Closed connected sets which remain connected upon the removal of certain connected subsets.

By


In § 1 of the present paper, it is proved that if \( M \) is a closed connected set which contains more than one point and which remains connected upon the removal of any connected subset, then \( M \) is a simple closed curve. Thus there does not exist an unbounded closed connected set \( M \) such that if \( g \) is any connected subset of \( M \), then \( M - g \) is connected. However we may ask:

(1) What types of unbounded closed connected sets (if any) are such that if \( g \) is any unbounded connected proper subset of \( M \), then \( M - g \) is connected?

(2) Are there any unbounded closed connected sets \( M \) such that if \( g \) is any bounded connected subset of \( M \), then \( M - g \) is connected?

In § 2 it is proved that any closed unbounded connected set satisfying the conditions set down in (1) belongs to one of three types.

In § 3 an example is given of an unbounded closed connected set satisfying the conditions of (2). This set is not a continuous curve. Moreover we further prove that no unbounded closed connected set satisfying the conditions of (2) can possibly be a continuous curve.

§ 1.

Theorem A. Suppose \( M \) is a closed connected set containing more than one point such that if \( g \) is any connected subset of \( M \), then \( M - g \) is connected. Under these conditions \( M \) is a simple closed curve.
Proof. The set \( M \) contains no continu of condensation \(^1\)). For suppose it contained a continu of condensation \( g \). Then \( M - g \) is connected. Let \( A \) and \( B \) be any two distinct points of \( g \). The set \( M - g + (g - A - B) \) is connected \(^2\). Hence \( M - [M - g + (g - A - B)] = A + B \) must be connected, which is impossible. Hence \( M \) contains no continu of condensation and is thus a continuous curve \(^3\).

Let \( A \) and \( B \) be any two distinct points of \( M \). Then there is a simple continuous arc \( AXB \) from \( A \) to \( B \) every point of which belongs to \( M \). As the set \( M \) contains no continu of condensation, it is clear that some point \( P \) of \( AXB - A - B \) is not a limit point of \( M - AXB \). Put about \( P \) as centre a circle \( K \) containing no point of \( M - AXB \). Let \( H \) be the first point of the subarc \( PA \) of the arc \( AXB \) going from \( P \) to \( A \) which is on \( K \) while \( T \) is the first point of the subarc \( PB \) of arc \( AXB \) going from \( P \) to \( B \) which is on \( K \). Now \( M - (HPT - H - T) \) is connected. It is also closed and contains no continu of condensation. Thus \( M - (HPT - H - T) \) is a continuous curve and contains an arc \( HQT \) from \( H \) to \( T \).

The point set \( HPT + HQT \) is a simple closed curve \( J \) belonging entirely to \( M \).

Let us suppose that there are points of \( M \) which are not on \( J \). As \( M \) contains no continu of condensation some point \( H \) of \( J \) is not a limit point of \( M - J \). Now \( J - H \) is connected. Hence \( M - (J - H) = H + (M - J) \) is connected. But \( H \) is not a limit point

\(^1\) A set \( g \) is said to be a continu of condensation of a set \( M \) if \( g \) is a closed connected subset of \( M \) containing more than one point such that every point of \( g \) is a limit point of \( M - g \).

\(^2\) F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914 p. 246, IV.

\(^3\) The term continuous curve is here used in the sense suggested by Professor R. L. Moore, who applies the term to sets which are closed, connected and connected im kleinen. Cf. R. L. Moore, Transactions of the American Mathematical Society, vol. XXI (1920) p. 347. A set of points is said to be connected im kleinen if for every point \( P \) of \( M \) and every circle \( K \) with centre at \( P \), there exists a circle \( K \), within \( K \) and with centre at \( P \) such that if \( X \) is a point of \( M \) within \( K \), then \( X \) and \( P \) lie in some connected subset of \( M \) that lies within \( K \). Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmengen die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 28 (1914) p. 318–22.

\(^4\) Cf. R. L. Moore, *A Theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917). While Professor Moore's theorem is proved only for the case where the continuous curve is bounded, it is clear that his methods suffice also for the case where it is unbounded.
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of \( M - J \) while no point of \( M - J \) is a limit point of a single point \( H \). Thus we are led to a contradiction if we suppose \( M - J \) is not vacuous. Thus \( M = J \).

\[\text{\S 2.}\]

**Theorem B.** If \( M \) is an unbounded closed connected set which remains connected upon the removal of any unbounded connected proper subset, then \( M \) is either an open curve \(^1\), a ray of an open curve or a simple closed curve \( J \) plus \( OP \), a ray of an open curve which has \( O \) and only \( O \) in common with \( J \).

**Proof.** The set \( M \) contains no continu of condensation. For suppose \( M \) contains a continu of condensation \( g \). Then as \( M \) is unbounded, \( M - g \) is unbounded. \( M - g \) cannot be connected. For suppose it were connected. Let \( A \) and \( B \) be any two points of \( g \). Then \( M - g + (g - A - B) \) is connected \(^2\). Hence, by hypothesis, \( M - \{M - g + (g - A - B)\} = A + B \) must be connected. But this is impossible.

Hence \( M - g = \overline{S}_1 + \overline{S}_2 \), two mutually separated sets \(^3\). At least one of the sets, \( \overline{S}_1 \) and \( \overline{S}_2 \), is unbounded. Let us call the unbounded set \( S \) (in case but one of the sets is unbounded) while \( S_2 \) will denote the other set. We shall now show that \( S \) is a connected set. For suppose it were not. Then \( S = T_1 + T_2 \); two mutually separated sets. One of these sets must be unbounded. Let us call the unbounded set (in case only one of the sets is unbounded) \( T_1 \), while \( T_2 \) denotes the other set. Now \( S + g \) is connected as \( M \) is connected. Now as \( (S + g) - g = T_1 + T_2 \), then \( T_1 + g \) is connected \(^4\). Then as \( T_1 + g \) is connected and unbounded, \( M - (T_1 + g) \) must be connected.

\(^1\) An open curve is defined by R. L. Moore as a closed connected set of points \( M \) such that if \( P \) is any point of \( M \), then \( M - P = M_1 + M_2 \), two mutually exclusive connected sets neither of which contains a limit point of the other one. Cf. R. L. Moore, *On the foundations of Plane Analysis Situs*, Transactions of the American Mathematical Society, vol. 17 (1916) p. 159. If \( P \) is a point of an open curve \( M \), the point set obtained by adding \( P \) to either of the two sets into which \( M \) is separated by the omission of \( P \) is called a ray of an open curve.

\(^2\) Cf. Hausdorff, loc. cit.

\(^3\) Two point sets are said to be mutually separated if neither contains a point or limit point of the other one.

But $M - (T_1 + g) = T_2 + S_2$, two mutually separated sets. Hence
the supposition that set $S_1$ is not connected has led to a contradiction.
Hence $S_1 + g$ is connected\(^1\). Hence, by hypothesis, $M - (S_1 + g) = S_2$
is also connected.

Let $L_1$ denote those points of $g$ which are limit points of $S_1$
while $L_2$ denotes those points of $g$ which are limit points of $S_2$.
Hence $S_1 + L_1$ and $S_2 + L_2$ are closed sets. As $g = L_1 + L_2$
and $M = (S_1 + L_1) + (S_2 + L_2)$ is a closed connected set it follows
that the sets $L_1$ and $L_2$ have a point $Q$ in common. Let $T$ and $U$
be two distinct points of $g$ each different from $Q$. As they do not
belong to $S_1$ nor to $S_2$, $S_1 + L_1 - T - U$ and $S_2 + L_2 - T - U$
are connected sets\(^2\). Their sum

$$(S_1 + L_1 - T - U) + (S_2 + L_2 - T - U) = S_1 + S_2 + g - T - U$$

is also connected and unbounded. Hence by hypothesis

$$M - (S_1 + S_2 + g - T - U) = T + U$$

is a connected set.

But this is impossible. Hence the supposition that $g$ is a continu
of condensation of $M$ has led to a contradiction.

As $M$ contains no continu of condensation, it follows that $M$ is
an unbounded continuous curve. Several cases may arise.

**Case I.** *For every point $A$ of $M$, $M - A = \overline{M_1} + \overline{M_2}$, two mutually separated sets.* Clearly one of these sets is unbounded. Let $M_1$
denote that one of the sets which is unbounded (in case that but
one of them is unbounded) while $M_2$ denotes the other one. Suppose
$M_1 = G_1 + G_2$, two mutually separated sets of which $G_1$ at least
is unbounded. Now as $M - A = M_1 + M_2$ and $M_1 = G_1 + G_2$, the
sets $M_1 + A$ and $A + G_1$ are connected\(^3\). Hence, as $G_1 + A$
is unbounded and connected, $M - (G_1 + A) = G_2 + M_2$ must be con-
nected. But $G_2$ and $M_2$ are mutually separated sets. Thus we are
led to a contradiction if we suppose $M_1$ not connected. Hence
$M - (M_1 + A) = M_2$ must be connected. Hence in Case I, $M$ satisfies
R. L. Moore's definition of an open curve.

**Case II.** *There is one and only one point $A$ such that $M - A$

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\(^1\) Cf. F. Hausdorff, loc. cit.

\(^2\) Ibid.

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is connected. Let us suppose $M$ were not a ray from $A$. Then there exists an unbounded connected set $Z$ of $M$ containing $A$ and a point $B$ in the set $M - Z^1$. But $M - B = S_1 + S_2$, two mutually separated sets of which $S_1$ contains $A$. Then $S_1$ must contain $Z$ and be unbounded.

Let us consider $S_1 + B - A$. If $S_1 + B - A$ is connected, then as it is also unbounded, $M - (S_1 + B - A) = A + S_2$ must be connected which is impossible as $A$ and $S_2$ are mutually separated sets. Thus $S_1 + B - A = H_1 + H_2$, two mutually separated sets of which $H_1$ contains $B$. Then $M - A = H_1 + S_2 + H_2$. But the sets $(H_1 + S_2)$ and $H_2$ are mutually separated and thus $M - A$ is not connected. Thus we are led to a contradiction if we suppose that in Case II $M$ is not a ray.

Case III. There is more than one point whose omission does not disconnect $M$. Let $A$ and $B$ be any two distinct points such that $M - A$ is connected and $M - B$ is connected. Then there is a ray $Z$ from $A$ situated on the set $M^2$.

But $M \neq Z$, otherwise $B$ would disconnect $M$. The point $A$ is a limit point of $M - Z$ for otherwise the set $M$ would be disconnected by the removal of the connected unbounded set $Z - A$. If $A$ was the only limit point of $M - Z$ in $Z$, then $M - A = (Z - A) + (M - Z)$, two mutually separated sets which is impossible. Then let $P[P \neq A]$ denote a point of $Z$ which is a limit point of $M - Z$. Let $V$ denote $M - Z$ plus its limit points. The set $V$ contains no continuin of condensation. Hence $V$ is a continuous curve. Thus there is an arc $AP$ lying entirely in $V$. We shall now show that $M = Z + AP$. As $AP$ contains no continuin of condensation there is a point $D$ of $AP$ [$A \neq D \neq P$] which is not a limit point of $M - AP$. The set $Z + AP - D$ is connected and unbounded. Hence $M - [Z + AP - D] = D + (M - Z - AP)$ is connected, which cannot be possible unless $M - Z - AP$ is a vacuous point set. Then $M = Z + AP$. As $M - Z$ is connected and $Z$ does not contain any connected part of arc $AP$ containing more than one point, then $A$ and $P$ are the only points of $AP$ on $Z$. Hence

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1) In order that a set $E$ should be a ray, it is necessary and sufficient that it be an unbounded closed connected set containing a point $P$ which is not situated on any proper connected unbounded subset of $E$. Cf. C. Kuratowski, *Quelques propriétés topologiques de la demi-droite*. Fund. Math., t. III. p. 62.

2) Cf. C. Kuratowski, ibid., p. 61.
$M$ is composed of a simple closed curve, formed by arc $AP$ of $M - Z + A + P$ and the subarc $AP$ of $Z$, plus the subray of ray $Z$ which starts from $P$. The simple closed curve and the ray have $P$ and only $P$ in common.

§ 3.

Let $M$ denote the set composed of the $Y$-axis, the curves $y = \frac{1}{x} \sin \frac{\pi}{x} \ [0 < x \leq 1]$, $y = 1 + \frac{1}{x} \sin \frac{\pi}{x} \ [0 < x \leq 1]$ and the interval $x = 1$, $0 \leq y \leq 1)$. The set $M$ is an unbounded closed connected set which remains connected upon the removal of any connected bounded subset. But at every point of the $Y$-axis the set $M$ fails to be connected im kleinen. We are in a position to prove the following theorem.

Theorem C. Suppose $M$ is an unbounded closed connected set such that if $g$ is any bounded connected subset of $M$, then $M - g$ is connected. Then $M$ is not a continuous curve.

Proof. Let us suppose that $M$ were a continuous curve. Let $A$ and $B$ denote 2 distinct points of $M$. Then there is an arc $AXB$ from $A$ to $B$ all points of which belong to $M$. Put about $A$ as centre a circle $R_1$ such that $B$ is without $R_1$. Put about $B$ a circle $R_2$ such that $R_1$ and its interior are entirely in the exterior of $R_2$. As $M$ is connected im kleinen there is a circle $R_1$ concentric with $R_1$ such that if $P$ is a point of $M$ within $R_1$, then $P$ and $A$ lie together on an arc of $M$ lying entirely in $R_1$. Likewise there exists a circle $R_2$ concentric with $R_2$ such that if $Q$ is any point of $M$ within $R_2$, then $Q$ and $B$ lie together on an arc of $M$ lying entirely within $R_2$. Let $K$ denote a point of $M$ which is not on $AXB$ and which lies in $R_1$. That such a point $K$ exists follows at once from the fact that $M - (AXB - A)$ is connected and hence $A$ is a limit point of $M - AXB$.

Let $M_1$ denote $K$ together with all points of $M - AXB$ which can be joined to $K$ by an arc of $M$ having no point in common with arc $AXB$. Let $M_2$ denote all other points of $M - AXB$. As $M_1 + M_2$ is connected, one of these sets must contain a limit point of the other one. Suppose $Q$ of $M_1$ is a limit point of $M_2$. Put

1) Cf. Fund. Math. II, p. 254, fig. 1,
about $Q$ a circle $S$ with centre at $Q$ such that all points of arc $AXB$ are without $S$. As $M$ is connected im kleinen there is a circle $\overline{S}$ such that if $W$ is any point of $M$ within $\overline{S}$, then $W$ and $Q$ are end points of an arc of $M$ lying wholly within $S$. Let $H$ be a point of $M_2$ within $S$. Then there is an arc $HQ$ of $M$ lying entirely within $S$ and thus having no point on arc $AXB$. By definition of $M_1$ there is an arc $KQ$ of $M$ having no point in common with $AXB$. The point set $KQ + QH$ contains as a subset an arc of $M$ from $K$ to $H$ having no point in common with $AXB$. Thus $H$ belongs to $M_1$ contrary to assumption. Thus $M_1$ contains no limit point of $M_2$. In like manner $M_2$ contains no limit point of $M_1$. Thus the division of $M — AXB$ into $M_1$ and $M_2$ is impossible; as $M_1$ exists, the set $M_2$ must be vacuous.

Let $T$ denote a point of $M — AXB$ lying in $\overline{R_2}$. That such a point $T$ exists follows at once from the fact that $M — (AXB — B)$ is connected and hence $B$ is a limit point of $M — AXB$. Let $TRK$ denote an arc of $M$ having no point in common with $AXB$. Let $TEB$ denote an arc of $M$ lying entirely within $R_2$ while $KFA$ denotes an arc of $M$ lying entirely within $R_1$. The sum of the arcs, $AXB$, $KFA$, $TEB$ and $TRK$ contains as a subset a simple closed curve $J$ belonging entirely to $M$. As $M$ is unbounded there are points of $M$ without $J$. As $M — J$ is connected, there can be no point of $M$ within $J$.

Let $A_1$ and $A_2$ denote distinct points of $J$. Put about $A_1$ as centre a circle $C_1$ such that $A_2$ is without $C_1$; with $A_2$ as centre describe a circle $C_2$ such that $C_1$ and its interior are without $C_2$. As $M$ is connected im kleinen there exist circles $C_i (i = 1, 2)$ such that if $B_i$ is any point of $M$ within $C_i$, then $B_i$ and $A_i$ lie together in an arc of $M$ lying entirely within $C_i$. Let $D_i$ be a point of $M — J$ within $C_i$. That such a point $D_i$ exists follows from the fact that $M — (J — A_i)$ is connected and hence $A_i$ is a limit point of $M — J$. As before we may prove that there is an arc $D_1 XD_2$ lying entirely in $M — J$ and hence entirely without $J$. Let $D_i Z_i A_i$ denote an arc of $M$ lying entirely within $C_i (i = 1, 2)$. The set $A_1 Z_1 D_1 + D_1 XD_2 + D_2 Z_2 A_2$ contains as a subset an arc $\overline{A_1XA_2}$ such that

1) $A_1$ and $A_2$ are on $J$
2) $\overline{A_1XA_2}$ is without $J$ and belongs to $M$

Let $E_1$ and $E_2$ denote 2 points of $J$ such that $A_1$ and $A_2$ separate $E_1$ and $E_2$ on $J$. It follows that either $E_1$ is within the clo-
sed curve $J_1$ formed by arcs $\overline{A_1X A_2}$ and $\overline{A_2 E_2 A_1}$ or $E_2$ is within the closed curve $J_2$ formed by the arcs $\overline{A_1X A_2}$ and $\overline{A_1 E A_2}$. In either case there is a closed curve of $M$ such that there is a point of $M$ without this closed curve and a point of $M$ within the closed curve. Then $M$ minus this closed curve is not connected. Thus the supposition that $M$ is connected im kleinen has led to a contradiction.

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