The derivates of functions of intervals.

By


1. The writer has recently communicated to the London Mathematical Society some properties of functions of intervals. A function of intervals is defined by a set of rules associating a definite number with each interval $I$ of a certain aggregate of intervals and is denoted by the symbol $g(I)$. Previous writers who have dealt with such functions have assumed them to be additive; we do not make this restriction. In this paper we consider intervals in one dimension only.

Every function of intervals gives rise to two functions of points the upper and lower derivates. We recall their definitions.

Given an interval $I$ and a point $x$, we define $\varrho$, the parameter of regularity of $I$ with respect to $x$, as $mI/mS$, where $S$ is the smallest interval with centre $x$ which contains $I$ (and $mI$ is the length of $I$).

We then define $u(\varrho, x)$ as the upper limit of $g(I)/mI$ as $mI \to 0$ where the parameter of regularity of $I$ with respect to $x$ is restricted to be greater than $\varrho$.

For fixed $x$, $u(\varrho, x)$ increases as $\varrho$ decreases, and we define $u(x)$, the upper derivate of $g(I)$ at $x$, as $\lim_{\varrho \to 0} u(\varrho, x)$.

We have corresponding definitions for $l(\varrho, x)$ and $l(x)$.

If $u(x) = l(x)$, we call their common value $g'(x)$, the derivative of $g(I)$ at $x$.

It may be proved that $u(\varrho, x)$, $l(\varrho, x)$, $u(x)$, $l(x)$ are measurable functions.

We also introduced the concept of the integral of a function of intervals.
Given an interval $R$, divide it into meshes $I_1, \ldots, I_n$. We define the upper and lower integrals

$$\int g(I) = \lim_{n \to \infty} \sum_{i=1}^{n} g(I_i), \quad \int g(I) = \lim_{n \to \infty} \sum_{i=1}^{n} g(I_i)$$

as the length of the greatest mesh tends to zero. If $\int g(I) = \int g(I)$, we write their common value $\int g(I)$.

The object of this paper is to investigate the possible values of the derivates of a function $g(I)$, assuming only that $\int g(I)$ exists. The arguments may be compared with those which Denjoy and M"{o} Young used in the corresponding problem for the derivates of a function $f(x)$.

To prove any theorem of this nature, we need a lemma on the covering of a set of points by intervals. An important covering lemma is that due to Vitali which states conditions under which a set of points may be approximately covered by a finite number of associated intervals. Thus Vitali's lemma is effective only when sets of intervals of arbitrarily small total measure can be neglected, that is to say, when $g(I)$ is absolutely continuous. To deal with the general $g(I)$ we need an extension of Vitali's lemma which provides for the exact covering of an interval $R$ by a system of intervals associated with points of $R$; this extension will be found in lemma 7.

2. We shall use the following lemmas.

Lemma 1. $E$ is a set of positive measure. Then, given $\delta > 1$, we can find a subset $F$ of $E$ and an interval $I$ containing $F$ such that

$$m I < \delta m F.$$ 

We can enclose $E$ in a denumerable set of non-overlapping intervals $I_1, \ldots, I_n, \ldots$, in such a manner that

$$\sum_{i=1}^{\infty} m I_i < \delta m E.$$ 


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Let \( F_i = EI_i \). Then \( E = \sum_{i=1}^{\infty} F_i \) and so \( mE = \sum_{i=1}^{\infty} mF_i \). Therefore

\[
\sum_{i=1}^{\infty} mI_i < \varepsilon \sum_{i=1}^{\infty} mF_i.
\]

Hence there is a value of \( i \) for which \( mI_i < \varepsilon mF_i \).

**Lemma 2.** (Lebesgue). The density of a set \( E \) is 1 almost everywhere in \( E \).


**Lemma 3** (Lusin). If \( E \) has positive measure, it contains a perfect subset which is throughout of positive measure (that is such that the part of it in every interval containing one of its points is of positive measure).

A proof is given by Mrs Young, loc. cit., page 365.

**Lemma 4** (Vitali). \( E \) is a measurable set of points contained in an interval \( R \). With each point \( P \) of \( E \) is associated a set of intervals whose lengths \( \to 0 \), having parameter of regularity with respect to \( P \) greater than \( q(P) > 0 \). Then, given \( \varepsilon \), we can choose a finite set \( \mathcal{S} \) of the associated intervals, non-overlapping and contained in \( R \), such that

\[
m(\mathcal{S} - \mathcal{S}E) < \varepsilon \text{ and } m(E - E\mathcal{S}) < \varepsilon.
\]


**Lemma 5.** In lemma 4, we can choose \( \mathcal{S} \) so that each interval complementary to the intervals of \( \mathcal{S} \) contains at least one point of \( E \).

Let \( F \) be the subset of \( E \) at which the density is 1. By lemma 2, \( mF = mE \).

Given \( \varepsilon(\ll mE) \), choose \( \eta < \varepsilon/8mE \).

Given any point \( P \) of \( F \), we can find an interval \( I \) with centre \( P \) such that if \( I_1 \) is any smaller concentric interval and \( F_1 = FI_1 \), then

\[
mF_1 > (1 - q\eta)mI_1.
\]

Let every interval associated with \( P \) be extended at each end by a fraction \( \eta \) of its length. Then if the extended interval is contained in \( I \), each of the two extensions must contain a point of \( F \) in its interior.

By lemma 4, we can find a set \( \mathcal{S} \) of a finite number of the extended intervals, non-overlapping and contained in \( R \), such that

\[
m(\mathcal{S} - \mathcal{S}E) < \varepsilon \text{ and } m(E - E\mathcal{S}) < \frac{1}{3}\varepsilon.
\]
Cutting off the extensions we have a set $\mathcal{S}$ of the original intervals such that
\[ m(\mathcal{F} - \mathcal{S}) = 2\eta m \mathcal{S} < 4\eta m \mathcal{L} < \frac{1}{2} \varepsilon \]
and therefore
\[ m(\mathcal{S} - \mathcal{S} \mathcal{E}) < \varepsilon \quad \text{and} \quad m(\mathcal{E} - \mathcal{E} \mathcal{S}) < \varepsilon. \]

Moreover each complementary interval contains a point of $\mathcal{E}$.

Lemma 6. $\mathcal{E}$ is a closed set. For each point $P$ of $\mathcal{E}$, let all sufficiently small intervals containing $P$ as an interior point possess a certain property $A$. Then we can find an interval $R$ (containing points of $\mathcal{E}$) such that any subinterval of $R$ which contains a point of $\mathcal{E}$ in its interior has the property $A$.

Suppose the result false. Then given any interval $R_1$, containing points of $\mathcal{E}$, we can find a subinterval $I_1$, containing a point $P_1$ of $\mathcal{E}$, which has not the property $A$.

Take an interval $R_2$ containing $P_1$ and contained in $I_1$ such that $mR_2 < \frac{1}{2} mR_1$.

Then we can find a subinterval $I_2$ of $R_2$, containing a point $P_2$ of $\mathcal{E}$, which has not the property $A$.

Repeat this argument; the points $P_1, P_2, \ldots$ have a limit point $P$ interior to every $I_n$.

Since $\mathcal{E}$ is closed, $P$ is a point of $\mathcal{E}$.

Therefore all sufficiently small intervals enclosing $P$ have the property $A$, and this is a contradiction.

Lemma 7. $\mathcal{E}$ is a closed set.

If $P$ is any point of $\mathcal{E}$, all sufficiently small intervals containing $P$ as an interior point have a property $A$.

Also, with each point $P$ of $\mathcal{E}$ is associated a set of intervals, having a property $B$, whose lengths $\to 0$ and which have parameter of regularity with respect to $P$ greater than $q(P) > 0$.

Then we can find an interval $R$, containing a part $\mathcal{E}_1$ of $\mathcal{E}$, such that, given $\varepsilon$, $R$ can be exactly covered by a set $\mathcal{S}_1$ of a finite number of the intervals $A$ together with a set $\mathcal{S}_2$ of a finite number of the intervals $B$, in such a way that
\[ m(\mathcal{S}_1 - \mathcal{S}_2 \mathcal{E}_1) < \varepsilon \quad \text{and} \quad m(\mathcal{E}_1 - \mathcal{E}_2 \mathcal{S}_2) < \varepsilon. \]

Choose $R$ as in lemma 6.

By lemma 5, we can find a set of intervals $\mathcal{S}_n$ such that
\[ m \mathcal{S}_n - \mathcal{S}_n \mathcal{E}_1 < \varepsilon \quad \text{and} \quad m(\mathcal{E}_1 - \mathcal{E}_1 \mathcal{S}_n) < \varepsilon, \]
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and such that each complementary interval contains a point of \( E \) in its interior.

Hence each complementary interval has the property \( A \), and these intervals form the set \( E_1 \).

3. We now prove the main theorems.

**Theorem 1.** If \( \int g(I) \) is finite, the points of \( R \) at which \( u(x) = +\infty \) and \( l(\frac{1}{2}, x) > -\infty \) form a set of measure zero.

Let \( E \) be the set at which \( u(x) = +\infty \) and \( l(\frac{1}{2}, x) > -\infty \). Let \( E_r \) be the set at which \( u(x) = +\infty \) and \( l(\frac{1}{2}, x) > -r \). Then \( E_{r-1} \subset E_r \) and \( E = \lim_{r \to \infty} E_r \).

Hence it is sufficient to prove that, for each \( r \), \( mE_r = 0 \). Writing \( g(I) = rmI \) in place of \( g(I) \), we have only to prove that the set \( E_0 \) at which \( u(x) = +\infty \) and \( l(\frac{1}{2}, x) > 0 \) has measure zero.

Suppose that \( mE_0 > 0 \).

Choose a perfect subset \( F \) of \( E_0 \) which is throughout of positive measure (lemma 3).

We say that \( I \) has the property \( A \) if \( g(I) \geq 0 \), and the property \( B \) if \( g(I) \geq kmI \) (where \( k \) is chosen later). By lemma 7, we can find an interval \( R \) (independent of \( k \)) containing a part \( F_1 \) of \( F \) which can be exactly covered by arbitrarily small intervals \( I_1, \ldots \) such that

\[
\sum_{i=1}^{n} g(I_i) \geq \frac{1}{2} kmF_1.
\]

Let \( \int g(I) = l \).

Choose \( k > 4|l|/mF_1 \), and take the covering intervals so small that

\[
\sum_{i=1}^{n} g(I_i) < 2|l|.
\]

This is a contradiction, and so the theorem is true.

**Corollary.** If \( \int g(I) \) and \( \int \overline{g}(I) \) are finite, then, except for a set of measure zero, the points at which \( u(x) = +\infty \) are the same as those at which \( l(x) = -\infty \).

**Theorem 2.** If \( \int g(I) \) exists, the points of \( R \) at which \( u(x) \) and \( l(x) \) are finite and unequal form a set of measure zero.
Let $E$ be the set of points at which $u(x)$ and $l(x)$ are finite and unequal.

Let $E_r$ ($r$ an integer) be the set at which

$$-r < l(x) < u(x) < r$$

and

$$u(x) - l(x) > \frac{1}{r}.$$ 

Then $E_{r-1} \subset E_r$ and $E = \lim_{r \to \infty} E_r$.

Hence it is sufficient to prove that, for each $r$, $mE_r = 0$. Suppose this untrue. Take the least $r$ for which $mE_r > 0$. $E$ is the sum of sets $S_r$ in which

$$\frac{y}{2r} - \frac{1}{2r} \leq l \leq \frac{y}{2r},$$

where $y$ takes integral values between $-2r^2 + 1$ and $2r^2 - 2$. Then there are one or more values of $y$ for which $mS_r > 0$; take the least such value.

Choose a perfect subset $T_r$ of $S_r$ which is throughout of positive measure (lemma 3).

We say that $I$ has the property $A$ if

$$-rmI < g(I) < rmI.$$ 

As in lemma 6, choose an interval $R_1$ containing a part $T$ of $T_r$. Take $\varepsilon$ satisfying

$$0 < (2y+1+8r^2)\varepsilon < \frac{1}{2}.$$ 

By lemma 1, choose a subset $F$ of $T$ and an interval $R_2$ containing $F$ and contained in $R_1$ such that

$$mR_2 \leq (1+\varepsilon)mF.$$ 

Since $\int_{R_2} g(I)$ exists, we can find $\delta$ such that if $\Sigma g(I_0)$ and $\Sigma g(I_0)$ are the sums corresponding to any two subdivisions of $R_2$ into finite sets of meshes of length less than $\delta$, then

$$(1) \quad |\Sigma g(I_0) - \Sigma g(I_0)| < \frac{mF'}{4r}.$$ 

Now define the property $B$ to be

$$\frac{g(I)}{mI} < \frac{y}{2r}.$$
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By lemma 7, we can exactly cover $R_x$ with intervals $I$, of types $A, B$, having lengths $< \delta$, in such a way that

$$m \delta_A < (1 + \varepsilon)mF \quad \text{and} \quad m \delta_B < 2 \varepsilon mF.$$ 

Hence

$$\Sigma g(I) < \frac{y}{2} (1 + \varepsilon)mF + 2 \varepsilon mF.$$ 

Again, taking the property $B$ to be

$$\frac{g(I)}{m} > \frac{y + 1}{2r}$$

and using lemma 7, we can exactly cover $R_x$ with intervals $I_i$, having lengths less than $\delta$, in such a way that

$$\Sigma g(I_i) > \frac{y + 1}{2r} (1 - \varepsilon)mF - 2 \varepsilon mF.$$ 

Then

$$\Sigma g(I_i) - \Sigma g(I) > \left(\frac{1}{2r} - \frac{2y + 1}{2r} \varepsilon - 4 \varepsilon \right) mF$$

$$> \frac{mF}{4r}, \text{ by choice of } \varepsilon.$$ 

This contradicts (1) and so the theorem is true. We deduce from Theorems 1 and 2 the result:

*If $\int g(l)$ exists, then except at a set of measure zero either

(1) \quad u(x) = + \infty, \quad l(x) = - \infty

or

(2) \quad \text{a point } g'(x) \text{ exists.}*

Any function $f(x)$ of points $x$ in $R$ generates a function of intervals $g(I)$ in $R$ defined by

$$g(I_i) = f(x_i) - f(x_{i-1})$$

where $x_{i-1}, x_i$ are the end points of an interval $l_i$ contained in $R$. $u(x), l(x)$ are then extensions of the ordinary upper and lower derivatives of $f(x)$, and we have the result:

*Except at a set of measure zero either

(1) \quad u(x) = + \infty, \quad l(x) = - \infty

or

(2) \quad a \text{ finite } f'(x) \text{ exists.}