is of at most dimension \( k \) and is not a local \( r \)-separating \( C \) for \( r < k \), and such that there exists an \( r \)-transformation of \( C \) into \( U \cup T \). For \( n = 2m \) or \( 2m+1, k = m-1 \), and in case \( n = 2m+1, H_m(C) \) is finitely generated. Then \( C \) is a closed, orientable \((n-1)\)-generalized closed manifold \(^1\).

Proof. Let \( U \) and \( V \) denote distinct domains of which \( C \) is given as the common boundary in the statement of the theorem. We shall show that \( U \) and \( V \) are \( u \)-local.

Let \( 0 < r < k \), e > 0, and \( p \) a point of \( C \). Since \( C \) is \( r \)-local, there exists \( d > 0 \) such that every \( r \)-cycle of \( C \cap S(p, d) \) bounds on \( C \cap S(p, e) \).

Suppose there exists a cycle \( C_r \) in \( U \cap S(p, d) \) which is non-bounding in \( U \cap S(p, e) \). Then \([2, \text{ p. 158, Lemma}]\) there exists \( s > 0 \) such that every \( \varepsilon \)-transformation of \( C \) is linked by \( C_r \) in \( S(p, s) \). By hypothesis, there exists an \( s \)-transformation \( f(U) = C \) into \( V \cup T \), where \( T \) is a closed subset of \( C \) of dimension at most \( k \), and which is not a local \( r \)-separating set of \( C \).

Now \( C_r \cap S(p, d) - T \), since \( r < k \) and \( \dim T < k \). Using the notation of \([4, \text{ p. 203, Lemma 1.13}]\), let \( M \) be a closed subset of \( S(p, d) - T \) carrying the homology \( C_r \cap -0 \), \( K \) a carrier of \( C_r \) in \( U \cap S(p, d) \), and \( L \) the closure of \( S(p, d) - U \). Then \( (\text{loc. cit.}) \) there exists \( M \cap F(L) \) and hence on \( C \cap S(p, d) - T \), \( C_r \cap -0 \), \( C \cap S(p, e) \). Moreover, \( C_r \cap -0 \) on \( C \cap S(p, e) \). Therefore, since \( T \) is not a local \( r \)-separating set of \( C \), \( C_r \cap -0 \) on \( C \cap S(p, e) \). But then \( (\text{by combining homologies}) \) \( C_r \cap -0 \) in \( S(p, s) - C \), contradicting the choice of \( s \) and \( f \). We conclude that such a cycle \( C \) cannot exist, and that \( U \) (and likewise \( V \)) is \( u \)-local.

When \( n = 2m \), then \( U \) and \( V \) are both \( u \)-local, and since \( (m-1) + (m-2) + \cdots + 1 = 2m - 2 = n - 2 \), it follows \([4, \text{ p. 306, Th. 7.1}]\) that \( C \) is an orientable \((n-1)\)-gmc. When \( n = 2m+1 \), \( 2m - 2 = n - 3 \), and since \( H_m(C) \) is finitely generated, it follows from the Alexander Duality Theorem and \([4, \text{ p. 306, Th. 7.3}]\) that \( C \) is an orientable \((n-1)\)-gmc.

Proof of a conjecture of S. Ružiewicz

by

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§ 1. Introduction. Let \( S \) be an infinite set of power \( m \). Let \( F(x) \) be a set-mapping defined on \( S \), i.e. a function which associates to every element \( x \) of \( S \) a subset \( F(x) \) of \( S \) such that \( x \notin F(x) \). Suppose that, for every \( x \in S \), \( |F(x)| < n \) where \( n \) is a given cardinal number less than \( m \) (finite or infinite). A subset \( S' \) of \( S \) is called a free set (with respect to the set-mapping \( F(x) \)) if, for every pair \( x, y \in S' \), \( x \notin F(y) \) and \( y \notin F(x) \).

The following proposition has been conjectured by S. Ružiewicz.

Under the above conditions \( S \) has a free subset \( S' \) of power \( m \).\(^1\)

This theorem was proved firstly for \( n = m \) and \( m \) either of the form \( 2^\alpha \) or of the form \( n_\alpha+1 \) (see \([2, \text{ see also 3}]\) ), secondly for \( m \) as a regular cardinal number, or as a countable sum of cardinals smaller than \( m \) (see \([4]\) ), and thirdly for \( m \) not the sum of \( n \) or fewer cardinal numbers less than \( m \) (see \([5]\) ).

Finally P. Erdős proved—using the generalized continuum hypothesis—that the conjecture is true in the general case (see \([6]\) ).

The aim of our paper is to prove the above mentioned conjecture without using the generalized continuum hypothesis.

§ 2. Theorem 1. Let \( S \) be an infinite set such that \( |S| = m \). Let \( F(x) \) be a set-mapping defined on \( S \) such that \( F(x) \) is a free set for every \( x \in S \), where \( n < m \). Then there exists a free subset \( S' \) of \( S \) such that \( |S'| = m \).

Proof. We distinguish two cases (i) \( m \) is regular and (ii) \( m \) is singular.

Case (i) \( m \). Let \( \varphi \) denote the initial cardinal of the cardinal number \( n \). We are going to define a sequence \( \{S_i\}_{i<n} \) of type \( \varphi \) of subsets of \( S \) by

\(^1\) For the definition and properties of a generalised closed manifold \((\varepsilon \text{-gcm})\), see \([4] \text{ VIII}\).

References


\(^1\) See \([1]\), Questions of this type have been first posed by P. Turán—see:


In some of the cited papers binary relations of form \( y \in x \) are investigated, where the corresponding set-mapping is to be defined by the stipulation \( F(x) = \{y : y \in x\} \).

\(^2\) In this case the theorem is well known (see the papers cited). However, for the convenience of the reader, we reconstruct here a simple proof of it. This proof is due to D. László.
transfinite induction on \( \nu \) as follows. Suppose that \( S_{\nu} \) is defined for every \( \nu' < \nu \) for a \( \nu < \varphi \).

(1) Let \( S_{\nu} \) be a maximal free subset of \( S - \bigcup_{\nu < \varphi} S_{\nu} \).

Such an \( S_{\nu} \) exists by Zorn's lemma.

If \( \bar{S}_{\nu} = m \) for a \( \nu < \varphi \), then the theorem is true, thus we may suppose that

\[
\bar{S}_{\nu} < m \quad \text{for every} \quad \nu < \varphi.
\]

Taking into consideration that \( \bar{\varphi} = n < m \) and that \( m \) is regular by the assumptions and by (1), we get from (2) that

\[
\bigcup_{\nu < \varphi} \bar{S}_{\nu} < m \quad \text{for every} \quad \nu < \varphi.
\]

Put \( S' = S - \bigcup_{\nu < \varphi} S_{\nu} \). It follows from (1) and (3) that

(4) \( S' \) is non-empty for every \( \nu < \varphi \); \( S_{\nu_1} \cap S_{\nu_2} = 0 \) for \( \nu_1 \neq \nu_2 < \varphi \)

and from (3) we get

\[
\bar{S}' = m. \tag{5}
\]

By the maximality of \( S_{\nu} \) and by (4) corresponding to every element \( x \) of \( S' \), we can single out an element \( y(x) \) of \( S_{\nu} \), such that

(6) \( y(x) \in F(x) \) or \( x \in F(y(x)) \) for every \( \nu < \varphi \).

Put \( S'' = \bigcup_{\nu < \varphi} F(x) \). We have

\[
\bigcup_{x \in S''} F(x) \leq \kappa_{\bar{\varphi}} < m \tag{7}
\]

by (2), hence by the regularity of \( m \) we get

\[
\bar{S}'' < m. \tag{8}
\]

It results from (5) and (8) that \( S' - S'' \) is non-empty. Let \( s_0 \) be an element of it. Then \( y(x) \in F(s_0) \) for every \( \nu < \varphi \) by (6), and \( y(x; s_0) \neq y(x; s_0) \) for every \( \nu_1 \neq \nu_2 < \varphi \) by (4), hence \( F(s_0) \supseteq S' \) in contradiction to the assumption.

Case (ii). Put \( m = \kappa_\varphi \). Then \( \alpha \) is an ordinal number of the second kind; \( c(\kappa_\varphi < \alpha) \). Let \( \beta \) be an ordinal number such that

\[
c(\alpha) < \beta + 1, \quad n < \kappa_{\beta+1} \quad \text{and} \quad \beta + 1 < \alpha \tag{9}.
\]

Let further \( (\alpha_i)_{i < \omega_{\alpha(\kappa_\varphi)}} \) be a monotone increasing sequence of type \( \omega_{\alpha(\kappa_\varphi)} \) of ordinal numbers less than \( \alpha \) cofinal with \( \alpha \).

We may suppose that the sequence \( (\alpha_i)_{i < \omega_{\alpha(\kappa_\varphi)}} \) satisfies the following conditions:

(11) \( \kappa_{\alpha_i} \) is regular for every \( \nu < \omega_{\alpha(\kappa_\varphi)} \)

(e.g. \( \alpha_i \) is of the form \( \beta_i + 1 \)),

(12) \( \alpha_i > \beta_i + 1 \) for every \( \nu < \omega_{\alpha(\kappa_\varphi)} \).

It follows from (10) that

\[
\sum_{\nu < \omega_{\alpha(\kappa_\varphi)}} \kappa_{\alpha_i} = \kappa_{\alpha_0} \tag{13}.
\]

Let now \( (S_i)_{i < \omega_{\alpha(\kappa_\varphi)}} \) be a sequence of type \( \omega_{\alpha(\kappa_\varphi)} \) of subsets of \( S \) such that

(14) \( S = \bigcup_{\nu < \omega_{\alpha(\kappa_\varphi)}} S_i, \quad S_i = \kappa_{\alpha_i} \) for every \( \nu < \omega_{\alpha(\kappa_\varphi)} \) and \( S_i \cap S_j = 0 \) for every \( \nu_1 \neq \nu_2 < \omega_{\alpha(\kappa_\varphi)} \).

Consider the set mapping \( F(x) \) induced by \( F(x) \) on \( S \), defined by the following stipulations

\[
F(x) = F(x) \bigcap S \quad \text{for every} \quad x \in S, \tag{15}
\]

where \( \nu < \omega_{\alpha(\kappa_\varphi)} \) is arbitrary.

Then \( S_{\nu} = \kappa_{\alpha_\nu} \) is regular by (11), \( F(x) < F(x) < F(x) < n \) by the assumption and by (9) and (12). It is obvious from the definitions that if a subset \( S' \) of \( S_{\nu} \) is free with respect to \( F(x) \) then it is also free with respect to \( F(x) \). Thus it follows from the case (i) of Theorem 1, which has already been proved, that there exists a subset \( S'_{\nu} \) of \( S_{\nu} \) satisfying the following conditions

(15) \( S'_{\nu} \subseteq S_{\nu}, \quad S'_{\nu} = \kappa_{\alpha_{\nu}} \) and \( S'_{\nu} \) is a free subset of \( S_{\nu} \) with respect to \( F(x) \).

The set \( \bigcup_{\nu < \omega_{\alpha(\kappa_\varphi)}} F(x) \) has power less than \( \kappa_{\alpha_{\nu}} \) for every \( \nu < \omega_{\alpha(\kappa_\varphi)} \), since \( \bigcup_{\nu < \omega_{\alpha(\kappa_\varphi)}} F(x) \subseteq \kappa_{\nu} < \kappa_{\nu} \) by (9) and (10), \( \kappa_{\nu} \) is regular by (11) and \( \nu < \kappa_{\nu} \) by (9).

Put \( S' = S'_{\nu} - \bigcup_{\nu < \omega_{\alpha(\kappa_\varphi)}} F(x) \).

It follows from (15) and from the definition that

(16) \( S' \subseteq \bigcup_{\nu < \omega_{\alpha(\kappa_\varphi)}} S'_{\nu}, \quad S' = \kappa_{\alpha_{\nu}} \) and \( x \in S'_{\nu}, \quad y \in S' \) implies \( y \notin F(x) \) for every \( \nu < \nu < \omega_{\alpha(\kappa_\varphi)} \).

(*) The construction of the sets \( S_{\nu}, S_{\nu}, S'_{\nu} \) is well known. The next step contains the main idea of our proof.
Now we are going to define a sequence \( \langle S^t \rangle \) of subsets of \( S \) by transfinite induction on \( t \) as follows.

(17) Put \( S^{\emptyset} = S^0 \).

Taking into consideration that by (12), \( S^0 = T = \kappa = \kappa_{\beta+1} \), there exists a sequence \( \langle S^t \rangle \) of subsets of \( S^t \) satisfying the conditions

\[
\bigcup_{t < \omega_{\beta+1}} S^t_{\kappa_{t+1}} = S^0 \backslash S^0_{\kappa_{t+1}} = T \setminus S^0 \quad S^t_{\kappa_{t+1}} \cap S^t_{\kappa_{t+2}} = 0
\]

for every \( \kappa_{t+1} < \kappa < \omega_{\beta+1} \).

Suppose that the sets \( S^t \) and \( S^t_{\kappa} \) are already defined for every \( \kappa < \omega_{\beta+1}, t \in \kappa \), in such a way that

\[
\bigcup_{t < \omega_{\beta+1}} S^t_{\kappa} = S^t \quad \text{and} \quad S^t_{\kappa_{t+1}} \cap S^t_{\kappa_{t+2}} = 0
\]

for every \( \kappa_{t+1} < \kappa < \omega_{\beta+1}, t < \kappa \).

Put \( Z_\kappa = \bigcup_{t \in \kappa} S^t_{\kappa} \), and \( Z_{\kappa_{t+1}} = \bigcup_{t \in \kappa} S^t_{\kappa_{t+1}} \). Then \( Z_{\kappa_{t+1}} \cap Z_{\kappa_{t+2}} = 0 \) for every \( \kappa_{t+1} < \kappa < \omega_{\beta+1} \). Let \( \alpha \) be an arbitrary element of \( S_{\kappa_{t+2}} \). Then \( F(\alpha) \cap Z_{\kappa_{t+1}} = \bigcup_{t \in \kappa_{t+1}} (F(\alpha) \cap Z_{\kappa_{t+2}}) \). Taking into consideration that by the assumption and by (9) \( F(\alpha) \cap Z_{\kappa_{t+1}} = 0 \), from the regularity of \( \kappa_{t+2} \) it follows that for every \( \alpha \in S^t_{\kappa_{t+1}} \) there exists an ordinal number \( \gamma(\alpha, \kappa_{t+2}) < \omega_{\beta+1} \), such that \( F(\alpha) \cap Z_{\kappa_{t+2}} = 0 \) if \( \gamma(\alpha, \kappa_{t+2}) < \kappa_{t+1} \).

Put \( T_{\kappa_{t+1}} = (x: \alpha \in S^t_{\kappa_{t+1}} \text{ and } \gamma(\alpha, \kappa_{t+2}) 

It follows from (9) and (11) that \( \kappa_{t+1} < \kappa \), and \( \kappa \) is regular, it results from (15) that there exists a \( \theta_{\kappa} = \theta_{\kappa}(\alpha) < \omega_{\beta+1} \) such that \( T_{\kappa_{t+1}} = \kappa_{\kappa_{t+1}} \).

(19) Put \( S^t_{\kappa} = T_{\kappa_{t+1}} \) for such a \( \theta_{\kappa} \).

Taking into consideration that by (12) \( S^0 = T \), there exists a sequence \( \langle S^t \rangle \) of subsets of \( S^t \) satisfying the conditions

\[
S^t = \bigcup_{t < \omega_{\beta+1}} S^t_{\kappa_{t+1}} \quad \text{and} \quad S^t_{\kappa_{t+1}} \cap S^t_{\kappa_{t+2}} = 0
\]

for every \( \kappa_{t+1} < \kappa < \omega_{\beta+1} \).

Thus \( S^t \) and \( S^t_{\kappa_{t+1}} \) are defined for every \( \kappa < \omega_{\beta+1}, \kappa_{t+1} < \kappa < \omega_{\beta+1} \), and it follows by induction from (17), (18), (19) and (20) that (19) and (20) holds for every \( \kappa < \omega_{\beta+1} \).

By the construction \( S^t \) has the following properties:

(21) \( S^t \subseteq S^t_{\kappa} \), there exists a \( \theta(\kappa) < \omega_{\beta+1} \) such that \( F(\kappa) \cap Z_{\kappa_{t+1}} = 0 \) for every \( \kappa \in \theta(\kappa) \) and for every \( \alpha \in S^t_{\kappa} \).

In fact, \( S^t \subseteq T \) and \( \theta(\kappa) = \theta(\kappa) \) satisfy the requirement of (21).

Put \( T_{\kappa_{t+1}} = (\theta(\kappa)) \) for every \( \kappa < \omega_{\beta+1} \).

Then \( T_{\kappa_{t+1}} < \kappa_{t+1} \) by (9). Hence by the hypothesis of \( \kappa_{t+1} \) there exists an ordinal number \( \theta(\kappa_{t+1}) \) such that

\[
\theta(\kappa_{t+1}) < \omega_{\beta+1} \quad \text{for every} \quad \kappa < \omega_{\beta+1}
\]

(22) Put \( T_{\kappa_{t+1}} = (\theta(\kappa_{t+1})) \) for every \( \kappa < \omega_{\beta+1} \).

Then \( T_{\kappa_{t+1}} < \kappa_{t+1} \). Hence by the hypothesis of \( \kappa_{t+1} \) there exists an ordinal number \( \theta(\kappa_{t+1}) \) such that

\[
\theta(\kappa_{t+1}) < \omega_{\beta+1} \quad \text{for every} \quad \kappa < \omega_{\beta+1}
\]

(23) Put \( T_{\kappa_{t+1}} = (\theta(\kappa_{t+1})) \) for every \( \kappa < \omega_{\beta+1} \).

Thus it follows from (20) and (23) that \( S^t \subseteq T \) for every \( \kappa < \omega_{\beta+1} \).

Taking into consideration that \( S^t \subseteq S^t \subseteq S^t \subseteq S \), it follows from (13) and (14) that

\[
S^t = \bigcup_{\kappa < \omega_{\beta+1}} S^t_{\kappa}
\]

On the other hand we have

(24) \( \forall \alpha \in S^t \) such \( \exists \alpha \in S^t \) implies \( \forall \alpha \in F(\alpha) \) for every \( \alpha < \omega_{\beta+1} \).

In fact, \( S^t \subseteq S^t \), hence if \( \alpha \in S^t \), then \( F(\alpha) \cap Z_{\kappa_{t+1}} = 0 \) for every \( \kappa_{t+1} \in \theta(\alpha) \) by (21), hence \( F(\alpha) \cap Z_{\kappa_{t+1}} = 0 \) for every \( \kappa_{t+1} \in \theta(\alpha) \) by (23). Thus

\[
F(\alpha) \cap S^t_{\kappa_{t+1}} = \bigcup_{\kappa_{t+1} < \kappa < \omega_{\beta+1}} (F(\alpha) \cap S^t_{\kappa_{t+1}}) = 0 \quad \text{for every} \quad \alpha < \omega_{\beta+1}
\]

Now we are going to prove that

(25) \( S^t = \bigcup_{\kappa < \omega_{\beta+1}} S^t_{\kappa} \) by (24).

This follows from (17), (18), and (19) that (19) and (20) holds for every \( \kappa < \omega_{\beta+1} \).

Thus \( S^t \) is a free set.

(26) \( S^t \subseteq S^t \) by (24).

Let \( x, y \) be two arbitrary distinct elements of \( S^t \). By (14) there exist uniquely determined ordinal numbers \( r_1, r_2 \) such that \( x \in S^t_{r_1}, y \in S^t_{r_2} \). By reason of symmetry, we may suppose \( r_1 \leq r_2 \).

Thus \( x \in S^t_{r_1}, y \in S^t_{r_2} \), so \( x \in F(\alpha) \) for every \( \alpha < \omega_{\beta+1} \) and \( F(\alpha) \cap Z_{\kappa_{t+1}} = 0 \) for every \( \kappa_{t+1} \in \theta(\alpha) \).

Thus \( S^t \) satisfies the requirements of Theorem 1 by (25) and (27). Q.e.d.

Remarks. In their paper [7] P. Erdős and G. Fodor prove the following theorem:
A new analytic approach to hyperbolic geometry

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Introduction

Hilbert was the first who constructed in plane hyperbolic geometry without the axiom of continuity a commutative ordered field \( \mathcal{E} = \langle E, +, \cdot, < \rangle \) and founded an analytic geometry over it (see [3] or [2], Appendix III). The field \( \mathcal{E} \) is known in the literature as the edelcalc since the class \( E \) consists of pencils of parallel half-lines, which Hilbert refers to as ends. The analytic geometry over \( \mathcal{E} \) is based upon a coordinate system for straight lines.

In this paper a new commutative ordered field \( \mathcal{S} = \langle S, +, \cdot, < \rangle \) is constructed in the same system of geometry. This field seems to be conceptually simpler and more adequate for the foundation of analytic geometry than \( \mathcal{E} \). It is generated by a hyperbolic calculus of segments, more precisely by an algebraic system \( \mathcal{S} = \langle S, +, \cdot, < \rangle \) in which the class \( S \) consists of the segments. The operations \( + \) and \( \cdot \) of \( \mathcal{S} \) are defined in terms of such simple notions as the Lambert quadrangle and the right triangle and are not relativized to any fixed geometrical objects, while the relation \( < \) coincides with the usual less-than relation for the segments. Finally a rectangular coordinate system over \( \mathcal{S} \) can be constructed (the two coordinates of a point being elements of \( S \)), and moreover the analytic geometry based on it is identical with that of the two-dimensional Klein space the absolute of which coincides with the unit circle.

Chapter I is algebraic. We introduce there the notion of a unit interval algebra and reduce the problem of constructing a commutative ordered field to that of constructing a unit interval algebra.

Chapter II is geometrical. In Section 1 we describe the axiomatic theory \( \mathcal{H} \) of the hyperbolic geometry in which the field \( \mathcal{S} \) is to be constructed. In Sections 2-13 we construct the system \( \mathcal{S} \), furthermore we prove it to be a unit interval algebra, and consequently, using the result of Chapter I, we obtain the ordered field \( \mathcal{S} \). In Sections 14-18 we outline the foundations of the analytic geometry over \( \mathcal{S} \).