is of at most dimension k and is not a local r-separating set of C for $r \leqslant k$, and such that there exists an e-transformation of C into $U \cup T$. For $n = 2m$ or $2m+1$, $k = m-1$, and in case $n = 2m+1$, $H_m(C)$ is finitely generated. Then C is a closed, orientable $(n-1)$-generalized closed manifold ([3]).

Proof. Let U and V denote distinct domains of which C is given as the common boundary in the statement of the theorem. We shall show that U and V are ulc^k.

Let $0 \leqslant r \leqslant k$, $e > 0$, and p a point of C. Since C is r-lc, there exists $d > 0$ such that every r-cycle of $C \cap S(p, d)$ bounds on $C \cap S(p, e/2)$. Suppose there exists a cycle C_r in $U \cap S(p, d)$ which is non-bounding in $U \cap S(p, e)$. Then ([2]; p. 159, Lemma) there exists $s > 0$ such that every s-transformation of C is linked by C_r in $S(p, 3e/2)$. By hypothesis, there exists an s-transformation $f(C) = C'$ into $V \cup T$, where T is a closed subset of C of dimension at most k, and which is not a local r-separating set of C.

Now $C_r \sim 0$ in $S(p, d) - T$, since $r \leqslant k$ and $\dim T \leqslant k$. Using the notation of [4]; p. 203, Lemma 1.13, let M be a closed subset of $S(p, d) - T$ carrying the homology $C_r \sim 0$, K a carrier of C_r in $U \cap S(p, d)$, and L the closure of $S(p, d) - U$. Then (loc. cit.) there exists on $M \cap F(L)$— and hence on $C \cap S(p, d)$—a cycle Z_r such that $C_r \sim Z_r$ on $M \cap \overline{U}$. As M does not meet T, Z_r is on $C - T$; and by the choice of d, $Z_r \sim 0$ on $C \cap S(p, e/2)$. Therefore, since T is not a local r-separating set of C, $Z_r \sim 0$ on $C \cap S(p, e/2) - T$. But then (by combining homologies) $C_r \sim 0$ in $S(p, 3e/2) - C'$, in contradiction to the choice of s and f. We conclude that such a cycle as C_r cannot exist, and that U (and likewise V) is ulc^k.

When $n = 2m$, then, U and V are both ulc^{m-1}, and since $(m-1) + (m-1) = 2m-2 = n-2$, it follows ([4]; p. 308, Th. 7.1) that C is an orientable $(n-1)$-gcm. When $n = 2m+1$, $2m-2 = n-3$, and since $H_m(C)$ is finitely generated, it follows from the Alexander Duality Theorem and [4]; p. 308, Th. 7.3, that C is an orientable $(n-1)$-gcm.

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([3]) For the definition and properties of a generalized closed manifold (= gcm), see [4]; VIII.

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Proof of a conjecture of S. Ruziewicz

by

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§ 1. Introduction. Let S be an infinite set of power m. Let $F(x)$ be a set-mapping defined on S, i.e. a function which associates to every element x of S a subset $F(x)$ of S such that $x \notin F(x)$. Suppose that, for every $x \in S$, $\overline{\overline{F(x)}} < n$ where n is a given cardinal number less than m (finite or infinite). A subset S' of S is called a *free set* (with respect to the set-mapping $F(x)$) if, for every pair $x, y \in S'$, $x \notin F(y)$ and $y \notin F(x)$.

The following proposition has been conjectured by S. Ruziewicz.

Under the above conditions S has a free subset S' of power m ([1]).

This theorem was proved firstly for $n = \aleph_0$ and m either of the form 2^p or of the form $\aleph_{\alpha+1}$ (see [2] and [3]), secondly for m a regular cardinal number, or m a countable sum of cardinals smaller than m (see [4]), and thirdly for m not the sum of n or fewer cardinal numbers less than m (see [5]).

Finally P. Erdös proved—using the generalized continuum hypothesis—that the conjecture is true in the general case (see [6]).

The aim of our paper is to prove the above mentioned conjecture without using the generalized continuum hypothesis.

§ 2. Theorem 1. *Let S be an infinite set such that $\overline{\overline{S}} = m$. Let $F(x)$ be a set-mapping defined on S such that $\overline{\overline{F(x)}} < n$ for every $x \in S$, where $n < m$. Then there exists a free subset S' of S such that $\overline{\overline{S'}} = m$.*

Proof. We distinguish two cases (i) m is regular and (ii) m is singular.

Case (i) ([2]). Let φ denote the initial number of the cardinal number n. We are going to define a sequence $\{S_\nu\}_{\nu<\varphi}$ of type φ of subsets of S by

([1]) See [1]. Questions of this type have been first posed by P. Turán — see: G. Grünwald, *Egy halmazelméleti tételről*, Math. Fiz. Lapok 44 (1937), pp. 51-53.

In some of the cited papers binary relations of form yRx are investigated, where the corresponding set-mapping is to be defined by the stipulation $F(x) = \{y: yRx\}$.

([2]) In this case the theorem is well known (see the papers cited). However, for the convenience of the reader, we reconstruct here a simple proof of it. This proof is due to D. Lázár.

transfinite induction on ν as follows. Suppose that $S_{\nu'}$ is defined for every $\nu' < \nu$ for a $\nu < \varphi$.

(1) Let S_ν be a maximal free subset of $S - \bigcup_{\nu'<\nu} S_{\nu'}$.

Such an S_ν exists by Zorn's lemma.

If $\overline{\overline{S_\nu}} = m$ for a $\nu < \varphi$, then the theorem is true, thus we may suppose that

(2) $$\overline{\overline{S_\nu}} < m \quad \text{for every} \quad \nu < \varphi .$$

Taking into consideration that $\overline{\overline{\varphi}} = n < m$ and that m is regular by the assumptions and by (i), we get from (2) that

(3) $$\overline{\overline{\bigcup_{\nu'<\nu} S_{\nu'}}} < m \quad \text{for every} \quad \nu \leqslant \varphi .$$

Put $S^* = S - \bigcup_{\nu<\varphi} S_\nu$. It follows from (1) and (3) that

(4) S_ν is non-empty for every $\nu < \varphi$, $S_{\nu_1} \cap S_{\nu_2} = 0$ for $\nu_1 \neq \nu_2 < \varphi$

and from (3) we get

(5) $$\overline{\overline{S^*}} = m .$$

By the maximality of S_ν and by (4) corresponding to every element x of S^*, we can single out an element $y_\nu(x)$ of S_ν such that

(6) either $y_\nu(x) \in F(x)$ or $x \in F\big(y_\nu(x)\big)$ for every $\nu < \varphi$.

Put $S^{**} = \bigcup_{\nu<\varphi} \bigcup_{x \in S_\nu} F(x)$. We have

(7) $$\overline{\overline{\bigcup_{x \in S_\nu} F(x)}} \leqslant n \overline{\overline{S_\nu}} < m$$

by (2), hence by the regularity of m we get

(8) $$\overline{\overline{S^{**}}} < m .$$

Its results from (5) and (8) that $S^* - S^{**}$ is non-empty. Let x_0 be an element of it. Then $y_\nu(x_0) \in F(x_0)$ for every $\nu < \varphi$ by (6), and $y_{\nu_1}(x_0) \neq y_{\nu_2}(x_0)$ for every $\nu_1 \neq \nu_2 < \varphi$ by (4), hence $\overline{\overline{F(x_0)}} \geqslant n$ in contradiction to the assumption.

Case (ii). Put $m = \aleph_\alpha$. Then α is an ordinal number of the second kind; $\mathrm{cf}(\alpha) < \alpha$. Let β be an ordinal number such that

(9) $$\mathrm{cf}(\alpha) < \beta+1 , \quad n < \aleph_{\beta+1} \quad \text{and} \quad \beta+1 < \alpha .$$

(10) Let further $(\alpha_\nu)_{\nu<\omega_{\mathrm{cf}(\alpha)}}$ be a monotone increasing sequence of type $\omega_{\mathrm{cf}(\alpha)}$ of ordinal numbers less than α cofinal with α.

We may suppose that the sequence $(\alpha_\nu)_{\nu<\omega_{\mathrm{cf}(\alpha)}}$ satisfies the following conditions:

(11) $\aleph_{\alpha_\nu}$ is regular for every $\nu < \omega_{\mathrm{cf}(\alpha)}$

(e.g. α_ν is of the form $\beta_\nu + 1$),

(12) $\alpha_\nu > \beta + 1$ for every $\nu < \omega_{\mathrm{cf}(\alpha)}$.

It follows from (10) that

(13) $$\sum_{\nu<\omega_{\mathrm{cf}(\alpha)}} \aleph_{\alpha_\nu} = \aleph_\alpha .$$

Let now $(S_\nu)_{\nu<\omega_{\mathrm{cf}(\alpha)}}$ be a sequence of type $\omega_{\mathrm{cf}(\alpha)}$ of subsets of S such that

(14) $S = \bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S_\nu$, $\overline{\overline{S_\nu}} = \aleph_{\alpha_\nu}$ for every $\nu < \omega_{\mathrm{cf}(\alpha)}$ and $S_{\nu_1} \cap S_{\nu_2} = 0$ for every $\nu_1 \neq \nu_2 < \omega_{\mathrm{cf}(\alpha)}$.

Consider the set mapping $F_\nu(x)$ induced by $F(x)$ on S_ν defined by the following stipulations

$$F_\nu(x) = F(x) S_\nu \quad \text{for every} \quad x \in S_\nu$$

where $\nu < \omega_{\mathrm{cf}(\alpha)}$ is arbitrary.

Then $\overline{\overline{S_\nu}} = \aleph_{\alpha_\nu}$ is regular by (11), $\overline{\overline{F_\nu(x)}} \leqslant \overline{\overline{F(x)}} < n$ by the assumption and by (9) and (12). It is obvious from the definitions that if a subset S' of S_ν is free with respect to $F_\nu(x)$ then it is also free with respect to $F(x)$. Thus it follows from the case (i) of Theorem 1, which has already been proved, that there exists a subset S_ν^1 of S_ν satisfying the following conditions

(15) $S_\nu^1 \subset S_\nu$, $\overline{\overline{S_\nu^1}} = \aleph_{\alpha_\nu}$ and S_ν^1 is a free subset of S with respect to $F(x)$.

The set $\bigcup_{\nu'<\nu} \bigcup_{x \in S_{\nu'}} F(x)$ has power less than $\aleph_{\alpha_\nu}$ for every $\nu < \omega_{\mathrm{cf}(\alpha)}$, since $\overline{\overline{\bigcup_{x \in S_{\nu'}} F(x)}} \leqslant n \overline{\overline{S_{\nu'}}} \leqslant \aleph_{\alpha_{\nu'}} < \aleph_{\alpha_\nu}$ by (9) and (10), $\aleph_{\alpha_\nu}$ is regular by (11) and $\overline{\overline{\nu}} < \aleph_{\mathrm{cf}(\alpha)} < \aleph_{\alpha_\nu}$ by (9).

Put $S_\nu^2 = S_\nu^1 - \bigcup_{\nu'<\nu} \bigcup_{x \in S_{\nu'}} F(x)$.

It follows from (15) and from the definition that

(16) $S_\nu^2 \subset S_\nu^1$, $\overline{\overline{S_\nu^2}} = \aleph_{\alpha_\nu}$ and $x \in S_{\nu'}^2$, $y \in S_\nu^2$ implies $y \notin F(x)$ for every $\nu' < \nu < \omega_{\mathrm{cf}(\alpha)}$ [3].

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[3] The construction of the sets S_ν, S_ν^1, S_ν^2 is well known. The next step contains the main idea of our proof.

Now we are going to define a sequence $(S^3_\nu)_{\nu<\omega_{\mathrm{cf}(\alpha)}}$ of type $\omega_{\mathrm{cf}(\alpha)}$ of subsets of S by transfinite induction on ν as follows.

(17) Put $S^3_0 = S^2_0$.

Taking into consideration that by (12), $\overline{\overline{S^3_0}} = \overline{\overline{S^2_0}} = \aleph_{\alpha_0} \geqslant \aleph_{\beta+1}$ there exists a sequence $(S^3_{0,\varrho})_{\varrho<\omega_{\beta+1}}$ of type $\omega_{\beta+1}$ of subsets of S^3_0 satisfying the conditions

(18) $$\bigcup_{\varrho<\omega_{\beta+1}} S^3_{0,\varrho} = S^3_0;\qquad \overline{\overline{S^3_{0,\varrho}}} = \aleph_{\alpha_0};\qquad S^3_{0,\varrho'}\cap S^3_{0,\varrho}=0$$

for every $\varrho' \neq \varrho < \omega_{\beta+1}$.

Suppose that the sets $S^3_{\nu'}$ and $S^3_{\nu',\varrho}$ are already defined for every $\varrho<\omega_{\beta+1}$, $\nu'<\nu$ for a $\nu<\omega_{\mathrm{cf}(\alpha)}$ in such a way that

$$\bigcup_{\varrho<\omega_{\beta+1}} S^3_{\nu',\varrho} = S^3_{\nu'} \qquad\text{and}\qquad S^3_{\nu',\varrho}\cap S_{\nu',\varrho'} = 0$$

for every $\varrho'\neq\varrho<\omega_{\beta+1}$, $\nu'<\nu$.

Put $Z_\nu = \bigcup_{\nu'<\nu} S^3_{\nu'}$, and $Z_{\nu,\varrho} = \bigcup_{\nu'<\nu} S^3_{\nu',\varrho}$. Then $Z_\nu = \bigcup_{\varrho<\omega_{\beta+1}} Z_{\nu,\varrho}$ and $Z_{\nu,\varrho'}\cap Z_{\nu,\varrho}=0$ for every $\varrho\neq\varrho'<\omega_{\beta+1}$. Let x be an arbitrary element of S^2_ν. Then $F(x)\cap Z_\nu = \bigcup_{\varrho<\omega_{\beta+1}} (F(x)\cap Z_{\nu,\varrho})$. Taking into consideration that by the assumption and by (9) $\overline{\overline{F(x)}} < n < \aleph_{\beta+1}$, from the regularity of $\aleph_{\beta+1}$ it follows that for every $x\in S^2_\nu$ there exists an ordinal number $\varrho(x,\nu)<\omega_{\beta+1}$ such that $F(x)\cap Z_{\nu,\varrho}=0$ if $\varrho(x,\nu)\leqslant\varrho<\omega_{\beta+1}$.

Put $T_{\nu,\varrho} = \{x\colon x\in S^2_\nu \text{ and } \varrho(x,\nu)\leqslant\varrho\}$ for every $\varrho<\omega_{\beta+1}$.

It follows that $S^2_\nu = \bigcup_{\varrho<\omega_{\beta+1}} T_{\nu,\varrho}$ and $T_{\nu,\varrho'} \subsetneq T_{\nu,\varrho}$ for every $\varrho'<\varrho<\omega_{\beta+1}$. Taking into consideration that by (11) and (12) $\aleph_{\beta+1}<\aleph_{\alpha_\nu}$ and $\aleph_{\alpha_\nu}$ is regular, it results from (16) that there exists a $\varrho_0=\varrho_0(\nu)<\omega_{\beta+1}$ such that $\overline{\overline{T_{\nu,\varrho_0}}} = \aleph_{\alpha_\nu}$.

(19) Put $S^3_\nu = T_{\nu,\varrho_0}$ for such a ϱ_0.

Taking into consideration that by (12) $\overline{\overline{S^3_\nu}} = \aleph_{\alpha_\nu}\geqslant\aleph_{\beta+1}$ there exists a sequence $(S^3_{\nu,\varrho})_{\varrho<\omega_{\beta+1}}$ of type $\omega_{\beta+1}$ of subsets of S^3_ν satisfying the following conditions

(20) $$S^3_\nu = \bigcup_{\varrho<\omega_{\beta+1}} S^3_{\nu,\varrho};\quad \overline{\overline{S^3_{\nu,\varrho}}} = \aleph_{\alpha_\nu} \text{ and } S^3_{\nu,\varrho}\cap S^3_{\nu,\varrho'}=0 \qquad\text{for every } \varrho'\neq\varrho<\omega_{\beta+1}.$$

Thus S^3_ν and $S^3_{\nu,\varrho}$ are defined for every $\nu<\omega_{\mathrm{cf}(\alpha)}$, $\varrho<\omega_{\beta+1}$ and it follows by induction from (17), (18), (19) and (20) that (19) and (20) holds for every $\nu<\omega_{\mathrm{cf}(\alpha)}$.

By the construction S^3_ν has the following properties:

(21) $S^3_\nu\subset S^2_\nu$; there exists a $\varrho(\nu)<\omega_{\beta+1}$ such that $F(x)\cap Z_{\nu,\varrho}=0$ for every $\varrho\geqslant\varrho(\nu)$ and for every $x\in S^3_\nu$.

In fact, $S^3_\nu = T_{\nu,\varrho_0}$ and $\varrho_0=\varrho(\nu)$ satisfy the requirement of (21).

(22) Put $R_\nu = \{\varrho(\nu')\}_{\nu<\nu'<\omega_{\mathrm{cf}(\alpha)}}$ for every $\nu<\omega_{\mathrm{cf}(\alpha)}$.

Then $\overline{\overline{R_\nu}} \leqslant \aleph_{\mathrm{cf}(\alpha)} < \aleph_{\beta+1}$ by (9). Hence by the regularity of $\aleph_{\beta+1}$ there exists an ordinal number $\varrho^*(\nu)$ such that

(23) $$\begin{aligned}&\varrho^*(\nu)<\omega_{\beta+1} &&\text{for every} && \nu<\omega_{\mathrm{cf}(\alpha)},\\ &\varrho(\nu')<\varrho^*(\nu) &&\text{for every} && \nu<\nu'<\omega_{\mathrm{cf}(\alpha)}.\end{aligned}$$

(24) Put $S^4_\nu = \bigcup_{\varrho^*(\nu)\leqslant\varrho<\omega_{\beta+1}} S^3_{\nu,\varrho}$ for every $\nu<\omega_{\beta+1}$ and $S'=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S^4_\nu$.

It follows from (20) and (23) that $\overline{\overline{S^4_\nu}}=\aleph_{\alpha_\nu}$ for every $\nu<\omega_{\mathrm{cf}(\alpha)}$. Taking into consideration that $S^4_\nu\subset S^3_\nu\subset S^2_\nu\subset S^1_\nu\subset S_\nu$ it follows from (13) and (14) that

(25) $$\overline{\overline{S'}} = \sum_{\nu<\omega_{\mathrm{cf}(\alpha)}} \aleph_{\alpha_\nu} = \aleph_\alpha .$$

On the other hand we have

(26) $y\in S^4_\nu$, $x\in S^4_{\nu'}$ implies $y\notin F(x)$ for every $\nu<\nu'<\omega_{\mathrm{cf}(\alpha)}$.

In fact, $S^4_{\nu'}\subset S^3_{\nu'}$, hence if $x\in S^4_{\nu'}$, then $F(x)\cap Z_{\nu',\varrho}=0$ for every $\varrho\geqslant\varrho(\nu')$ by (21), hence $F(x)\cap Z_{\nu',\varrho}=0$ for every $\varrho\geqslant\varrho^*(\nu)$ by (23). Thus

$$F(x)\cap S^4_\nu = \bigcup_{\varrho^*(\nu)\leqslant\varrho<\omega_{\beta+1}} (F(x)\cap S^3_{\nu,\varrho}) = 0 \qquad\text{for every}\qquad \nu<\nu' .$$

Now we are going to prove that

(27) S' is a free set.

$S'=\bigcup_{\nu<\omega_{\mathrm{cf}(\alpha)}} S^4_\nu$ by (24). Let x,y be two arbitrary distinct elements of S'. By (14) there exist uniquely determined ordinal numbers ν_1,ν_2 such that $x\in S^4_{\nu_1}$, $y\in S^4_{\nu_2}$. By reason of symmetry, we may suppose $\nu_1\leqslant\nu_2$. If $\nu_1=\nu_2$ then $y\notin F(x)$ and $x\notin F(y)$ follows from (15) taking into consideration that $S^4_{\nu_1}\subset S^1_{\nu_1}$. If $\nu_1<\nu_2$ then $y\notin F(x)$ by (16) and $x\notin F(y)$ by (26). Hence $x\notin F(y)$ and $y\notin F(x)$ in any cases.

S' satisfies the requirements of Theorem 1 by (25) and (27). Q.e.d.

Remarks. In their paper [7] P. Erdös and G. Fodor prove the following theorem.

Let S be a set, $\overline{\overline{S}} = m \geqslant \aleph_0$, $F(x)$ a set-mapping defined on S such that $\overline{\overline{F(x)}} < n < m$ for every $x \in S$, for an $n < m$. Let further $\{S_\nu\}_{\nu<\varphi}$ be a system of disjoint subsets of S satisfying the conditions: $\overline{\overline{S_\nu}} = m$ for every $\nu < \varphi$ and $\overline{\overline{\varphi}} < m$. Then there exists a free subset $S' \subset S$ such that $\overline{\overline{S' \cap S_\nu}} = m$ for every $\nu < \varphi$.

The proof given in [7] makes use of the generalized continuum hypothesis in the case when m is singular.

It is easy to see that using the idea of the proof of our Theorem 1 this generalization of the Ruziewicz conjecture can also be proved without using the generalized continuum hypothesis.

On the other hand in his paper [8] G. Fodor states the following generalization of the Ruziewicz conjecture.

Let S be a set, $\overline{\overline{S}} = m \geqslant \aleph_0$, and $F(x)$ a set-mapping defined on S, satisfying the condition $\overline{\overline{F(x)}} < n < m$ for every $x \in S$ for some $n < m$. Let further $\Pi(S')$ denote the set $\bigcup_{x \neq y, x, y \in S'} (F(x) \cap F(y))$ for every $S' \subset S$. Then there exists a subset $S' \subset S$, $\overline{\overline{S'}} = m$ such that $\overline{\overline{\Pi(S')}} < m$.

Fodor proves this theorem for singular m using the generalized continuum hypothesis; our method does not enable us to prove this theorem without using this hypothesis. The simplest unsolved problem here is: Is it possible to prove Fodor's theorem without using this hypotheses for $m = \aleph_{\omega_0}$ or for $m = \aleph_{\omega_1}$?

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A new analytic approach to hyperbolic geometry

by

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Introduction

Hilbert was the first who constructed in plane hyperbolic geometry without the axiom of continuity a commutative ordered field $\overline{\overline{\mathfrak{E}}} = \langle \overline{\overline{E}}, +, \cdot, < \rangle$ and founded an analytic geometry over it (see [3] or [2], Appendix III). The field $\overline{\overline{\mathfrak{E}}}$ is known in the literature as the *end-calculus* since the class $\overline{\overline{E}}$ consists of pencils of parallel half-lines, which Hilbert refers to as *ends*. The analytic geometry over $\overline{\overline{\mathfrak{E}}}$ is based upon a coordinate system for straight lines.

In this paper a new commutative ordered field $\overline{\overline{\mathfrak{S}}} = \langle \overline{\overline{S}}, +, \cdot, < \rangle$ is constructed in the same system of geometry. This field seems to be conceptually simpler and more adequate for the foundation of analytic geometry than $\overline{\overline{\mathfrak{E}}}$. It is generated by a *hyperbolic calculus of segments*, more precisely by an algebraic system $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ in which the class S consists of the segments. The operations $+$ and $\cdot$ of $\mathfrak{S}$ are defined in terms of such simple notions as the Lambert quadrangle and the right triangle and are not relativized to any fixed geometrical objects, while the relation $<$ coincides with the usual less-than relation for the segments. Finally a rectangular coordinate system over $\overline{\overline{\mathfrak{S}}}$ can be constructed (the two coordinates of a point being elements of $\overline{\overline{S}}$), and moreover the analytic geometry based on it is identical with that of the two-dimensional Klein space the absolute of which coincides with the unit circle.

Chapter I is algebraic. We introduce there the notion of a *unit interval algebra* and reduce the problem of constructing a commutative ordered field to that of constructing a unit interval algebra.

Chapter II is geometrical. In Section 1 we describe the axiomatic theory $\mathcal{H}'$ of the hyperbolic geometry in which the field $\overline{\overline{\mathfrak{S}}}$ is to be constructed. In Sections 2-13 we construct the system $\mathfrak{S}$, furthermore we prove it to be a unit interval algebra, and consequently, using the result of Chapter I, we obtain the ordered field $\overline{\overline{\mathfrak{S}}}$. In Sections 14-18 we outline the foundations of the analytic geometry over $\overline{\overline{\mathfrak{S}}}$.