Residue class fields of lattice-ordered algebras *

by

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This paper is a continuation of [5], and is concerned with the structure of the residue class fields of the Φ-algebras introduced and studied in that paper. These are archimedean lattice-ordered algebras with a multiplicative identity that is a weak order unit. The lattice-ordered ring $C(\mathcal{Y})$ of all continuous real-valued functions on a topological space $\mathcal{Y}$ is a Φ-algebra, and it is shown in [5] that every Φ-algebra A is isomorphic to a ring of continuous functions from a compact space $\mathcal{X}$ into the two-point compactification of the real line R such that every $f \in A$ is real-valued on an (open) dense subset of $\mathcal{X}$.

If $A = C(\mathcal{Y})$, and M is a maximal l-ideal of A, it is known that A/M is a real-closed field that is either the real field, or an η_1-set in its unique ordering. We show that for any uniformly closed Φ-algebra A, the residue-class fields are real-closed. This result seems to be new even for Φ-algebras of real-valued functions. Stronger assumptions must be made to guarantee that if A/M is not the real field, then it is an η_1-set. We show that if A is closed under countable composition (i.e. if $\{f_n\}$ is a sequence of elements of A, and $g \in C(R^\infty)$, then there is an $h \in A$ such that $h(x) = g(f_1(x), \ldots, f_n(x), \ldots)$ whenever all of the f_n are real-valued), then A is closed under uniform convergence, and A/M is an η_1-set if it is not the real field. In fact, under this hypothesis, A is a homomorphic image of $C(\mathcal{Y})$, for some topological space $\mathcal{Y}$.

It is shown also that every Φ-algebra A is a homomorphic image of a Φ-algebra B of real-valued functions; moreover, B can be chosen so that it is closed under countable composition, (finite) composition, uniform convergence, or bounded inversion, provided that A is.

An example is given of a uniformly closed Φ-algebra A that is closed under (finite) composition, with a maximal l-ideal M such that A/M contains R properly, and has a countable cofinal subset. This serves to correct an error in [6].

The notation and terminology is that of [5]. An effort has been made to keep the exposition reasonably self-contained.

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1. Residue class fields of uniformly closed Φ-algebras.

Recall from [5] that a Φ-algebra A is said to be closed under *bounded inversion* provided $1/a \in A$ whenever $a \geqslant 1$ in A.

1.1. LEMMA. *A Φ-algebra A is closed under bounded inversion if and only if every maximal ideal of A is an l-ideal.*

Proof. If $a \geqslant 1$ in A, then a is in no proper l-ideal of A. Hence, if every maximal ideal of A is an l-ideal, then A is closed under bounded inversion.

For the converse, let M be a maximal ring ideal of A and suppose $b \notin M$, $|a| \geqslant |b|$. Since A/M is a field, there is an $x \in A$ such that $bx+m = 1$. Squaring, we obtain $b^2x^2+m' = 1$, where $m' = 2bxm+m^2 \in M$. Then, since $b^2 \leqslant a^2$, and $x^2 \geqslant 0$, we must have $a^2x^2+m' \geqslant 1$. If A is closed under bounded inversion, there is a $z \in A$ such that $(a^2x^2+m')z = 1$, so that $a(ax^2z) \equiv 1 (\mathrm{mod}\, M)$. Thus $a \notin M$. Hence M is an l-ideal of A.

As in [5], we say that A is *uniformly closed* if every Cauchy sequence of elements of A converges to an element of A.

If $\mathcal{X}$ is any compact space, let $D(\mathcal{X})$ denote the set of all continuous functions defined on $\mathcal{X}$ with values in the two point compactification $\gamma R = [-\infty, +\infty]$ of the real line R that are real-valued on a dense (open) subset of $\mathcal{X}$. If lattice operations are defined coordinatewise, then $D(\mathcal{X})$ forms a lattice. Let $f, g \in D(\mathcal{X})$. If there is an $h \in D(\mathcal{X})$ such that $h(x) = f(x)+g(x)$ whenever $f(x)$ and $g(x)$ are real, we write $h = f+g$, and similarly for multiplication. In general, neither $f+g$, nor $f \cdot g$ is defined. It is true, however, that every Φ-algebra A can be isomorphically represented in $D(\mathfrak{M}(A))$, where $\mathfrak{M}(A)$ is the space of maximal l-ideals of with the Stone (= hull-kernel) topology. $\mathfrak{M}(A)$ is always a compact Hausdorff space ([5], Theorem 2.3). We will regard A as represented this way whenever it is convenient to do so.

We will also utilize the following, proved in [5], 3.2 and 3.7.

1.2. *The following properties of a Φ-algebra A are equivalent.*

(i) *A is uniformly closed.*

(ii) *A^* and $C(\mathfrak{M}(A))$ are isomorphic.*

(iii) *A is an order-convex subset of $D(\mathfrak{M}(A))$.*

From (ii), it is evident that every uniformly closed Φ-algebra is closed under bounded inversion.

If $a \in A$, let $\mathcal{R}(a) = \{x \in \mathfrak{M}(A): |a(x)| < \infty\}$, let $\mathcal{Z}(a) = \{x \in \mathfrak{M}(A): a(x) = 0\}$ and let $\mathcal{N}(a) = \mathfrak{M}(A) \sim \mathcal{R}(a)$. Finally, let $\mathcal{R}(A) = \bigcap_{a \in A} \mathcal{R}(a)$. If $\mathcal{R}(A)$ is dense in $\mathfrak{M}(A)$, then A is called *an algebra of real-valued functions.*

Let A be any Φ-algebra, and let $g \in C(R^n)$. If, for every $f_1, \ldots, f_n \in A$, there is an $h \in A$ such that $h(x) = g(f_1(x), \ldots, f_n(x))$, whenever $x \in \bigcap_{i=1}^{n} \mathcal{R}(f_i)$,

we say that A is *closed under composition with* g, or that A *admits* g. Evidently h is unique; we shall write $h = g(f_1, \ldots, f_n)$. Every Φ-algebra admits the constant functions and the projection functions p_i, where $p_i(\lambda_1, \ldots, \lambda_n) = \lambda_i$ $(i = 1, 2, \ldots, n)$.

We let $F(A, n)$ denote the family of all $g \in C(R^n)$ that A admits. It is easily verified that $F(A, n)$ is a Φ-algebra if operations are defined in the usual coordinatewise fashion.

If A is uniformly closed, so is $F(A, n)$. For if $\{g_i\}$ is a Cauchy sequence in $F(A, n)$, then it converges to some $g \in C(R^n)$. If $f_1, \ldots, f_n \in A$, then $\{g_i(f_1, \ldots, f_n)\}$ is a Cauchy sequence of elements of A whose limit must be $g(f_1, \ldots, f_n)$.

Let A be uniformly closed and let $p = \left(\sum_{i=1}^{n} p_i^2\right)^{1/2}$. Note that $R^n \subset \mathfrak{M}(F(A, n))$, and that $\mathcal{R}(p) = R^n$. Hence by [5], Lemma 3.5, every $g \in C^*(R^n)$ has a continuous extension over $\mathfrak{M}(F(A, n))$, so $\mathfrak{M}(F(A, n))$ and βR^n are homeomorphic. By 1.2 (iii), $F(A, n)$ is an order-convex sub-Φ-algebra of $C(R^n)$. Thus, we have established

1.3. LEMMA. *If A is a uniformly closed Φ-algebra, then, for $n = 1, 2, \ldots$, $F(A, n)$ is a uniformly closed sub-Φ-algebra of $C(R^n)$ containing all $g \in C(R^n)$ such that $|g| \leqslant \lambda(1+p^2)^m$ for some $\lambda \in R^+$, and some positive integer m.*

Recall that a totally ordered field F is called *real-closed* if every $a \in F^+$ has a square root and every polynomial of odd degree with coefficients in F has a zero in F.

1.4. THEOREM. *If A is a uniformly closed Φ-algebra, and $M \in \mathfrak{M}(A)$, then A/M is a real-closed field.*

Proof. Since A is closed under bounded inversion, Lemma 1.1 shows that A/M is a field. By [5] Theorem 3.8, every $a \in A^+$ has a square root, so we need only show that polynomials of odd degree with coefficients in A/M have zeros.

Let $p_\lambda(w) = w^{m+1}+\lambda_m w^m+\ldots+\lambda_0$ denote a monic polynomial with real coefficients of positive degree. Let $r_1(\lambda), r_2(\lambda), \ldots, r_{m+1}(\lambda)$ denote the real parts of the complex zeros of $p_\lambda(w)$ indexed so that $r_1(\lambda) \leqslant r_2(\lambda) \leqslant \ldots \leqslant r_{m+1}(\lambda)$. This serves to define $m+1$ real-valued functions on R^{m+1}. It is known that each of these functions is continuous ([4]). Moreover, by [9], p. 96, $|r_i(\lambda)| < 1+|\lambda_0| \vee |\lambda_1| \vee \ldots \vee |\lambda_m|$ for each $\lambda = (\lambda_0, \ldots, \lambda_m) \in R^{m+1}$, and $i = 1, \ldots, m+1$. Hence, by Lemma 1.3, A is closed under composition with r_i.

Let $q(w) = w^{2n+1}+f_{2n}w^{2n}+\ldots+f_0$ denote a monic polynomial of odd degree with coefficients in A. By the above, $s_i = r_i(f_0, \ldots, f_{2n}) \in A$. Since $q(w)$ has odd degree, for each $x \in \bigcap_{i=1}^{2n+1} \mathcal{R}(f_i)$, there is an i such that $q(s_i)(x) = 0$.

Hence $q(s_1)q(s_2)\dots q(s_{2n+1}) = 0$. Since M is a prime ideal, there is an i_0 such that $q(s_{i_0}) \in M$. Hence A/M is a real-closed field.

The argument just given enables us to reach the following slightly stronger conclusion. *If A is uniformly closed Φ-algebra, and P is a prime l-ideal of A, then every positive element of A/P has a square root, and every monic polynomial of odd degree with coefficients in A/P has a zero in A/P.* Also, as we will show next, the assumption that P is an l-ideal is redundant.

1.5. LEMMA. *Every prime ideal P of a uniformly closed Φ-algebra A is an l-ideal.*

Proof. Since $|c|^2 = c^2$, we know that $c \in P$ if and only if $|c| \in P$. Thus, since $|c| = (|c| \wedge 1)(|c| \vee 1)$, and since, by Lemma 1.2, A is closed under bounded inversion, $c \in P$ if and only if $|c| \wedge 1 \in P$.

Suppose now that $|b| \leqslant |a|$, and $a \in P$. Then $|b| \wedge 1 \leqslant |a| \wedge 1 \in P \cap A^*$. But, by Lemma 1.2, A^* and $C(\mathcal{M}(A))$ are isomorphic, and by [3], Chapt. 14, every prime ideal of the latter is an l-ideal. So $|b| \wedge 1 \in P$, whence $b \in P$. Hence P is an l-ideal.

1.6. REMARK. It is remarked in [3], Chapt. 13, that any totally ordered field containing R properly in which exponentiation of positive elements to real powers can be defined has degree of transcendency at least c over R. It follows that if A is a uniformly closed Φ-algebra, and $M \in \mathcal{M}(A)$ is hyper-real, then A/M has degree of transcendency at least c over R.

If S and T are subsets of a totally ordered set L, and $s < t$ whenever $s \in S$ and $t \in T$, we will write $S < T$.

1.7. THEOREM. *Let P be a prime ideal of a uniformly closed Φ-algebra A. If S and T are countably infinite subsets of A/P such that S has no largest element, T has no smallest element, and $S < T$, then there is an $a \in A/P$ such that $S < a < T$.*

Proof. Since, by 1.5, P is a prime l-ideal, A/P is totally ordered, and by 1.2 ff. we may assume that $0 \leqslant S < T \leqslant 1$. By Lemma 1.2, $A^* \cong C(\mathcal{M}(A))$. Kohls has shown that the conclusion follows in case $A \cong C(\mathcal{Y})$ for any space $\mathcal{Y}$ ([8], Theorem 2.6). Since

$$\frac{A^*}{P \cap A^*} \cong \frac{A^* + P}{P} \subset A/P\,,$$

the conclusion holds in this case as well.

A totally ordered set L is called an *η_1-set* if whenever S and T are countable subsets of L such that $S < T$, then there is an $a \in L$ such that $S < a < T$. In particular, an η_1-set has no countable cofinal or coinitial subset.

For any topological space $\mathcal{Y}$, and any hyper-real maximal ideal M of $C(\mathcal{Y})$, it is known that $C(\mathcal{Y})/M$ is an η_1-set. Example 1.9 below shows strongly that no comparable conclusion holds for arbitrary uniformly closed Φ-algebras.

Most of the remainder of the paper will be devoted to a discussion of the extra hypotheses needed to conclude that A/M is an η_1-set.

A Φ-algebra A is said to be *closed under (finite) composition* if $F(A, n) = C(R^n)$ for $n = 1, 2, \dots$; that is, if A admits every $g \in C(R^n)$.

As in [5], A is said to be *closed under l-inversion* if $\langle a \rangle = A$ whenever $\mathcal{Z}(a) \subset \mathcal{N}(b)$ for some $b \in A$. (Recall that $\langle a \rangle$ is the smallest l-ideal of A containing a.)

1.8. LEMMA. *Let A be a Φ-algebra.*

(i) *If $F(A, 2) = C(R^2)$ (in particular, if A is closed under composition), then A is closed under l-inversion.*

(ii) *If A is closed under uniform convergence and l-inversion, then A is closed under composition.*

Proof. (i) Let $a, b \in A$, and suppose that $\mathcal{Z}(a) \subset \mathcal{N}(b)$. Let $h = |a| \vee |b|$, let $B_h = \{f \in A\colon \mathcal{R}(f) \supset \mathcal{R}(h)\}$, and let $\mathcal{H} = \{(a(x), b(x)) \in R^2\colon x \in \mathcal{R}(h)\}$. If $(0, q) \in \mathcal{H}^-$, then there is a sequence $\{x_n\}$ of points of $\mathcal{R}(h)$ such that $a(x_n) \to 0$ and $b(x_n) \to q$. Since $\mathcal{M}(A)$ is compact, $\{x_n\}$ has a limit point $x \in \mathcal{M}(A)$. Clearly $a(x) = 0$, and $b(x) = q$, contrary to the assumption that $\mathcal{Z}(a) \subset \mathcal{N}(b)$. Thus, the function g defined on $\mathcal{H}^-$ by letting $g(p, q) = 1/p$ is continuous. By the Tietze extension theorem, it has an extension $\bar{g} \in C(R^2)$. Since $F(A, 2) = C(R^2)$, this shows that $1/a \in A$.

(ii) Suppose that $f_1, \dots, f_n \in A$, let $h = |f_1| \vee \dots \vee |f_n|$, and let $B_h = \{a \in A\colon \mathcal{R}(a) \supset \mathcal{R}(h)\}$. By [5], Theorem 5.8, since A is closed under uniform convergence and l-inversion, B_h and $C(\mathcal{R}(h))$ are isomorphic. Hence, for any $g \in C(R^n)$, $g(f_1, \dots, f_n) \in A$ $(n = 1, 2, \dots)$. Thus, A is closed under composition.

In [6], Theorem 1.28, Isbell states that if A is an algebra of real-valued functions closed under uniform convergence and composition, and $M \in \mathcal{M}(A)$ is hyper-real, then A/M is an η_1-set. While he establishes correctly the conclusion of Theorem 1.7 above, A/M may have a countable cofinal subset, as is shown by the following. For $a \in A$, the image of a in A/M is denoted by $M(a)$.

1.9. EXAMPLE. *There exists a uniformly closed Φ-algebra A, closed under composition, and a hyper-real $M \in \mathcal{M}(A)$ such that A/M has a countable cofinal subset.*

Proof. Let $\mathcal{I}$ denote the space of irrational numbers in $(0, 1)$ with its usual topology. Since $\beta\mathcal{I}$ is the largest compactification of $\mathcal{I}$, there is a continuous mapping π of $\beta\mathcal{I}$ onto $[0, 1]$ keeping $\mathcal{I}$ pointwise fixed. Let $\mathcal{Z}_0 = \pi^{-1}(0)$, and for $i = 1, 2, \dots$, let $\mathcal{S}_i = \{1/p^j \in (0, 1)\colon p \text{ a prime}; j \text{ a positive integer}, j \leqslant i\}$, let $\mathcal{Z}_i = \mathcal{Z}_0 \cup \pi^{-1}(\mathcal{S}_i)$, and let $\mathcal{Y}_i = \beta\mathcal{I} \sim \mathcal{Z}_i$.

Observe that $\mathcal{S} \subset \mathcal{Y}_{i+1} \subset \mathcal{Y}_i$ for $i = 1, 2, \ldots$, and let $A_i = \{f \in D(\beta\mathcal{S}) : \mathcal{P}(f) \supset \mathcal{Y}_i\}$. Since $\mathcal{Y}_i$ contains $\mathcal{S}$, it is C^*-imbedded in $\beta\mathcal{S}$, so A_i and $C(\mathcal{Y}_i)$ are isomorphic. Finally, let $A = \bigcup_{i=1}^{\infty} A_i$.

If $\{f_n\}$ is a Cauchy sequence of elements of A, then (as is noted in [5], 3.1) $\mathcal{P}(f_n) = \mathcal{P}(f_{n+1}) = \ldots$ for all but finitely many of the f_n. Thus we may assume that $\{f_n\} \subset A_i$ for some i, whence $\{f_n\}$ converges. Similarly, any finite number of elements of A is contained in some A_i. Thus A is closed under uniform convergence and l-inversion.

If every point of the compact space $\mathcal{Z}_0$ had a neighborhood meeting only finitely many of the sets $\{\mathcal{Z}_{i+1} \sim \mathcal{Z}_i\}$, then $\mathcal{Z}_0$ itself would have such a neighborhood. But every neighborhood of 0 meets infinitely many of the sets $\mathcal{S}_{i+1} \sim \mathcal{S}_i$, so this cannot be the case. Hence, there is an $x \in \mathcal{Z}_0$ such that every neighborhood of x meets infinitely many of the sets $\{\mathcal{Z}_{i+1} \sim \mathcal{Z}_i\}$. By a suitable change of notation, we may assume that every neighborhood of x meets all such sets.

Now each $\mathcal{Z}_i$ is the inverse image of a closed subset of a metrizable space, and hence is a closed G_δ. Hence there is an $f_i \in A_i^+$ such that $\mathcal{N}(f_i) = \mathcal{Z}_i$. Now, $M_x(f_1)$ is greater than all the constant functions, so M_x is hyper-real. If $g \in A$, there is an i such that $\mathcal{N}(g) \subset \mathcal{Z}_i$. Suppose there were an $h \in M_x$ such that $g + h \geqslant f_{i+1}$. Then $\mathcal{N}(f_{i+1}) \subset \mathcal{N}(g) \cup \mathcal{N}(h)$, and hence $\mathcal{N}(h) \supset \mathcal{N}(f_{i+1}) \sim \mathcal{N}(g) \supset \mathcal{Z}_{i+1} \sim \mathcal{Z}_i$. But this latter set has x as a limit point, contrary to the fact that $h \in M_x$. We conclude that $\{M_x(f_i) : i = 1, 2, \ldots\}$ is a countable cofinal subset of A/M_x.

2. Φ-algebras closed under countable composition. The example of the last section motivates the consideration of a more restricted class of Φ-algebras.

We designate a countable product of copies of R as R^∞.

Let A be a Φ-algebra, and suppose that for every $g \in C(R^\infty)$, and every sequence $\{f_n : n = 1, 2, \ldots\}$ of elements of A, there is an $h \in A$ such that $h(x) = g(f_1(x), \ldots, f_n(x), \ldots)$ whenever $x \in \bigcap_{n=1}^{\infty} \mathcal{P}(f_n)$; we say that A is *closed under countable composition*. By the Baire category theorem, $\bigcap_{n=1}^{\infty} \mathcal{P}(f_n)$ is dense in $\mathcal{M}(A)$, so h is unique. We denote it by $g(f_1, f_2, \ldots, f_n, \ldots)$.

Clearly, if A is closed under countable composition, it is closed under composition, and hence, by Lemma 1.8, it is closed under l-inversion. This motivates the consideration of the following concept.

A Φ-algebra A is said to be *closed under countable l-inversion* provided that $\langle g \rangle = A$ for each $g \in A$ for which there is a sequence $\{f_n : n = 1, 2, \ldots\}$ of elements of A such that $\mathcal{Z}(g) \subset \bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$.

The relationship between these two latter concepts is given by

2.1. THEOREM. *A Φ-algebra A is closed under countable composition if and only if it is uniformly closed and closed under countable l-inversion.*

Proof of necessity. Suppose that A is closed under countable composition, and that $Z(g) \subset \bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$ for some $g, f_1, \ldots, f_n, \ldots$ in A. Let $g = f_0$, $\mathcal{Y} = \bigcap_{n=0}^{\infty} \mathcal{P}(f_n)$, and define $\psi : \mathcal{Y} \to R^\infty$ by letting $\psi(y) = (f_0(y), f_1(y), \ldots, f_n(y), \ldots)$ for all $y \in \mathcal{Y}$. Let $\mathcal{H}$ denote the closure in R^∞ of $\psi[\mathcal{Y}]$. If $x = (x_0, x_1, \ldots, x_n, \ldots) \in \mathcal{H}$, then $x_0 \neq 0$. For, otherwise there would be a sequence $\{y_n\}$ of points of $\mathcal{Y}$ such that $\psi(y_n)$ converges to x. Since $\mathcal{M}(A)$ is compact, $\{y_n\}$ has an accumulation point in $\mathcal{M}(A)$, which is a point of $Z(g)$ not in $\bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$.

Hence the function $r : \mathcal{H} \to R$ defined by letting $r(x_0, x_1, \ldots, x_n, \ldots) = 1/x_0$ is well-defined and continuous. By the Tietze extension theorem ([7], p. 242), r has an extension $s \in C(R^\infty)$. Since A is closed under countable composition, $s(f_0, f_1, \ldots, f_n, \ldots)$ is an element h of A such that $gh = 1$ on the dense subset $\mathcal{Y}$ of $\mathcal{M}(A)$. Thus h is the inverse of g, whence $\langle g \rangle = A$.

Suppose next that $\{f_n\}$ is a Cauchy sequence of elements of A, define $\mathcal{Y}$ as above, define $\psi : \mathcal{Y} \to R^\infty$ by letting $\psi(y) = (f_1(y), \ldots, f_n(y), \ldots)$ for all $y \in \mathcal{Y}$, and let $\mathcal{H}$ denote the closure of $\psi[\mathcal{Y}]$ in R^∞.

Since $\{f_n\}$ is a Cauchy sequence, for every $\varepsilon > 0$ there is a positive integer m such that for every $x = (x_1, x_2, \ldots, x_n, \ldots)$ of $\psi[\mathcal{Y}]$, $|x_p - x_q| < \varepsilon$ whenever $p, q \geqslant m$. For any $z \in \mathcal{H}$, if $p, q \geqslant m$, then $|z_p - z_q| \leqslant \varepsilon$. For, if not, for some such z, p, and q, there is a $\delta > 0$ such that $|z_p - z_q| = \varepsilon + 2\delta$. Then $\{w \in R^\infty : |w_p - z_p| < \delta$ and $|w_q - z_q| < \delta\}$ is a neighborhood of z in R^∞ that contains no point of $\psi[\mathcal{Y}]$, contrary to the fact that $z \in \mathcal{H}$. Hence, for each $z \in \mathcal{H}$, $\{z_n\}$ is a Cauchy sequence. Define $s : \mathcal{H} \to R$ by letting $s(z) = \lim_{n \to \infty} z_n$. It is easily verified that $s \in C(\mathcal{H})$. By the Tietze extension theorem, s has a continuous extension $t \in C(R^\infty)$. Since A is closed under countable composition, $h = t(f_1, f_2, \ldots, f_n, \ldots) \in A$. Clearly $\{f_n\}$ converges to h. This completes the proof of the necessity.

Before proving the sufficiency, we prove two lemmas that are of independent interest.

Recall that a topological space $\mathcal{Y}$ is called a *Lindelöf space* if every open cover of $\mathcal{Y}$ has a countable subcover.

2.2. LEMMA. *Let $\mathcal{Y}$ be a subspace of a compact space $\mathcal{X}$ such that for some countable family Ψ of closed subsets of $\mathcal{X}$, for every pair of points $p \in \mathcal{Y}$, $q \in \mathcal{X} \sim \mathcal{Y}$ there is a set in Ψ containing p but not q. Then $\mathcal{Y}$ is a Lindelöf space.*

Proof. Let $\{\mathcal{U}_\alpha: \alpha \in \Gamma\}$ denote an open cover of $\mathcal{Y}$. For each $\alpha \in \Gamma$, let $\mathcal{R}_\alpha = \mathcal{Y} \sim \mathcal{U}_\alpha$, $\mathcal{S}_\alpha$ denote the closure of $\mathcal{R}_\alpha$ in $\mathcal{X}$, and let $\mathcal{V}_\alpha = \mathcal{X} \sim \mathcal{S}_\alpha$. Clearly $\mathcal{V}_\alpha \cap \mathcal{Y} = \mathcal{U}_\alpha$, and the sets $\mathcal{V}_\alpha$ cover $\mathcal{X} \sim \mathcal{K}$, where $\mathcal{K} = \bigcap \{\mathcal{S}_\alpha: \alpha \in \Gamma\}$. Clearly $\mathcal{K}$ is a compact subset of $\mathcal{X} \sim \mathcal{Y}$.

Let $\mathcal{F}$ denote the union of all those subsets of $\mathcal{X}$ that are disjoint from $\mathcal{K}$, and are finite intersections of elements of Ψ. Then $\mathcal{F}$ is σ-compact, and hence is a Lindelöf space. Thus, it suffices to show that $\mathcal{Y} \subset \mathcal{F}$. But, for each $p \in \mathcal{Y}$, by hypothesis, the intersection of all the elements of Ψ containing p is disjoint from $\mathcal{K} \subset \mathcal{X} \sim \mathcal{Y}$. Hence some finite intersection of them is disjoint from $\mathcal{K}$. Hence $\mathcal{Y} \subset \mathcal{F}$.

2.3. Corollary. *Every subset of a compact space $\mathcal{X}$ that is in the smallest family of subsets of $\mathcal{X}$ containing the closed subsets and closed under countable union and intersection, is a Lindelöf space. In particular, for any Φ-algebra A and any sequence $\{f_n\}$ of elements of A, $\bigcap_{n=1}^{\infty} \mathcal{R}(f_n)$ is a Lindelöf space.*

Proof. Every closed subspace of $\mathcal{X}$ satisfies the hypothesis of Lemma 2.2, so it sufficies to show that if $\mathcal{E}_n$ satisfies this latter condition with associated countable family of closed sets Ψ_n for $n = 1, 2, \dots$, then so does $\mathcal{Y} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$, and $\mathcal{Z} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$. If $p \in \mathcal{Y}$, $q \in \mathcal{X} \sim \mathcal{Y}$, then $p \in \mathcal{E}_n$ for some n, and $q \notin \mathcal{E}_n$, so there is an element of Ψ_n that contains p and not q. Thus, $\mathcal{Y}$ satisfies the hypothesis of Lemma 2.2 with associated countable family of closed sets $\bigcup_{n=1}^{\infty} \Psi_n$. The proof for $\mathcal{Z}$ is similar.

2.4. Lemma. *Let A be a uniformly closed Φ-algebra that is closed under countable l-inversion, let $\{f_n\}$ be a sequence of elements of A, let $\mathcal{Y} = \bigcap_{n=1}^{\infty} \mathcal{R}(f_n)$, and let $B = \{g \in A: \mathcal{Y} \subset \mathcal{R}(g)\}$. Then B and $C(\mathcal{Y})$ are isomorphic.*

Proof. Clearly B is a sub-Φ-algebra of A. Since $B \supset A^*$, $\mathcal{M}(B) = \mathcal{M}(A)$, and since A is uniformly closed, so is B. Since $f_n \in B$ for $n = 1, 2, \dots$, $\mathcal{R}(B) = \mathcal{Y}$. B is also closed under inversion of elements without zeros in $\mathcal{R}(B)$. For, if $Z(g) \cap \mathcal{R}(B) = \emptyset$, then $\mathcal{Z}(g) \subset \bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$, so, since A is closed under countable l-inversion, $1/g$ is in A and is real-valued on $\mathcal{Y}$. By Corollary 2.3, $\mathcal{Y}$ is a Lindelöf space, so by [5], Lemma 5.3, for every $h \in C(\mathcal{Y})$, there is a $b \in C^*(\mathcal{M}(B))$ such that $h^{-1}(0) = \mathcal{Z}(b) \cap \mathcal{Y}$. It follows from [5], Theorem 5.2 that B and $C(\mathcal{R}(B))$ are isomorphic.

The proof of sufficiency for Theorem 2.1 is now easy in view of Lemma 2.4. If $\{f_n\}$ is any sequence of elements of a uniformly closed Φ-algebra that is closed under countable l-inversion, then, by Lemma 2.4,

if $\mathcal{Y} = \bigcap_{n=1}^{\infty} \mathcal{R}(f_n)$, then $C(\mathcal{Y})$ is a subalgebra of A. So, for any $g \in C(R^\infty)$, $g(f_1, \dots, f_n, \dots)$ is in A.

Before returning to hyper-real residue class fields, we prove

2.5. Theorem. *Every Φ-algebra A can be obtained as a homomorphic image of a Φ-algebra B of real-valued functions in such a way that if A is uniformly closed, or closed under bounded inversion, or composition, or countable composition, then so is B.*

Proof. B will be defined as an algebra of continuous real-valued functions on $\mathcal{M}(A) \times \mathcal{N}$ where $\mathcal{N}$ is the discrete space of positive integers. Every element g of B will be regarded as a sequence $\{g_n\}$ of functions on $\mathcal{M}(A)$, where $g_n(p) = g(p, n)$, for all $p \in \mathcal{M}(A)$. B consists precisely of all those $\{g_n\}$ which converge pointwise to an element of A on a dense G_δ; i.e. those $g \in C(\mathcal{M}(A) \times \mathcal{N})$ such that for some $f \in A$, and for some dense G_δ-set $\mathcal{Y} \subset \mathcal{M}(A)$, for each $p \in \mathcal{Y}$, the sequence $\{g(p, n)\}$ of real numbers converges in $\gamma R = [-\infty, +\infty]$ to $f(p)$.

Since the intersection of two dense G_δ-sets is dense, each $g \in B$ converges to a unique $\lambda(g) \in A$. Similarly, it is easily verified that B is a Φ-algebra, and that λ is a homomorphism of B into A. Moreover, if $f \in A$, and $g_n = (f \wedge n) \vee (-n)$ for $n = 1, 2, \dots$, then $g_n(p)$ converges to $f(p)$ for all $p \in \mathcal{M}(A)$. Hence $\lambda(g) = f$, so λ is a homomorphism of B onto A.

Suppose that A is closed under countable composition, that $\{g_n\}$ is a sequence of elements of B, and that $h \in C(R^\infty)$. For $n = 1, 2, \dots$, there is a dense G_δ-set $\mathcal{Y}_n$ in $\mathcal{M}(A)$ such that for each $p \in \mathcal{Y}_n$, $g_n(p, m)$ converges to $\lambda(g_n)(p)$. Let $\mathcal{Y}$ denote the intersection of all the $\mathcal{Y}_n$ and all $\mathcal{R}(\lambda(g_n))$, for $n = 1, 2, \dots$ Since $\mathcal{M}(A)$ is compact, this countable intersection of dense G_δ-sets is a dense G_δ-set. Moreover, each $\lambda(g_n)$ is real-valued on $\mathcal{Y}$, and so is $h(\lambda(g_1), \dots, \lambda(g_n), \dots) \in A$. For each $p \in \mathcal{Y}$, and for each n, the real numbers $g_n(p, m)$ converge to $\lambda(g_n)(p)$. Then the points $\{x_m\}$ of R^∞, whose n-th coordinates are $g_n(p, m)$, form a convergent sequence in R^∞, whose limit z has as n-th coordinate $\lambda(g_n)(p)$. Since h is continuous, $h(x_m) \to h(z)$. Thus, the well-defined continuous function $h(g_1, \dots, g_n, \dots)$ in $C(\mathcal{M}(A) \times \mathcal{N})$ is in B, since it converges pointwise on $\mathcal{Y}$ to $h(\lambda(g_1), \dots, \lambda(g_n), \dots)$. That is, B is closed under countable composition.

Simplified versions of the preceding establish the remaining assertions.

In [1], 2.1, Corson and Isbell show that if an algebra A of real-valued functions is closed under countable composition, then it is closed under composition for all higher cardinals. This fact may be used to establish the following.

2.6. THEOREM. *Every Φ-algebra A closed under countable composition is a homomorphic image of $C(\mathcal{Y})$ for some topological space $\mathcal{Y}$.*

For, by Theorem 2.5, we may assume without loss of generality that A is an algebra of real-valued functions. Let $\mathcal{Y}$ denote the cartesian product of as many copies R_f of R as there are elements f of A. Let e denote the mapping of $\mathcal{R}(A)$ into $\mathcal{Y}$ such that the f-th coordinate $e(x)_f$ of $e(x)$ is $f(x)$. Finally, let $\tau g = g \cdot e$ for each $g \in C(\mathcal{Y})$. By the result cited above, since A is closed under countable composition, and hence unlimited composition, $\tau g \in A$ for all $g \in C(\mathcal{Y})$. Clearly τ is a homomorphism of $C(\mathcal{Y})$ onto A.

In [2], it is shown that if M is a hyper-real maximal ideal of $C(\mathcal{Y})$, for some topological space $\mathcal{Y}$, then $C(\mathcal{Y})/M$ is an η_1-set. Hence, by Theorems 1.4 and 2.6, we have immediately

2.7. COROLLARY. *If A is a Φ-algebra closed under countable composition, and $M \in \mathcal{M}(A)$ is hyper-real, then A/M is real-closed field that is an η_1-set.*

2.8. COROLLARY. *If $A = D(\mathcal{M}(A))$ is a Φ-algebra, and $M \in \mathcal{M}(A)$ is hyper-real, then A/M is a real-closed field that is an η_1-set.*

Proof. By 2.1 and 2.7, it suffices to show that the Φ-algebra $A = D(\mathcal{M}(A))$ is closed under countable l-inversion and uniform convergence. The latter follows immediately from Lemma 1.2. Let $\{f_n\}$ be a sequence of elements of A such that $Z(g) \subset \bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$. Then $\mathcal{Z}(g)$ is nowhere dense and g cannot be a divisor of zero. Thus, by [5], Theorem 3.9, $1/g \in A$.

In [2], it is shown that all real-closed η_α-fields of power $\aleph_\alpha$ are isomorphic, if $\alpha > 0$. It follows from Corollary 2.8 that, if $\aleph_1 = c$, then all of the residue class fields of the Φ-algebra of all Lebesgue measurable functions on R, modulo the ideal of functions vanishing off sets of measure zero, are isomorphic. See [5], Corollary 3.10.

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