The derivates of functions of intervals.

By

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1. The writer has recently communicated to the London Mathematical Society some properties of *functions of intervals*. A function of intervals is defined by a set of rules associating a definite number with each interval I of a certain aggregate of intervals and is denoted by the symbol $g(I)$. Previous writers who have dealt with such functions have assumed them to be *additive*; we do not make this restriction. In this paper we consider intervals in one dimension only.

Every function of intervals gives rise to two *functions of points* the upper and lower derivates. We recall their definitions.

Given an interval I and a point x, we define ϱ, the *parameter ot regularity* of I with respect to x, as mI/mS, where S is the smallest interval with centre x which contains I (and mI is the length of I).

We then define $u(\varrho, x)$ as the upper limit of $g(I)/mI$ as $mI \to 0$ where the parameter of regularity of I with respect to x is restricted to be greater than ϱ.

For fixed x, $u(\varrho, x)$ increases as ϱ decreases, and we define $u(x)$, the upper derivate of $g(I)$ at x, as $\lim\limits_{\varrho \to 0} u(\varrho, x)$.

We have corresponding definitions for $l(\varrho, x)$ and $l(x)$.

If $u(x) = l(x)$, we call their common value $g'(x)$, the derivative of $g(I)$ at x.

It may be proved that $u(\varrho, x)$, $l(\varrho, x)$, $u(x)$, $l(x)$ are measurable functions

We also introduced the concept of the *integral* of a function of intervals.

Given an interval R, divide it into meshes $I_1, \dots I_n$. We define the upper and lower integrals

$$\overline{\int_R} g(I) = \overline{\lim} \sum_{i=1}^{n} g(I_i), \quad \underline{\int_R} g(I) = \underline{\lim} \sum_{i=1}^{n} g(I_i)$$

as the length of the greatest mesh tends to zero. If $\overline{\int_R} g(I) = \underline{\int_R} g(I)$, we write their common value $\int_R g(I)$.

The object of this paper is to investigate the possible values of the derivates of a function $g(I)$, assuming only that $\int_R g(I)$ exists. The arguments may be compared with those which Denjoy and M^rs^ Young[1]) use in the corresponding problem for the derivates of a function $f(x)$.

To prove any theorem of this nature, we need a lemma on the *covering* of a set of points by intervals. An important covering lemma is that due to Vitali[2]) which states conditions under which a set of points may *approximately* covered by a finite number of associated intervals. Thus Vitali's lemma is effective only when sets of intervals of arbitrarily small total measure can be neglected, that is to say, when $g(I)$ is absolutely continuous. To deal with the general $g(I)$ we need an extension of Vitali's lemma which provides for the *exact* covering of an interval R by a system of intervals associated with points of R; this extension will be found in lemma 7.

2. We shall use the following lemmas.

Lemma 1. *E is a set of positive measure. Then, given $\vartheta > 1$, we can find a subset F of E and an interval I containing F such that*

$$mI < \vartheta mF.$$

We can enclose E in a denumerable set of non-overlapping intervals $I_1, \dots I_n, \dots$, in such a manner that

$$\sum_{i=1}^{\infty} mI_i < \vartheta mE.$$

1) G. C. Young, Proceedings of the London Mathematical Society, Vol. 15 (1916), page 360, and references there given.

2) Carathéodory, Vorlesungen über reelle Funktionen, page 299.

Let $F_i = EI_i$. Then $E = \sum_{i=1}^{\infty} F_i$ and so $mE = \sum_{i=1}^{\infty} mF_i$. Therefore

$$\sum_{i=1}^{\infty} mI_i < \vartheta \sum_{i=1}^{\infty} mF_i.$$

Hence there is a value of i for which $mI_i < \vartheta mF_i$.

Lemma 2. (Lebesgue). *The density of a set E is 1 almost everywhere in E.*

An elegant proof is given by Sierpiński, Fundamenta. Vol. IV, page 167.

Lemma 3 (Lusin). *If E has positive measure, it contains a perfect subset which is throughout of positive measure (that is such that the part of it in every interval containing one of its points is of positive measure).*

A proof is given by M[rs] Young, loc. cit., page 365.

Lemma 4 (Vitali). *E is a measurable set of points contained in an interval R. With each point P of E is associated a set of intervals whose lengths $\to 0$, having parameter of regularity with respect to P greater than $\varrho(P) > 0$. Then, given ε, we can choose a finite set $\mathcal{S}$ of the associated intervals, non-overlapping and contained in R, such that*

$$m(\mathcal{S} - \mathcal{S}E) < \varepsilon \quad \textit{and} \quad m(E - E\mathcal{S}) < \varepsilon.$$

See Carathéodory, loc. cit. Banach, *Fund. Math.*, Vol. V, p. 130.

Lemma 5. *In lemma 4, we can choose $\mathcal{S}$ so that each interval complementary to the intervals of $\mathcal{S}$ contains at least one point of E.*

Let F be the subset of E at which the density is 1. By lemma 2, $mF = mE$.

Given $\varepsilon (< mE)$, choose $\eta < \varepsilon/8\,mE$.

Given any point P of F, we can find an interval I with centre P such that if I_1 is any smaller concentric interval and $F_1 = FI_1$, then

$$mF_1 > (1 - \varrho\eta) mI_1.$$

Let every interval associated with P be extended at each end by a fraction η of its length. Then if the extended interval is contained in I, each of the two extensions must contain a point of F in its interior.

By lemma 4, we can find a set $\mathcal{F}$ of a finite number of the *extended* intervals, non-overlapping and contained in R, such that

$$m(\mathcal{F} - \mathcal{F}E) < \varepsilon \quad \text{and} \quad m(E - E\mathcal{F}) < \tfrac{1}{2}\varepsilon.$$

Culting off the extensions we have a set $\mathcal{E}$ of the original intervals such that

$$m(\mathcal{F}-\mathcal{E}) = 2\eta m\mathcal{E} < 4\eta mE < \tfrac{1}{2}\varepsilon$$

and therefore

$$m(\mathcal{E}-\mathcal{E}E) < \varepsilon \quad \text{and} \quad m(E-E\mathcal{E}) < \varepsilon.$$

Moreover each complementary interval contains a point of E.

Lemma 6. *E is a closed set. For each point P of E, let all sufficiently small intervals containing P as an interior point possess a certain property A. Then we can find an interval R (containing points of E) such that any subinterval of R which contains a point of E in its interior has the property A.*

Suppose the result false. Then given any interval R_1, containing points of E, we can find a subinterval I_1, containing a point P_1 of E, which has not the property A.

Take an interval R_2 containing P_1 and contained in I_1 such that $mR_2 < \frac{1}{2} mR_1$.

Then we can find a subinterval I_2 of R_2, containing a point P_2 of F, which has not the property A.

Repeat this argument; the points $P_1, P_2 \ldots$ have a limit point P interior to every I_n.

Since E is closed, P is a point of E.

Therefore all sufficiently small intervals enclosing P have the property A, and this is a contradiction.

Lemma 7. *E is a closed set.*

If P is any point of E, all sufficiently small intervals containing P as an interior point have a property A.

Also, with each point P of E is associated a set of intervals, having a property B, whose lengths $\to 0$ and which have parameter of regularity with respect to P greater tham $\varrho(P) > 0$.

Then we can find an interval R, containing a part E_1 of E such that, given ε, R can be exactly covered by a set $\mathcal{E}_A$ of a finite number of the intervals A together with a set $\mathcal{E}_B$ of a finite number of the intervals B, in such a way that

$$m(\mathcal{E}_B-\mathcal{E}_B E_1) < \varepsilon \quad \textit{and} \quad m(E_1 - E_1\mathcal{E}_B) < \varepsilon.$$

Choose R as in lemma 6.

By lemma 5, we can find a set of intervals $\mathcal{E}_B$ such that

$$m\,\mathcal{E}_B-\mathcal{E}_B E_1) < \varepsilon \quad \text{and} \quad m(E_1 - E_1\mathcal{E}_B) < \varepsilon,$$

and such that each complementary interval contains a point of E in its interior.

Hence each complementary interval has the property A, and these intervals form the set $\mathcal{E}_A$.

3. We now prove the main theorems.

Theorem 1. *If $\overline{\int_R} g(I)$ is finite, the points of R at which $u(x)=+\infty$ and $l(\frac{1}{2}, x) > -\infty$ form a set of measure zero.*

Let E be the set at which $u(x)=+\infty$ and $l(\frac{1}{2}, x) > -\infty$. Let E_r be the set at which $u(x)=+\infty$ and $l(\frac{1}{2}, x) > -r$. Then $E_{r-1} \subset E_r$, and $E = \lim_{r\to\infty} E_r$.

Hence it is sufficient to prove that, for each r, $mE_r=0$. Writing $g(I)-rmI$ in place of $g(I)$, we have only to prove that the set E_0 at which $u(x)=+\infty$ and $l(\frac{1}{2}, x) > 0$ has measure zero.

Suppose that $mE_0 > 0$.

Choose a perfect subset F of E_0 which is troughout of positive measure (lemma 3).

We say that I has the property A if $g(I)>0$, and the property B if $g(I) > kmI$ (where k is chosen later). By lemma 7, we can find an interval R (independent of k) containing a part F_1 of F which can be exactly covered by arbitrarily small intervals $I_1, ..$ such that

$$\sum_{i=1}^{n} g(I_i) > \tfrac{1}{2} k m F_1 .$$

Let $\overline{\int_R} g(I) = l$.

Choose $k > 4|l|/mF_1$, and take the covering intervals so small hat

$$\sum_{i=1}^{n} g(I_i) < 2|l|.$$

This is a contradiction, and so the theorem is true.

Corollary. *If $\overline{\int_R} g(I)$, $\underline{\int_R} g(I)$ are finite, then, except for a set of measure zero, the points at which $u(x)=+\infty$ are the same as those at which $l(x)=-\infty$.*

Theorem 2. *If $\int_R g(I)$ exists, the points of R at which $u(x)$ and $l(x)$ are finite and unequal form a set of measure zero.*

Let E be the set of points at which $u(x)$ and $l(x)$ are finite and unequal.

Let E_r (r an integer) be the set at which

$$-r < l(x) < u(x) < r$$

and

$$u(x) - l(x) > \frac{1}{r}.$$

Then $E_{r-1} \subset E_r$ and $E = \lim_{r\to\infty} E_r$.

Hence it is sufficient to prove that, for each r, $mE_r = 0$. Suppose this untrue. Take the least r for which $mE_r > 0$. E is the sum of sets S_y in which

$$\frac{y-1}{2r} \leqslant l < \frac{y}{2r}$$

where y takes integral values between $-2r^2+1$ and $2r^2-2$. Then there are one or more values of y for which $mS_y > 0$; take the least such value.

Choose a perfect subset T_y of S_y which is throughout of positive measure (lemma 3).

We say that I has the property A if

$$-rmI < g(I) < rmI.$$

As in lemma 6, choose an interval R_1 containing a part T of T_y. Take ε satisfying

$$0 < (2y+1+8r^2)\varepsilon < \tfrac{1}{2}.$$

By lemma 1, choose a subset F of T and an interval R_2 containing F and contained in R_1 such that

$$mR_2 \leqq (1+\varepsilon)mF.$$

Since $\int R_2\, g(I)$ exists, we can find δ such that if $\Sigma g(I_i)$ and $\Sigma g(I_j)$ are the sums corresponding to any two subdivisions of R_2 into finite sets of meshes of length less than δ, then

$$(1) \qquad |\Sigma g(I_i) - \Sigma g(I_i)| < \frac{mF}{4r}.$$

Now define the property B to be

$$\frac{g(I)}{mI} < \frac{y}{2r}.$$

By lemma 7, we can exactly cover R_2 with intervals I_i of types A, B, having lengths $<\delta$, in such a way that

$$m\mathcal{E}_A < (1+\varepsilon)mF \quad \text{and} \quad m\mathcal{E}_B < 2\varepsilon mF.$$

Hence

$$\Sigma g(I_i) < \frac{y}{2r}(1+\varepsilon)mF + 2r\varepsilon mF.$$

Again, taking the property B to be

$$\frac{g(I)}{mI} > \frac{y+1}{2r}$$

and using lemma 7, we can exactly cover R_2 with intervals I_k, having lengths less than δ, in such a way that

$$\Sigma g(I_k) > \frac{y+1}{2r}(1-\varepsilon)mF - 2r\varepsilon mF.$$

Then

$$\Sigma g(I_k) - \Sigma g(I_i) > \left(\frac{1}{2r} - \frac{2y+1}{2r}\varepsilon - 4r\varepsilon\right)mF$$

$$> \frac{mF}{4r}, \text{ by choice of } \varepsilon.$$

This contradicts (1) and so the theorem is true. We deduce from Theorems 1 and 2 the result:

If $\int_R g(I)$ *exists, then except at a set of measure zero either*

(1) $u(x) = +\infty, \; l(x) = -\infty$

or (2) *a point* $g'(x)$ *exists.*

Any function $f(x)$ of points x in R generates a function of intervals $g(I)$ in R defined by

$$g(I_i) = f(x_i) - f(x_{i-1})$$

where x_{i-1}, x_i are the end points of an interval I_i contained in R.

$u(x)$, $l(x)$ are then extensions of the ordinary upper and lower derivates of $f(x)$, and we have the result:

Except at a set of measure zero either

(1) $u(x) = +\infty, \; l(x) = -\infty$

or (2) *a finite* $f'(x)$ *exists.*