A converse of a theorem of R. H. Bing and its generalization *

by

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Recently R. H. Bing announced ([1]) that if C is a 2-sphere in three-dimensional euclidean space E^3, U a component of E^3-C, and e an arbitrary given positive number, then C contains a Cantor set T such that there is an e-transformation f of C into $U \cup T$ such that f is a homeomorphism on $C-T$. We propose to show that the converse of this result holds, thus providing, in conjunction with Bing's result, a new positional characterization of the 2-sphere in E^3. In addition, we obtain an analogous result for the 2-manifold, and in § 2 we give an n-dimensional generalization.

This work stands in narrow relation to our earlier papers [2] and [3], in which a study was made of subsets of E^n which are "free" in various senses. None of the results obtained in the latter papers afforded manifold characterizations, however, since the types of "freeness" considered are not necessarily characteristic of manifolds imbedded in E^n when $n > 2$.

1. The three-dimensional case. We treat this case first partly because of its simplicity, but, of greater importance, because of the fact that the 2-sphere case has a special feature that allows of a simpler hypothesis. We refer to the fact that if a Cantor set (i.e. a closed, totally disconnected set) does not separate a 1-acyclic Peano continuum, then it does not locally separate it. As a consequence, for the 2-sphere case no hypothesis regarding local separation need be made, while for the case of the general 2-manifold such a hypothesis must be made. We shall make these remarks more precise:

DEFINITION 1. In any space X, a point set M *locally separates* X if there exists a domain (= open, connected set) D of X such that $D-M$ is not connected.

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LEMMA. *Let X be a normal, locally compact, locally connected and connected space such that $h_1(X) = 0$ (1). If T is a closed, totally disconnected subset of X that does not separate X, then T does not locally separate X.*

Proof. Let D be a domain of X, and x_1, x_2 distinct points of D. Since X is locally compact and locally connected, there exists in D a continuum K_1 containing x_1 and x_2. Let Q be an open subset of D such that $D \supset \bar{Q} \supset Q \supset K_1$ and $\bar{Q}$ is compact. Then, since T is closed and totally disconnected, there exists an open set R such that $D \supset \bar{R} \supset R \supset \bar{Q}$ and such that $T \cap F(R) = \emptyset$ (2).

Since T does not separate X, there exists in $X-T$ a continuum K_2 containing x_1 and x_2. Then T and $F(R)$ are disjoint, closed subsets of X such that x_1 and x_2 lie in a single constituant of $X-T$ and a single constituant of $X-F(R)$. And since $h_1(X) = 0$, it follows ([4], p. 242, Th. 9.2) that x_1 and x_2 lie in a single constituant of $X-T-F(R)$. We conclude that $D-T$ must be connected.

COROLLARY. *Let C be a locally connected subset of E^n, $n > 2$, forming a common boundary of (at least) two domains of E^n, and T a closed, totally disconnected subset of C. If $H_1(C) = 0$, then T does not locally separate C.*

EXAMPLE. The torus in E^3, pinched to a point p along one meridional circle (or a 2-sphere with two distinct points identified) is a common boundary, C, of two domains, satisfying all conditions of the Corollary except "$H_1(C) = 0$". The set $T = \{p\}$ locally separates C.

In stating the converse of Bing's theorem, we shall omit the requirement that f be a homeomorphism on $C-T$, since it turns out to be sufficient that f be simply a continuous mapping:

THEOREM 1. *In E^3, let C be a locally connected continuum forming a common boundary of (at least) two domains, and such that $H_1(C) = 0$. If for arbitrary given $e > 0$ and U either of the given domains complementary to C, there exists a closed and totally disconnected subset T of C, and an e-transformation $f(C) = C'$ into $H \cup T$, then C is a 2-sphere.*

Proof. Let U denote a fixed one of the two given complementary domains. We shall show that U is ulc (= uniformly locally connected).

Let x be a point of C and e a given positive number. Since C is locally connected, there exists a positive number d such that all points of $C \cap S(x, d)$ lie in one component, M, of $C \cap S(x, e/2)$. Suppose that p and q are points of $U \cap S(x, d)$ lying in different components of $U \cap S(x, e)$. Then ([2]; p. 159, Lemma) there exists a positive number s such that

(1) We use the symbol "$h_r(X)$" to denote the r-dimensional homology group of X determined by chains with compact carriers, using a field of coefficients. When X is itself compact, it agrees with $H_r(X)$.

(2) By $F(R)$ we denote the boundary of R; $\emptyset$ denotes the null set.

every s-transformation of C separates p and q in $S(x, e/2)$. By hypothesis, there exists a closed and totally disconnected subset T of C and an s-transformation $f(C) = C'$ such that $C' \subset V \cup T$, where V denotes the other given complementary domain.

Let A be an arc of $S(p, d)$ joining p and q and not meeting T. On A, in the order from p to q, let a be the first point of M and b the last point of M. Then, denoting by A_1 and A_2 the subarcs of A from p to a and from b to q, respectively, the set $K = A_1 \cup (M-T) \cup A_2$ is connected. For A_1 and A_2 both meet $M-T$ and the latter set is connected by virtue of the Corollary. However, $K \cap C' = \emptyset$, since $C' \subset V \cup T$ and $K \subset U \cup (C-T)$. This contradicts the fact that by the choice of s, C' must separate p and q in $S(x, e/2)$.

We conclude that U and V are ulc. By the Alexander Duality Theorem ([4]; p. 263, Th. 6.4) $h_1(U) = 0$. It follows ([4]; p. 309, Corollary 7.5) that C is an orientable closed 2-manifold and hence, since $H_1(C) = 0$, a 2-sphere.

The extension of Theorem 1 to cover the case of the general closed 2-manifold in E^3 will, of course, necessitate replacing the condition "$H_1(C) = 0$" by "$H_1(C)$ is finitely generated". But that this is not sufficient is shown by the example of the "pinched torus" above.

THEOREM 2. *In E^3, let C be a locally connected continuum forming a common boundary of (at least) two domains and such that $H_1(C)$ is finitely generated. If, for arbitrary given $e > 0$ and U either of the given complementary domains of C, there exists a closed and totally disconnected subset T of C which does not locally separate C, and an e-transformation $f(C) = C'$ into $U \cup T$, then C is an orientable closed 2-manifold.*

Proof. The proof given for Theorem 1 may be repeated, since we have provided in the new hypothesis for the connectedness of the set $M-T$ used in that proof. That C is an orientable closed 2-manifold follows again from [4]; p. 309, Corollary 7.5.

Remark. Compare Theorem 2 with [2]; p. 161, Th. 3.

2. The n-dimensional case. Here we may expect that the local connectedness of C will have to be strengthened to local connectedness of higher dimensions. On the other hand, we may hope that the dimension of T may be increased.

DEFINITION 2. A subset T of a space X will be called a *local r-separating* set of X if for some open subset U of X and compact cycle Z_r of $U-T$ which bounds on a compact subset of U, the cycle Z_r fails to bound on any compact subset of $U-T$.

THEOREM 3. *In E^n, $n > 1$, let C be an lc^k set which is a common boundary of (at least) two domains and such that if U is one of these domains and e an arbitrary positive number, then C contains a closed set T which* *is of at most dimension k and is not a local r-separating set of C for $r \leqslant k$, and such that there exists an e-transformation of C into $U \cup T$. For $n = 2m$ or $2m+1$, $k = m-1$, and in case $n = 2m+1$, $H_m(C)$ is finitely generated. Then C is a closed, orientable $(n-1)$-generalized closed manifold* [3].

Proof. Let U and V denote distinct domains of which C is given as the common boundary in the statement of the theorem. We shall show that U and V are ulc^k.

Let $0 \leqslant r \leqslant k$, $e > 0$, and p a point of C. Since C is r-lc, there exists $d > 0$ such that every r-cycle of $C \cap S(p, d)$ bounds on $C \cap S(p, e/2)$. Suppose there exists a cycle C_r in $U \cap S(p, d)$ which is non-bounding in $U \cap S(p, e)$. Then ([2]; p. 159, Lemma) there exists $s > 0$ such that every s-transformation of C is linked by C_r in $S(p, 3e/2)$. By hypothesis, there exists an s-transformation $f(C) = C'$ into $V \cup T$, where T is a closed subset of C of dimension at most k, and which is not a local r-separating set of C.

Now $C_r \sim 0$ in $S(p, d) - T$, since $r \leqslant k$ and $\dim T \leqslant k$. Using the notation of [4]; p. 203, Lemma 1.13, let M be a closed subset of $S(p, d) - T$ carrying the homology $C_r \sim 0$, K a carrier of C_r in $U \cap S(p, d)$, and L the closure of $S(p, d) - U$. Then (loc. cit.) there exists on $M \cap F(L)$— and hence on $C \cap S(p, d)$—a cycle Z_r such that $C_r \sim Z_r$ on $M \cap \overline{U}$. As M does not meet T, Z_r is on $C - T$; and by the choice of d, $Z_r \sim 0$ on $C \cap S(p, e/2)$. Therefore, since T is not a local r-separating set of C, $Z_r \sim 0$ on $C \cap S(p, e/2) - T$. But then (by combining homologies) $C_r \sim 0$ in $S(p, 3e/2) - C'$, in contradiction to the choice of s and f. We conclude that such a cycle as C_r cannot exist, and that U (and likewise V) is ulc^k.

When $n = 2m$, then, U and V are both ulc^{m-1}, and since $(m-1)+(m-1) = 2m-2 = n-2$, it follows ([4]; p. 308, Th. 7.1) that C is an orientable $(n-1)$-gcm. When $n = 2m+1$, $2m-2 = n-3$, and since $H_m(C)$ is finitely generated, it follows from the Alexander Duality Theorem and [4]; p. 308, Th. 7.3, that C is an orientable $(n-1)$-gcm.

References

[1] R. H. Bing, *Pushing a 2-sphere into its complement*, Amer. Math. Soc. Notices 6 (1959), p. 838, Abstract No. 564-165.

[2] R. L. Wilder, *Concerning a problem of K. Borsuk*, Fund. Math. 21 (1933), pp. 157-167.

[3] — *On free subsets of E_n*, Fund. Math. 25 (1935), pp. 200-208.

[4] — *Topology of Manifolds*, Amer. Math. Soc. Coll. Pub. 32, 1949.

(3) For the definition and properties of a generalized closed manifold (= gcm), see [4]; VIII.

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Proof of a conjecture of S. Ruziewicz

by

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§ 1. Introduction. Let S be an infinite set of power m. Let $F(x)$ be a set-mapping defined on S, i.e. a function which associates to every element x of S a subset $F(x)$ of S such that $x \notin F(x)$. Suppose that, for every $x \in S$, $\overline{\overline{F(x)}} < n$ where n is a given cardinal number less than m (finite or infinite). A subset S' of S is called a *free set* (with respect to the set-mapping $F(x)$) if, for every pair $x, y \in S'$, $x \notin F(y)$ and $y \notin F(x)$.

The following proposition has been conjectured by S. Ruziewicz.

Under the above conditions S has a free subset S' of power m (1).

This theorem was proved firstly for $n = \aleph_0$ and m either of the form 2^p or of the form $\aleph_{\alpha+1}$ (see [2] and [3]), secondly for m a regular cardinal number, or m a countable sum of cardinals smaller than m (see [4]), and thirdly for m not the sum of n or fewer cardinal numbers less than m (see [5]).

Finally P. Erdös proved—using the generalized continuum hypothesis—that the conjecture is true in the general case (see [6]).

The aim of our paper is to prove the above mentioned conjecture without using the generalized continuum hypothesis.

§ 2. THEOREM 1. *Let S be an infinite set such that $\overline{\overline{S}} = m$. Let $F(x)$ be a set-mapping defined on S such that $\overline{\overline{F(x)}} < n$ for every $x \in S$, where $n < m$. Then there exists a free subset S' of S such that $\overline{\overline{S'}} = m$.*

Proof. We distinguish two cases (i) m is regular and (ii) m is singular.

Case (i) (2). Let φ denote the initial number of the cardinal number n. We are going to define a sequence $\{S_\nu\}_{\nu<\varphi}$ of type φ of subsets of S by

(1) See [1]. Questions of this type have been first posed by P. Turán — see: G. Grünwald, *Egy halmazelméleti tételröl*, Math. Fiz. Lapok 44 (1937), pp. 51-53.

In some of the cited papers binary relations of form yRx are investigated, where the corresponding set-mapping is to be defined by the stipulation $F(x) = \{y\colon yRx\}$.

(2) In this case the theorem is well known (see the papers cited). However, for the convenience of the reader, we reconstruct here a simple proof of it. This proof is due to D. Lázár.