q}_0, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_b$ (quantifiers). We denote by p_s the number of arguments of $\mathfrak{F}_s$ $(s = 0, 1, \ldots, a)$. Formulas are expressions which belong to the smallest class K such that: (i) atomic expressions $F_n^j x_{i_1} \ldots x_{i_j}$ belong to K $(n, j = 0, 1, \ldots,\ i_s = 0, 1, \ldots$ for $s = 1, 2, \ldots, j)$; (ii) if $0 \leqslant s \leqslant a$ and $\Phi_1, \ldots, \Phi_{p_s}$ belong to K, then so does $\mathfrak{F}_s \Phi_1 \ldots \Phi_{p_s}$; (iii) if $0 \leqslant s \leqslant b$ and Φ belongs to K, then so does $\mathfrak{Q}_s x_q \Phi$, $q = 0, 1, \ldots$