Closed connected sets which remain connected upon the removal of certain connected subsets.

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In § 1 of the present paper, it is proved that if M is a closed connected set which contains more than one point and which remains connected upon the removal of any connected subset, then M is a simple closed curve. Thus there does not exist an unbounded closed connected set M such that if g is any connected subset of M, then $M-g$ is connected. However we may ask:

(1) What types of unbounded closed connected sets (if any) are such that if g is any unbounded connected proper subset of M, then $M-g$ is connected?

(2) Are there any unbounded closed connected sets M such that if g is any bounded connected subset of M, then $M-g$ is connected?

In § 2 it is proved that any closed unbounded connected set satisfying the conditions set down in (1) belongs to one of three types.

In § 3 an example is given of an unbounded closed connected set satisfying the conditions of (2). This set is not a continuous curve. Moreover we further prove that no unbounded closed connected set satisfying the conditions of (2) can possibly be a continuous curve.

§ 1.

Theorem *A*. *Suppose M is a closed connected set containing more than one point such that if g is any connected subset of M, then $M-g$ is connected. Under these conditions M is a simple closed curve.*

Proof. The set M contains no continu of condensation [1]). For suppose it contained a continu of condensation g. Then $M-g$ is connected. Let A and B be any two distinct points of g. The set $M-g+(g-A-B)$ is connected [2]). Hence $M-[M-g+(g-A-B)]= =A+B$ must be connected, which is impossible. Hence M contains no continu of condensation and is thus a continuous curve [3]). Let A and B be any two distinct points of M. Then there is a simple continuous arc AXB from A to B every point of which belongs to M [4]). As the set M contains no continu of condensation, it is clear that some point P of $AXB-A-B$ is not a limit point of $M-AXB$. Put about P as centre a circle K containing no point of $M-AXB$. Let H be the first point of the subarc PA of the arc AXB going from P to A which is on K while T is the first point of the subarc PB of arc AXB going from P to B which is on K. Now $M-(HPT-H-T)$ is connected. It is also closed and contains no continu of condensation. Thus $M-(HPT-H-T)$ is a continuous curve and contains an arc HQT from H to T. The point set $HPT+HQT$ is a simple closed curve J belonging entirely to M.

Let us suppose that there are points of M which are not on J. As M contains no continu of condensation some point H of J is not a limit point of $M-J$. Now $J-H$ is connected. Hence $M- -(J-H)=H+(M-J)$ is connected. But H is not a limit point

[1]) A set g is said to be a *continu of condensation* of a set M if g is a closed connected subset of M containing more than one point such that every point of g is a limit point of $M-g$.

[2]) F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914 p. 246, IV.

[3]) The term *continuous curve* is here used in the sense suggested by Professor R. L. Moore, who applies the term to sets which are closed, connected and connected im kleinen. Cf. R. L. Moore, Transactions of the American Mathematical Society, vol. XXI (1920) p. 347. A set of points is said to be *connected im kleinen* if for every point P of M and every circle K with centre at P, there exists a circle K_1 within K and with centre at P such that if X is a point of M within K_1 then X and P lie in some connected subset of M that lies within K. Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmenge die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914) p. 318—22.

[4]) Cf. R. L. Moore, *A Theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol 23 (1917). While Professor Moore's theorem is proved only for the case where the continuous curve is bounded, it is clear that his methods suffice also for the case where it is unbounded.

of $M-J$ while no point of $M-J$ is a limit point of a single point H. Thus we are led to a contradiction if we suppose $M-J$ is not vacuous. Thus $M\equiv J$.

§ 2.

Theorem *B*. *If M is an unbounded closed connected set which remains connected upon the removal of any unbounded connected proper subset, then M is either an open curve*[1]*, a ray of an open curve or a simple closed curve J plus OP, a ray of an open curve which has O and only O in common with J.*

Proof. The set M contains no continu of condensation. For suppose M contains a continu of condensation g. Then as M is unbounded, $M-g$ is unbounded. $M-g$ cannot be connected. For suppose it were connected. Let A and B be any two points of g. Then $M-g+(g-A-B)$ is connected[2]. Hence, by hypothesis, $M-\{M-g+(g-A-B)\}=A+B$ must be connected. But this is impossible.

Hence $M-g=\overline{S}_1+\overline{S}_2$, two mutually separated sets[3]. At least one of the sets, $\overline{S}_1$ and $\overline{S}_2$, is unbounded. Let us call the unbounded set S_1 (in case but one of the sets is unbounded) while S_2 will denote the other set. We shall now shaw that S_1 is a connected set. For suppose it were not. Then $S_1=\overline{T}_1+\overline{T}_2$; two mutually separated sets. One of these sets must be unbounded. Let us call the unbounded set (in case only one of the sets is unbounded) T_1, while T_2 denotes the other set. Now S_1+g is connected as M is connected. Now as $(S_1+g)-g=T_1+T_2$, then T_1+g is connected[4]. Then as T_1+g is connected and unbounded, $M-(T_1+g)$ must be connected.

[1]) An *open curve* is defined by R. L. Moore as a closed connected set of points M such that if P is any point of M, then $M-P=M_1+M_2$, two mutually exclusive connected sets neither of which contains a limit point of the other one. Cf. R. L. Moore, *On the foundations of Plane Analysis Situs*, Transactions of the American Mathematical Society, vol. 17 (1916) p. 159. If P is a point of an open curve M, the point set obtained by adding P to either of the two sets into which M is separated by the omission of P is called a *ray* of an open curve.

[2]) Cf. Hausdorff, loc. cit.

[3]) Two point sets are said to be *mutually separated* if neither contains a point or limit point of the other one.

[4]) Cf. B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fund. Math., t. II (1921) p. 210 Theorem VI.

But $M-(T_1+g)=T_2+S_2$, two mutually separated sets. Hence the supposition that set S_1 is not connected has led to a contradiction. Hence S_1+g is connected [1]. Hence, by hypothesis, $M-(S_1+g)=S_2$ is also connected.

Let L_1 denote those points of g which are limit points of S_1 while L_2 denotes those points of g which are limit points of S_2. Hence S_1+L_1 and S_2+L_2 are closed sets. As $g=L_1+L_2$ and $M=(S_1+L_1)+(S_2+L_2)$ is a closed connected set it follows that the sets L_1 and L_2 have a point Q in common. Let T and U be two distinct points of g each different from Q. As they do not belong to S_1 nor to S_2, S_1+L_1-T-U and S_2+L_2-T-U are connected sets [2]. Their sum

$$(S_1+L_1-T-U)+(S_2+L_2-T-U)=S_1+S_2+g-T-U$$

is also connected and unbounded. Hence by hypothesis

$$M-(S_1+S_2+g-T-U)=T+U$$

is a connected set.

But this is impossible. Hence the supposition that g is a continu of condensation of M has led to a contradiction.

As M contains no continu of condensation, it follows that M is an unbounded continuous curve. Several cases may arise.

Case I. *For every point A of M, $M-A=\overline{M}_1+\overline{M}_2$, two mutually separated sets.* Clearly one of these sets is unbounded. Let M_1 denote that one of the sets which is unbounded (in case that but one of them is unbounded) while M_2 denotes the other one. Suppose $M_1=G_1+G_2$, two mutually separated sets of which G_1 at least is unbounded. Now as $M-A=M_1+M_2$ and $M_1=G_1+G_2$, the sets M_1+A and $A+G_1$ are connected [3]. Hence, as G_1+A is unbounded and connected, $M-(G_1+A)=G_2+M_2$ must be connected But G_2 and M_2 are mutually separated sets. Thus we are led to a contradiction if we suppose M_1 not connected. Hence $M-(M_1+A)=M_2$ must be connected. Hence in Case I, M satisfies R. L. Moore's definition of an open curve.

Case II. *There is one and only one point A such that $M-A$*

[1] Cf. F. Hausdorff, loc. cit.

[2] Ibid.

[3] Cf. B. Knaster and C. Kuratowski, loc. cit. p. 210.

is connected. Let us suppose M were not a ray from A. Then there exists an unbounded connected set Z of M containing A and a point B in the set $M-Z$ [1]). But $M-B=S_1+S_2$, two mutually separated sets of which S_1 contains A. Then S_1 must contain Z and be unbounded.

Let us consider S_1+B-A. If S_1+B-A is connected, then as it is also unbounded, $M-(S_1+B-A)=A+S_2$ must be connected which is impossible as A and S_2 are mutually separated sets. Thus $S_1+B-A=H_1+H_2$, two mutually separated sets of which H_1 contains B. Then $M-A=H_1+S_2+H_2$. But the sets (H_1+S_2) and H_2 are mutually separated and thus $M-A$ is not connected. Thus we are led to a contradiction if we suppose that in Case II M is not a ray.

Case III. *There is more than one point whose omission does not disconnect* M. Let A and B be any two distinct points such that $M-A$ is connected and $M-B$ is connected. Then there is a ray Z from A situated on the set M [2]).

But $M\neq Z$, otherwise B would disconnect M. The point A is a limit point of $M-Z$ for otherwise the set M would be disconnected by the removal of the connected unbounded set $Z-A$. If A was the only limit point of $M-Z$ in Z, then $M-A=$ $=(Z-A)+(M-Z)$, two mutually separated sets which is impossible. Then let $P[P\neq A]$ denote a point of Z which is a limit point of $M-Z$. Let V denote $M-Z$ plus its limit points. The set V contains no continu of condensation. Hence V is a continuous curve. Thus there is an arc AP lying entirely in V. We shall now show that $M=Z+AP$. As AP contains no continu of condensation there is a point D of AP $[A\neq D\neq P]$ which is not a limit point of $M-AP$. The set $Z+AP-D$ is connected and unbounded. Hence $M-[Z+AP-D]=D+(M-Z-AP)$ is connected, which cannot be possible unless $M-Z-AP$ is a vacuous point set. Then $M=Z+AP$. As $M-Z$ is connected and Z does not contain any connected part of arc AP containing more than one point, then A and P are the only points of AP on Z. Hence

[1]) In order that a set E should be a ray, it is necessary and sufficient that it be an unbounded closed connected set containing a point P which is not situated on any proper connected unbounded subset of E. Cf. C. Kuratowski, *Quelques propriétés topologiques de la demi-droite.* Fund. Math., t. III. p. 62.

[2]) Cf. C. Kuratowski, ibid, p. 61.

M is composed of a simple closed curve, formed by arc AP of $M-Z+A+P$ and the subarc AP of Z, plus the subray of ray Z which starts from P. The simple closed curve and the ray have P and only P in common.

§ 3.

Let M denote the set composed of the Y-axis, the curves $y=\frac{1}{x}\sin\frac{\pi}{x}$ $[0<x\leqslant 1]$, $y=1+\frac{1}{x}\sin\frac{\pi}{x}$ $[0<x\leqslant 1]$ and the interval $x=1$, $0\leqslant y\leqslant 1$ [1]). The set M is an unbounded closed connected set which remains connected upon the removal of any connected bounded subset. But at every point of the Y-axis the set M fails to be connected im kleinen. We are in a position to prove the following theorem.

Theorem C. *Suppose M is an unbounded closed connected set such that if g is any bounded connected subset of M, then $M-g$ is connected. Then M is not a continuous curve.*

Proof. Let us suppose that M were a continuous curve. Let A and B denote 2 distinct points of M. Then there is an arc AXB from A to B all points of which belong to M. Put about A as centre a circle R_1 such that B is without R_1. Put about B a circle R_2 such that R_1 and its interior are entirely in the exterior of R_2. As M is connected im kleinen there is a circle $\overline{R}_1$ concentric with R_1 such that if P is a point of M within $\overline{R}_1$, then P and A lie together on an arc of M lying entirely in R_1. Likewise there exists a circle $\overline{R}_2$ concentric with R_2 such that if Q is any point of M within $\overline{R}_2$, then Q and B lie together on an arc of M lying entirely within R_2. Let K denote a point of M which is not on AXB and which lies in $\overline{R}_1$. That such a point K exists follows at once from the fact that $M-(AXB-A)$ is connected and hence A is a limit point of $M-AXB$.

Let M_1 denote K together with all points of $M-AXB$ which can be joined to K by an arc of M having no point in common with arc AXB. Let M_2 denote all other points of $M-AXB$. As M_1+M_2 is connected, one of these sets must contain a limit point of the other one. Suppose Q of M_1 is a limit point of M_2. Put

[1]) Cf. Fund. Math. II, p. 254, fig. 1.

about Q a circle S with centre at Q such that all points of arc AXB are without S. As M is connected im kleinen there is a circle $\overline{S}$ such that if W is any point of M within $\overline{S}$, then W and Q are end points of an arc of M lying wholly within S. Let H be a point of M_2 within $\overline{S}$. Then there is an arc HQ of M lying entirely within S and thus having no point on arc AXB. By definition of M_1 there is an arc KQ of M having no point in common with AXB. The point set $KQ+QH$ contains as a subset an arc of M from K to H having no point in common with AXB. Thus H belongs to M_1 contrary to assumption. Thus M_1 contains no limit point of M_2. In like manner M_2 contains no limit point of M_1. Thus the division of $M-AXB$ into M_1 and M_2 is impossible; as M_1 exists, the set M_2 must be vacuous.

Let T denote a point of $M-AXB$ lying in $\overline{R}_2$. That such a point T exists follows at once from the fact that $M-(AXB-B)$ is connected and hence B is a limit point of $M-AXB$. Let TRK denote an arc of M having no point in common with AXB. Let TEB denote an arc of M lying entirely within R_2 while KFA denotes an arc of M lying entirely within R_1. The sum of the arcs, AXB, KFA, TEB and TRK contains as a subset a simple closed curve J belonging entirely to M. As M is unbounded there are points of M without J. As $M-J$ is connected, there can be no point of M within J.

Let A_1 and A_2 denote distinct points of J. Put about A_1 as centre a circle C_1 such that A_2 is without C_1; with A_2 as centre describe a circle C_2 such that C_1 and its interior are without C_2. As M is connected im kleinen there exist circles $\overline{C}_i (i=1, 2)$ such that if $\overline{B}_i$ is any point of M within $\overline{C}_i$, then $\overline{B}_i$ and A_i lie together in an arc of M lying entirely within C_i. Let D_i be a point of $M-J$ within $\overline{C}_i$. That such a point D_i exists follows from the fact that $M-(J-A_i)$ is connected and hence A_i is a limit point of $M-J$. As before we may prove that there is an arc D_1XD_2 lying entirely in $M-J$ and hence entirely without J. Let $D_iZ_iA_i$ denote an arc of M lying entirely within $C_i (i=1, 2)$. The set $A_1Z_1D_1+D_1XD_2+$ $+D_2Z_2A_2$ contains as a subset an arc $\overline{A}_1\overline{X}\overline{A}_2$ such that

1) $\overline{A}_1$ and $\overline{A}_2$ are on J

2) $\overline{A}_1\overline{X}\overline{A}_2-\overline{A}_1-\overline{A}_2$ is without J and belongs to M.

Let E_1 and E_2 denote 2 points of J such that $\overline{A}_1$ and $\overline{A}_2$ separate E_1 and E_2 on J. It follows that either E_1 is within the closed curve J_1 formed by arcs $\overline{A_1 X A_2}$ and $\overline{A_2 E_2 A_1}$ or E_2 is within the closed curve J_2 formed by the arcs $\overline{A_1 X A_2}$ and $\overline{A_1 E A_2}$. In either case there is a closed curve of M such that there is a point of M without this closed curve and a point of M within the closed curve. Then M minus this closed curve is not connected. Thus the supposition that M is connected im kleinen has led to a contradiction.

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