An application of games to the completeness problem for formalized theories

by

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1. Let A be a set of predicates. Let η be a one variable predicate such that $\eta \in A$. We will denote by $\mathfrak{H}(A)$ the set of all formulas of the lower predicate calculus with identity ι which contains the binary predicate ϵ and predicates from A only. As models of $\mathfrak{H}(A)$ we will admit those models for the set of formulas in which

(1) $\hat{x}\eta(x)$ is a set of individuals.

(2) $|\mathfrak{M}|$ (the set of elements of model $\mathfrak{M}$) is the smallest set X such that $\hat{x}\eta(x) \subset X$ and if $x_1 \in X, \ldots, x_k \in X$ then the finite set $\{x_1, \ldots, x_k\} \in X$, $k = 1, 2, \ldots$

(3) Predicates from A are interpreted as relations in $\hat{x}\eta(x)$.

(4) Predicate ι is interpreted as the identity relation $=$.

(5) ϵ is the set-theoretical ϵ-relation in $|\mathfrak{M}|$.

By $\mathfrak{G}(A)$ we denote the set of those formulas in $\mathfrak{H}(A)$ which do not contain ϵ and have all quantifiers restricted to η.

Speaking more intuitively, $\mathfrak{G}(A)$ is a part of the lower predicate calculus and $\mathfrak{H}(A)$ is obtained by the addition of the set-theoretical notion of finite set. Conditions (1)-(5) ensure that the sets we are speaking about in $\mathfrak{H}(A)$ are interpreted as true finite sets.

We will denote by $\mathfrak{H}_n(A)$ the set of formulas which are of the form

$$(q_k x_k)(q_{k-1}x_{k-1})\ldots(q_1 x_1)\Psi(x_1, \ldots, x_n)$$

where q_i is either the existential or the universal quantifier, and Ψ is quantifier-free.

The intersection $\mathfrak{G}(A) \cap \mathfrak{H}_n(A)$ will be denoted by $\mathfrak{G}_n(A)$. Suppose that we are given two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for $\mathfrak{H}(A)$.

Definition 1 $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *indiscernible by means of finite sets* (briefly *indiscernible*) if for any closed formula $\alpha \in \mathfrak{H}(A)$

$$\mathrm{stsf}_{\mathfrak{M}_1}\alpha \equiv \mathrm{stsf}_{\mathfrak{M}_2}\alpha .$$

DEFINITION 2. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *elementarily indiscernible* if for any closed formula $\alpha \in \mathfrak{G}(A)$

$$\mathrm{stsf}_{\mathfrak{M}_1} \alpha \equiv \mathrm{stsf}_{\mathfrak{M}_2} \alpha .$$

2. Games G_n and H_n. In this chapter we define games (in the ordinary sense of the word) which will be used in the formulation of a sufficient condition for indiscernibility of models and a necessary and sufficient condition for elementary indiscernibility. The last one is only a new formulation of the condition given by Fraïssé [3].

We are given two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for $\mathfrak{H}(A)$, and two players I and II. In the game $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ each player makes n moves. In his ith move ($i = 1, \ldots, n$) player I first chooses one of the models $\mathfrak{M}_{l_i}$ ($l_i = 1, 2$) and then points out, in the chosen model, an arbitrary finite sequence of elements $a_{i1}^{l_i}, a_{i2}^{l_i}, \ldots, a_{ik_i}^{l_i}$, such that $\mathrm{stsf}_{\mathfrak{M}_{l_i}} \eta(a_{ij}^{l_i})$. In his ith move player II chooses in the model $\mathfrak{M}_{3-l_i}$ a sequence of k_i elements $a_{i1}^{3-l_i}, \ldots, a_{ik_i}^{3-l_i}$, where $\mathrm{stsf}_{\mathfrak{M}_{3-l_i}} \eta(a_{ij}^{3-l_i})$. Then after n moves we have $k_1 + \ldots + k_n$ pairs

$$\left.\begin{array}{l} a_{11}^1 \leftrightarrow a_{11}^2 \\ a_{12}^1 \leftrightarrow a_{12}^2 \\ \ldots\ldots\ldots \\ a_{1k_1}^1 \leftrightarrow a_{1k_1}^2 \end{array}\right\} \text{1st move}$$

$$\ldots\ldots\ldots$$

$$\left.\begin{array}{l} a_{n1}^1 \leftrightarrow a_{n1}^2 \\ \ldots\ldots\ldots \\ a_{nk_n}^1 \leftrightarrow a_{nk_n}^2 \end{array}\right\} n\text{th move}$$

Player II wins if after the game the correspondence written above is an isomorphism of these sequences with respect to relations from A. It means that II wins if and only if for any $\alpha \in A$, and for any sequence of pairs of positive integers $(i_1, j_1), \ldots, (i_s, j_s)$, where s is the number of variables in α and $1 \leqslant i_t \leqslant n$, $1 \leqslant j_t \leqslant k_{i_t}$, $t = 1, \ldots, n$,

$$\mathrm{stsf}_{\mathfrak{M}_1} \alpha(a_{i_1 j_1}^1, \ldots, a_{i_s j_s}^1) \equiv \mathrm{stsf}_{\mathfrak{M}_2} \alpha(a_{i_1 j_1}^2, \ldots, a_{i_s j_s}^2) .$$

Player I wins if and only if player II does not win.

The difference between $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ and $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ is that in the game $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ player I points out only one element $a_i^{l_i}$ in his ith move, $i = 1, \ldots, n$. Then after n moves there are only n pairs:

$$\begin{array}{l} a_1^1 \leftrightarrow a_1^2 \ \} \text{ 1st move} \\ a_2^1 \leftrightarrow a_2^2 \ \} \text{ 2nd move} \\ \ldots\ldots \\ a_n^1 \leftrightarrow a_n^2 \ \} \ n\text{th move} \end{array}$$

Besides that restriction, all the rules of the game and the definition of winning are the same as in $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$.

EXAMPLE. Let A contain only two binary predicates, ϱ and ι, and let $\mathfrak{M}_1$ be a set of integers with relations $<$ (less than) and $=$, and let $\mathfrak{M}_2$ be a set of rationals with the same relations.

(a) In the game $H_1(\mathfrak{M}_1, \mathfrak{M}_2)$ player II can always win: Each player has only one move; when player I points out the sequence $a_1^{l_1}, \ldots, a_n^{l_1}$ in $\mathfrak{M}_{l_1}$, then player II can find in $\mathfrak{M}_{3-l_1}$ a sequence of numbers $a_1^{3-l_1}, \ldots, a_n^{3-l_1}$ such that

$$a_i^{l_1} < a_j^{l_1} \equiv a_i^{3-l_1} < a_j^{3-l_1}$$

and

$$a_i^{l_1} = a_j^{l_1} \equiv a_i^{3-l_1} = a_j^{3-l_1} .$$

(b) In the game $G_3(\mathfrak{M}_1, \mathfrak{M}_2)$ player I can win if he points out in the first two moves integers 0 and 1 in $\mathfrak{M}_1$ and in the third move an element $\frac{1}{2}(w_1 + w_2)$ where w_1 and w_2 are rationals pointed out by player II in the latter's 1st and 2nd move. Now whatever is chosen by player II, he loses because either $w_1 < w_2$ and $w_1 < \frac{1}{2}(w_1 + w_2) < w_2$ or $w_2 \geqslant w_1$ and for any integer x, $x \leqslant 0$ or $x \geqslant 1$ and $0 < 1$

$$\text{1st move of I } 0 \leftrightarrow w_1 \text{ 1st move of II},$$

$$\text{2nd move of I } 1 \leftrightarrow w_2 \text{ 2nd move of II},$$

$$\text{3rd move of II } x \leftrightarrow \frac{w_1 + w_2}{2} \text{ 3rd move of I}.$$

In the example (a) we say that player II has a *winning method* (or *winning strategy*) in the game H_1, and in (b) we say that player I has a *winning method* in G_3. Generally by *method* (or *strategy*) for player I in a game $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ we understand a sequence of functions $\varphi_1, \ldots, \varphi_n$, $f_1, \ldots, f_n$ such that each function φ_i ($i = 1, \ldots, n$) correlates with any finite sequence of pairs

$$(*) \qquad \begin{array}{c} a_1^1 \leftrightarrow a_1^2 \\ \ldots\ldots \\ a_i^1 \leftrightarrow a_i^2 , \end{array}$$

where a^1 are in $\mathfrak{M}_1$ and a^2 are in $\mathfrak{M}_2$, a number l_i equal to 1 or 2; and f_i correlates with $(*)$ a finite sequence $b_1, \ldots, b_s$ of elements in $\mathfrak{M}_{l_i}$. Intuitively speaking, φ_i chooses the model in which player I will point out elements in his ith move and f_i gives the chosen sequence of elements. The strategies in $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ for player I are those strategies in $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ in which sequences $b_1, \ldots, b_s$ have only one element. The strategy for player II in $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ and $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ is the sequence of function $g_1, \ldots, g_n$ such that for every sequence of pairs $(*)$ and the sequence $b_1^l, \ldots, b_k^l$ (b_i^l in $\mathfrak{M}_l$, $l = 1, 2$) the function g_i attaches a sequence of k elements $b_1^{3-l}, \ldots, b_k^{3-k}$ in the model $\mathfrak{M}_{3-k}$. (g_i describes the choice of player II in his ith move.)

Strategy a is called the *winning strategy* (*winning method*) if for any strategy of the other player the first player wins using strategy a.

Obviously in the games $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ and $G_1(\mathfrak{M}_1, \mathfrak{M}_2)$ either player I or II has the winning strategy.

Suppose we are given a model $\mathfrak{M}$ for $\mathfrak{H}(A)$; we will define a function $S(x)$ for $x \in |\mathfrak{M}|$

$$S(x) = \begin{cases} \{x\} \text{ if } \mathrm{stsf}_{\mathfrak{M}}\eta(x)\,, \\ \text{if non } \mathrm{stsf}_{\mathfrak{M}}\eta(x)\,, \text{ then } S(x) \text{ is the least set } Y \\ \quad \text{such that } x \in Y \text{ and if } z \in Y \text{ and } t \in z \text{ then } t \in Y\,. \end{cases}$$

Now let $s(x) = S(x) \cap \hat{x}\,\mathrm{stsf}_{\mathfrak{M}}\eta(x)$. It is easy to see that the correspondence

$$(**) \qquad a_1 \leftrightarrow b_1, \ldots, a_m \leftrightarrow b_m\,,$$

$(a_i \in \mathfrak{M}_1, b_i \in \mathfrak{M}_2)$ such that for every predicate $a \in A$

$$\mathrm{stsf}_{\mathfrak{M}_1} a(a_{i_1}, \ldots, a_{i_s}) \equiv \mathrm{stsf}_{\mathfrak{M}_2} a(b_{i_1}, \ldots, b_{i_s})$$

uniquely determines an isomorphism f between a family of such $x \in |\mathfrak{M}_1|$ that $s(x) \subset \{a_1, \ldots, a_n\}$ and a family of such $y \in |\mathfrak{M}_2)|$ that $s(y) \subset \{b_1, \ldots, b_n\}$ (an isomorphism under relations from A, and $\in$). We will say that such a correspondence $(**)$ *establishes* an isomorphism between sets $x \in |\mathfrak{M}_1|$ and $y \in |\mathfrak{M}_2|$ if $s(x) \subset \{a_1, \ldots, a_n\}$, $s(y) \subset \{b_1, \ldots, b_n\}$ and f maps x onto y.

THEOREM 1. *Suppose* $x_1, \ldots, x_k \in |\mathfrak{M}_1|$, $y_1, \ldots, y_k \in |\mathfrak{M}_2|$, $s(x_i) \subset \{x_i^1, \ldots, x_i^{n_i}\}$, $s(y_i) \subset \{y_i^1, \ldots, y_i^{n_i}\}$. *If player* II *has the winning method in a game* $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ *after* k*-moves:*

$$\left.\begin{matrix} x_1^1 \leftrightarrow y_1^1 \\ \cdots\cdots \\ x_1^{n_1} \leftrightarrow y_1^{n_1} \end{matrix}\right\} \text{1st move}$$

$$\cdots\cdots$$

$$\left.\begin{matrix} x_k^1 \leftrightarrow y_k^1 \\ \cdots\cdots \\ x_k^{n_k} \leftrightarrow y_k^{n_k} \end{matrix}\right\} k\text{-th move}$$

and this correspondence establishes an isomorphism between x_i *and* y_i $(i = 1, \ldots, k)$, *then for each formula* $\Psi \in \mathfrak{H}_n(A)$ *with* k *free variables*

$$\mathrm{stsf}_{\mathfrak{M}_1}\Psi(x_1, \ldots, x_k) \equiv \mathrm{stsf}_{\mathfrak{M}_2}\Psi(y_1, \ldots, y_k)\,.$$

Proof. The proof is made by induction with respect to $n-k$. If $n-k = 0$, then Ψ is quantifier-free and the theorem is true by the definition of isomorphism between x_i and y_i.

Let us assume that the theorem is true for $n-k-1 \geqslant 0$. Suppose that the assumptions of the theorem are satisfied but, for example,

$\mathrm{stsf}_{\mathfrak{M}_1}\Psi(x_1, \ldots, x_k)$ and not $\mathrm{stsf}_{\mathfrak{M}_2}\Psi(y_1, \ldots, y_k)$, i. e. $\mathrm{stsf}_{\mathfrak{M}_2}\sim\Psi(y_1, \ldots, y_k)$. Suppose that Ψ is a formula $(\exists u)\Phi(u_1, \ldots, u_k, u)$, $u_1, \ldots, u_k$ free variables of Ψ; if it were not, then we should consider the formula $\sim\Psi$.

Then in $\mathfrak{M}_1$ there is such an x_{k+1} that

$$\mathrm{stsf}_{\mathfrak{M}_1}\Phi(x_1, \ldots, x_k, x_{k+1})\,.$$

Let player I point out in the $(k+1)$-st move:

$$\begin{matrix} x_{k+1}^1 \\ \cdots \\ x_{k+1}^p \end{matrix} \quad \text{where} \quad \{x_{k+1}^1, \ldots, x_{k+1}^p\} = s(x_{k+1})\,.$$

By assumption, player II has the winning method; therefore he can find a sequence $y_{k+1}^1, \ldots, y_{k+1}^p$ in $\mathfrak{M}_2$ such that the correspondence

$$\begin{matrix} x_1^1 \leftrightarrow y_1^1 \\ \cdots\cdots\cdots \\ x_k^{m_k} \leftrightarrow y_k^{m_k} \end{matrix}$$

$$\left.\begin{matrix} x_{k+1}^1 \leftrightarrow y_{k+1}^1 \\ \cdots\cdots\cdots \\ x_{k+1}^p \leftrightarrow y_{k+1}^p \end{matrix}\right\} (k+1)\text{-th move}$$

is an isomorphism, i. e. for any $a \in A$

$$\mathrm{stsf}_{\mathfrak{M}_1} a(x_{i_1}^{j_1}, \ldots, x_{i_s}^{j_s}) \equiv \mathrm{stsf}_{\mathfrak{M}_2} a(y_{i_1}^{j_1}, \ldots, y_{i_s}^{j_s})$$

for $0 \leqslant i_e \leqslant k+1$, $0 \leqslant j_e \leqslant m_{i_e}$. Then there is such an element $y_{k+1} \in |\mathfrak{M}_2|$ that the correspondence given above establishes an isomorphism between x_{k+1} and y_{k+1}, and therefore, by the inductive assumption,

$$\mathrm{stsf}_{\mathfrak{M}_1}\Phi(x_1, \ldots, x_{k+1}) \equiv \mathrm{stsf}_{\mathfrak{M}_2}\Phi(y_1, \ldots, y_{k+1})\,,$$

which means that $\mathrm{stsf}_{\mathfrak{M}_2}(\exists u)\Phi(y_1, \ldots, y_k, u)$, which contradicts the assumption that $\sim\mathrm{stsf}_{\mathfrak{M}_2}\Psi(y_1, \ldots, y_k)$, q. e. d.

THEOREM 2. *If player* II *has the winning method in the game* $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ *then for any closed formula* $\Psi \in \mathfrak{H}_n(A)$.

$$\mathrm{stsf}_{\mathfrak{M}_1}\Phi \equiv \mathrm{stsf}_{\mathfrak{M}_2}\Psi\,.$$

Proof. It is enough to put $k = 0$ in the assumption of Theorem 1.

THEOREM 3. *If player* II *has the winning strategy for every* n *in the game* $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$, *then models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are indiscernible by means of finite sets.*

Proof. The above follows immediately by Theorem 2.

We will show in the next chapter that the converse of Theorem 3 is not true, i. e. that there exist models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for some $\mathfrak{H}(A)$ such that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are indiscernible by means *of finite sets* though for some n_0 player I has a winning strategy in the game $H_{n_0}(\mathfrak{M}_1, \mathfrak{M}_2)$.

THEOREM 4. *If $\mathfrak{M}$ is a model for $\mathfrak{H}(A)$ then there is an arithmetical submodel $\mathfrak{M}' < \mathfrak{M}$ such that $|\overline{\overline{\mathfrak{M}'}}| = \aleph_0$.*

Proof. We can introduce in the model Skolem functions $f_1, .., f_n, ...$ eliminating quantifiers. The number of the functions is $\aleph_0$. Let us now take the least subset $X \subset |\mathfrak{M}|$, which contains an arbitrarily given element x_0, closed under

(1) the construction of finite sets $\{x_1 ... x_k\}$ where $x_1 \in X, ..., x_k \in X$, $k = 1, 2, ...$,

(2) the construction of subsets: $x \in X$ and $y \in x$ then $y \in X$,

(3) functions $f_1, ..., f_n, ...$

As $\mathfrak{M}'$ we now take X, defining relations in X according to relations in $\mathfrak{M}$. It is easy to see that $\mathfrak{M}'$ is an arithmetical submodel of $\mathfrak{M}$, and that $|\overline{\overline{\mathfrak{M}'}}| = \overline{\overline{X}} = \aleph_0$ because X is obtained from one element by the application of constructions, which give a finite number of new elements only countable many times, q. e. d.

3. Example of two models which are indiscernible but for which player I has a winning strategy in the game H_3.

Let A contain (besides ι and η) four predicates $\alpha(x)$, $\beta(x)$, $\gamma(x, y)$, $\delta(x, y, z)$. Let us take as a model $\mathfrak{M}_1$ the set of all infinite sequences with values 0, 1, and all natural numbers (and the set of all finite subsets, the sets of the subsets and so on of those two sets).

Let $\alpha(x)$ be interpreted as the relation "x is a sequence with values 0, 1",

$\beta(x)$ as "x is a natural number",

$\gamma(x, y)$ as "x, y are natural numbers and $x < y$",

and $\delta(x, y, z)$ as "z is 0 or 1, y is a natural number, x is a sequence and the yth term of sequence x is equal to z".

Let us notice that we can *define* in this theory all natural numbers (by means of γ) and that each 0-1 sequence $x = \{x_n\}$ is characterized by an infinite sequence of sentences $\delta(x, 0, x_0)$, $\delta(x, 1, x_1)$, ...

Let us take as $\mathfrak{M}_2$ a countable arithmetical submodel (Theorem 4) of $\mathfrak{M}$. Since $|\overline{\overline{\mathfrak{M}_2}}| = \aleph_0$, there exists a sequence x in $\mathfrak{M}_1$ which is not in $\mathfrak{M}_2$.

Let player I point out this sequence in his first move and let player II point out the element y in $\mathfrak{M}_2$. If y is not a sequence but a number, then $\mathrm{stsf}_{\mathfrak{M}_1}\alpha(x)$ but $\mathrm{stsf}_{\mathfrak{M}_2}\sim\alpha(y)$ and player I has won. Then let us assume that y is a sequence; since $y \neq x$, there is such an n that $y_n \neq x_n$, i. e.

$$(***) \qquad \mathrm{stsf}_{\mathfrak{M}_1}\delta(x, n, 0) \equiv \mathrm{stsf}_{\mathfrak{M}_2}\sim\delta(x, n, 0) .$$

Player I now chooses the sequence of integers $0, 1, ..., n$ in $\mathfrak{M}_1$ and II has to choose some elements $a_0, ..., a_n$ in $\mathfrak{M}_2$.

$$x \leftrightarrow y \;\}\; \text{1st move}$$
$$\left.\begin{array}{l} 0 \leftrightarrow a_0 \\ 1 \leftrightarrow a_1 \\ \cdot\;\cdot\;\cdot\;\cdot \\ n \leftrightarrow a_n \end{array}\right\} \text{2nd move}$$

Case (a): if $a_0 = 0, a_1 = 1, ...$ then I wins by (***);

Case (b): if some a_i is not a natural number then I also wins because $\mathrm{stsf}_{\mathfrak{M}_1}\beta(i)$ and $\mathrm{stsf}_{\mathfrak{M}_2}\sim\beta(a_i)$;

Case (c): if $a_0, ..., a_n$ are natural numbers and non $a_i < a_j$ for some $i < j$ then $\mathrm{stsf}_{\mathfrak{M}_1}\gamma(i, j)$ and $\mathrm{stsf}_{\mathfrak{M}_2}\sim\gamma(a_i, a_j)$;

Case (d): if $a_0, ..., a_n$ are natural numbers and $a_0 < ... < a_n$ and non (a), then there exists such b, that $b < a_n$ and $b \neq a_i$ for $i = 1, ..., n$.

In cases (a), (b), (c) the 3rd move of player I is immaterial.

In case (d) player I can choose the number b in $\mathfrak{M}_2$.

$$\begin{array}{ll} \text{1st move of I} & x \leftrightarrow y \\ \text{2nd move of I} & \left\{\begin{array}{l} 0 \leftrightarrow a_0 \\ \cdot\;\cdot\;\cdot\;\cdot \\ n \leftrightarrow a_n \end{array}\right. \\ & \quad \leftrightarrow b \quad \text{3rd move of I} \end{array}$$

and then after any move of II, player I wins. Thus we have shown that player I has a winning strategy in $H_3(\mathfrak{M}_1, \mathfrak{M}_2)$ though $\mathfrak{M}_2$ as an arithmetical submodel of $\mathfrak{M}_1$ is indiscernible from $\mathfrak{M}_1$, q. e. d.

THEOREM 5. *If player* II *has a winning strategy in $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ after k moves*

$$\begin{array}{c} x_1 \leftrightarrow y_1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_k \leftrightarrow y_k , \end{array}$$

then for every $\Psi \in \mathfrak{G}_n(A)$ with k free variables

$$\mathrm{stsf}_{\mathfrak{M}_1}\Psi(x_1, ..., x_k) \equiv \mathrm{stsf}_{\mathfrak{M}_1}\Psi(y_1, ..., y_k) .$$

Proof. It is enough to notice that for formulas in $\mathfrak{G}(A)$ (in which all quantifiers are restricted to η) player I, in the proof of Theorem 1, points out only 1-term sequences. So, we can repeat this proof. (See also Fraïssé [3].)

THEOREM 6. *If player* II *has a winning strategy in $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$, then for every closed formula $\Psi \in \mathfrak{G}_n(A)$*

$$\mathrm{stsf}_{\mathfrak{M}_1}\Psi \equiv \mathrm{stsf}_{\mathfrak{M}_2}\Psi .$$

Proof. The above follows from Theorem 5 for $k = 0$.

THEOREM 7. *If player* II *has for every* n *a winning strategy in the game* $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$, *then* $\mathfrak{M}_1, \mathfrak{M}_2$ *are elementarily indiscernible.*

Proof. The above follows immediately from Theorem 6 (see Fraïssé [3]). The converse theorem is not true:

EXAMPLE. Let A contain infinitely many predicates of one variable $a_1, a_2, \ldots$ Let us take as $\mathfrak{M}_1$ the set of natural numbers where $\mathrm{stsf}_{\mathfrak{M}_1} a_i(m)$ if and only if $m > i$, and as $\mathfrak{M}_2$ the set of ordinals less than or equal to ω with the same interpretation of a_i. Player I wins in $G_1(\mathfrak{M}_1, \mathfrak{M}_2)$ pointing out ω in $\mathfrak{M}_2$ though these two models are elementarily indiscernible.

Remark. One can easily show that existence of a winning method for player II in $H_n(\mathfrak{M}_1, \mathfrak{M}_2)$ does not follow from the existence of a corresponding method in $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$. (It is enough to apply Theorem 8 to Example 1.) Also, two models can be elementarily indiscernible but discernible by means of finite sets. It is enough to take an ordering relation of type ω and an ordering relation of type $\omega+\omega^*+\omega$.

Let us now assume that A contains only a finite number of predicates.

THEOREM 8. *For every* $\mathfrak{M}_1$ *and every sequence of elements* $x_1, \ldots, x_k \in \mathfrak{M}_1$ *there exists such a formula* $\Phi \in \mathfrak{S}(A)$ *that for any* $\mathfrak{M}_2$ *and* $y_1, \ldots, y_k \in \mathfrak{M}_2$ *player* II *has a winning method after* k *moves*

$$\begin{array}{c} x_1 \leftrightarrow y_1 \\ \cdots\cdots \\ x_k \leftrightarrow y_k \end{array}$$

in the game $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ *if and only if*

$$\mathrm{stsf}_{\mathfrak{M}_2} \Phi(y_1, \ldots, y_k)\,.$$

Proof. Let us define a family of finite sequences Ω_{kl} of formulas in $\mathfrak{S}(A)$, where $k = 0, \ldots, n$; $l = 0, \ldots, l_k$.

(i) Let $\{\Omega_{nl}\}$, $l = 0, \ldots, l_n$, be an arbitrary sequence of quantifier-free formulas in $\mathfrak{S}(A)$ which satisfies the following conditions.

(a) Ω_{nl} contains variables $u_1, \ldots, u_n$ only,

(b) $\Omega_{n0} \vee \Omega_{n1} \vee \ldots \vee \Omega_{nl_n}$, $\sim(\Omega_{ni} \wedge \Omega_{nj})$ are tautologies of the predicate calculus (for $i \neq j$),

(c) for any quantifier-free formula Γ in $\mathfrak{S}(A)$ which contains variables $u_1, \ldots, u_n$ only, and for $i < l_n$, either $\Omega_{ni} \supset \Gamma$ or $\Omega_{ni} \supset \sim\Gamma$ is a tautology.

Intuitively speaking, Ω_{nl} is a sequence of atomic formulas of n variables in $\mathfrak{S}(A)$. Such finite sequence exists by the finiteness of A.

(ii) Let a sequence $\{\Omega_{k-1,l}\}$ be an arbitrary sequence of all formulas

$$(*) \qquad A_1 \wedge A_2 \wedge \ldots \wedge A_{l_k}$$

where A_i denotes either $(\exists u_k)\Omega_{ki}$ or $\sim(\exists u_k)\Omega_{ki}$. It is easy to see that condition (i) (b) holds for formulas $\Omega_{k,0}, \ldots, \Omega_{k,l_k}$.

LEMMA. *A formula* Ω_{ki} *such that*

$$\mathrm{stsf}_{\mathfrak{M}_1} \Omega_{ki}(x_1, \ldots, x_k)$$

satisfies the conditions of Theorem 7.

The proof of the lemma is given by induction with respect to $n-k$.

(1) for $n-k = 0$, the lemma is true by the definition of winning of player II and by the definition of sequence $\{\Omega_{ni}\}$.

(2) Let as assume that the lemma is true for some $n-k$ and let

$$\mathrm{stsf}_{\mathfrak{M}_1} \Omega_{k-1,i_0}(x_1, \ldots, x_{k-1})\,.$$

Let us suppose that

$$(**) \qquad \mathrm{stsf}_{\mathfrak{M}_2} \Omega_{k-1,i_0}(y_1, \ldots, y_{k-1})\,.$$

If for example player I points out in the kth move an element x_k in $\mathfrak{M}_1$ then for some i_1

$$\mathrm{stsf}_{\mathfrak{M}_1} \Omega_{ki_1}(x_1, \ldots, x_k)$$

but by $(**)$, $(*)$ $\mathrm{stsf}_{\mathfrak{M}_2}(\exists u_k)\Omega_{ki_1}(y_1, \ldots, y_{k-1}, u_k)$ and (by the inductive assumption) player II pointing out an arbitrary element y such that

$$\mathrm{stsf}_{\mathfrak{M}_2} \Omega_{ki_1}(y_1, \ldots, y_{k-1}, y)\,.$$

If conversely to $(**)$,

$$\sim \mathrm{stsf}_{\mathfrak{M}_2} \Omega_{k-1,i_0}(y_1, \ldots, y_{k-1})$$

then there is an i_1 such that

$$\mathrm{stsf}_{\mathfrak{M}_1}(\exists u_k)\Omega_{ki_1}(x_1, \ldots, x_{k-1}, u_k) \equiv \sim \mathrm{stsf}_{\mathfrak{M}_2}(\exists u_k)\Omega_{k,i_1}(y_1, \ldots, y_{k-1}, u_k)\,.$$

For example, let $\mathrm{stsf}_{\mathfrak{M}_1}(\exists u_k)\Omega_{k,i_1}(y_1, \ldots, y_{k-1}, u_k)$; then plyaer I wins pointing out in the kth move such an elementy y in $\mathfrak{M}_2$ that

$$\mathrm{stsf}_{\mathfrak{M}_2} \Omega_{k,i_1}(y_1, \ldots, y_{k-1}, y)\,.$$

The lemma is then true for $n-k+1$, q. e. d.

THEOREM 9. *If* A *is finite, then for any* n *there is a sequence of closed formulas* $\Phi_1, \ldots, \Phi_n$ *such that player* II *has a winning method in* $G_n(\mathfrak{M}_1, \mathfrak{M}_2)$ *if and only if*

$$\mathrm{stsf}_{\mathfrak{M}_1} \Phi_i \quad \textit{if and only if} \quad \mathrm{stsf}_{\mathfrak{M}_2} \Phi_i\,.$$

The proof follows by Theorem 3 and the lemma of Theorem 8.

THEOREM 10. *If* A *is finite, then player* II *has a winning strategy for each* n *in* $G_n(\mathfrak{M}_1 \mathfrak{M}_2)$ *if and only if models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *are elementarily indiscernible.*

Proof by Theorem 7 and 9.

II

In this section we will apply the results of the first section to the arithmetic of ordinal numbers.

Let $\mathcal{T}_1$ denote $\mathfrak{H}(A)$ in the case where A contains one binary predicate ν.

Let $\mathcal{T}_2$ denote $\mathfrak{H}(A)$ in the case where A contains one binary predicate ν and one ternary σ,

Let $\mathcal{T}_3$ denote $\mathfrak{H}(A)$ in the case where A contains one binary predicate ν and two ternary predicates σ, μ.

Let us denote by T_1, T_2, T_3 the corresponding sets $\mathfrak{G}(A)$.

We denote by $\mathfrak{M}_a$ (where a are ordinals) models for $\mathcal{T}_i$ $(i = 1, 2, 3)$ in which the set of individuals is the set of all ordinals $x < a$ and ν stands for the inequality $(x < y)$, σ for addition $(x+y = z)$, and μ for multiplication $(x\cdot y = z)$ of ordinals. Similarly by $\mathfrak{M}_{ab}$ we denote the models which have as elements those x for $a < x < b$.

THEOREM 11. *If player* II *has a winning strategy in the games*

$$H_{n-1}(\mathfrak{M}_{a_1}, \mathfrak{M}_{b_1}), H_{n-1}(\mathfrak{M}_{a_1a_2}, \mathfrak{M}_{b_1b_2}), \ldots, H_{n-1}(\mathfrak{M}_{a_ka}, \mathfrak{M}_{b_kb}),$$

where $a_1 < \ldots < a_k < a$, $b_1 < \ldots < b_k < b$ *and* $\mathfrak{M}_{a_i}, \mathfrak{M}_{b_i}$ *are models for* $\mathcal{T}_1$, *then player* II *has a winning strategy in the game* $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$, *after the first move*:

$$\begin{matrix} a_1 \leftrightarrow b_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_k \leftrightarrow b_k \end{matrix}$$

The proof is immediate. Player II can point out elements according to his strategies in the games $H_n(\mathfrak{M}_{a_ia_{i+1}}, \mathfrak{M}_{b_ib_{i+1}})$ for "segments" of $\mathfrak{M}_a$ and $\mathfrak{M}_b$ (1).

DEFINITION. We say that $a \equiv b (\mathrm{mod}\, c)$ (where c is not *cofinal* with any smaller number) if either $a = b$ or $a = cd+e$, $b = cd'+e$ where $d, d' \neq 0$.

THEOREM 12. *If* $a \equiv b(\mathrm{mod}\,\omega^n)$ *then in the game* $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$ *(where* $\mathfrak{M}_a, \mathfrak{M}_b$ *are models for* $\mathcal{T}_1$*) player* II *has a winning strategy.*

Proof (by induction). (i) For $n = 1$ the theorem is true because, by the definition of $\equiv$, either both models are identical or both are infinite. Besides we can assume, without restriction of generality, that the sequence of elements chosen by player I in the 1st move is ordered according to magnitude. In the first case the strategy for player II is obvious. In the second case he can point out in the other model any sequence ordered according to magnitude which has the same number of elements as the sequence pointed out by player I.

(1) See also S. Feferman, *Summaries of tables at Cornell University*, 1956, pp. 201-209.

(ii) Let us assume that the theorem is true for some k. Now let player I point out in his first move elements

$$a_1, a_2, \ldots, a_l$$

in $\mathfrak{M}_a$, and let $a_1 < a_2 < \ldots < a_l$. Let $a = \omega^{k+1}c+d$ and $b = \omega^{k+1}c'+d$.

LEMMA. *If player* II *has a winning strategy in the game* $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$ *then he has a winning strategy in the game* $H_n(\mathfrak{M}_{a+d}, \mathfrak{M}_{b+d})$.

The proof is obvious: see Theorem 11.

By this lemma we can consider only the case in which $c, c' \neq 0$ and $d = 0$. Let us consider the ordinal types $\overline{(0, a_1)} = \omega^k c_1+d_1, \ldots, \overline{(a_{l-1}, a_l)} = \omega^k c_l+d_l$, $\overline{(a_l, \omega^{k+1}c)} = \omega^k c_{l+1}$ (where $\overline{(x, y)}$ denotes an ordinal type of a set of those z for which $x \leqslant z < y$). Let us notice that there are such $b_1 < \ldots < b_l < \omega^{k+1} \leqslant b$ that

$$\overline{(0, b_1)} = \omega^k\cdot \mathrm{sgn}\, c_1 + a d_1\,, \quad \ldots, \quad \overline{(b_{l-1}, b_l)} = \omega^k\cdot \mathrm{sgn}\, c_l + d_l\,,$$

where

$$\mathrm{sgn}\, x = \begin{cases} 0 & \text{if} \quad x = 0\,, \\ 1 & \text{if} \quad x > 0\,. \end{cases}$$

Let player II point out in his 1st move a sequence $b_1, \ldots, b_l$.

$$(*) \qquad \begin{cases} a_1 \leftrightarrow b_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_l \leftrightarrow b_l \end{cases}$$

By the inductive assumption player II has a winning strategy in the games $H_{n-1}(\mathfrak{M}_{a_ia_{i+1}}, \mathfrak{M}_{b_ib_{i+1}})$ and then by Theorem 11, he has a winning strategy in $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$ *after the first move* $(*)$. But since there was no restriction for the 1st move of I, player II has a winning strategy in $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$, q. e. d.

THEOREM 13. *If* $\mathfrak{M}_a$ *and* $\mathfrak{M}_b$ *are models for* $\mathcal{T}_1$ *and* $a \equiv b(\mathrm{mod}\,\omega^\omega)$ *then* $\mathfrak{M}_a$ *and* $\mathfrak{M}_b$ *are indiscernible by finite sets.*

This follows by Theorems 2 and 12.

THEOREM 14. *Let* $\mathfrak{M}_a$ *and* $\mathfrak{M}_b$ *be models for* $\mathcal{T}_1$ *such that player* II *has a winning method in* $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$, *let* c *be the least number greater than* $\omega^{a_1}\cdot n_1+\ldots+\omega^{a_k}\cdot n_k$ *for* $a_1, \ldots, a_n$ *in* $\mathfrak{M}_a$, $n_1, \ldots, n_k < \omega$, *and let* d *be the corresponding number for* $\mathfrak{M}_b$; *let* $\mathfrak{M}_c$ *and* $\mathfrak{M}_d$ *be models for* $\mathcal{T}_2$; *then player* II *has a winning method in the game* $H_n(\mathfrak{M}_c, \mathfrak{M}_d)$.

Proof. The method for player II is the following: if player I points out in his kth move elements $a_{k1}, \ldots, a_{kn_k}$ in one model, then player II considers Cantor's representations of those numbers

$$a_{ki} = \omega^{a_{ki1}} n_{ki1} + \ldots + \omega^{a_{kil_i}} n_{kil_i}$$

and considers the game $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$ where $\mathfrak{M}_a, \mathfrak{M}_b$ are models for $\mathcal{T}_1$ assuming that in the kth move player I points out

$$a_{k11}, \ldots, a_{k1l_1}, \ldots, a_{kn_k1}, \ldots, a_{kn_kl_{n_k}}.$$

In this game he has (by Theorem 12) a winning method. Let

$$b_{k11}, \ldots, b_{k1l_1}, \ldots, b_{kn_k1}, \ldots, b_{kn_kl_{n_k}}$$

be a sequence pointed out by player II in the game $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$ according to this method. Now player II in $H_n(\mathfrak{M}_c, \mathfrak{M}_d)$ points out in this kth move a sequence $b_{k1}, \ldots, b_{k,n_k}$ where $b_{ki} = \omega^{b_{ki1}}n_{ki1} + \ldots + \omega^{b_{kil_i}}n_{kil_i}$. It is easy to see that this is the winning method for II, q. e. d.

THEOREM 15. *If* $a \equiv b (\mathrm{mod}\, \omega^\omega)$, $\mathfrak{M}_{\omega^a}, \mathfrak{M}_{\omega^b}$ *are models for* $\mathcal{T}_2$ *then* $\mathfrak{M}_{\omega^a}$ *and* $\mathfrak{M}_{\omega^b}$ *are indiscernible by finite sets.*

This follows by Theorem 12 and 13.

THEOREM 16. *If* $\mathfrak{M}_a, \mathfrak{M}_b$ *are models for* $\mathcal{T}_2$, *player* II *has a winning strategy in* $H_n(\mathfrak{M}_a, \mathfrak{M}_b)$, c, d *being defined as in Theorem* 13; $\mathfrak{M}_c, \mathfrak{M}_d$ *are models for* $\mathcal{T}_3$, *then player* II *has a winning method in* $H_n(\mathfrak{M}_c, \mathfrak{M}_d)$.

The proof is the same as in Theorem 13.

THEOREM 17. *If* $a \equiv b (\mathrm{mod}\, \omega^\omega)$, $\mathfrak{M}_{\omega^{\omega^a}}, \mathfrak{M}_{\omega^{\omega^b}}$ *are models for* $\mathcal{T}_3$ *then* $\mathfrak{M}_{\omega^{\omega^a}}$ *and* $\mathfrak{M}_{\omega^{\omega^b}}$ *are indiscernible by finite sets.*

DEFINITION. An ordinal number a is *definable in a model* $\mathfrak{M}$ for the $\mathcal{T}_i$ (or T_i) if and only if there is a formula $\Psi \in \mathcal{T}_i$ ($\in T_i$) with one free variable such that

$$\mathrm{stsf}_{\mathfrak{M}} \Psi(b) \equiv b = a.$$

An ordinal number is *definable* in $\mathcal{T}_i$ (T_i) if it is definable in the *class of all ordinals*. Let us denote ω^ω by a_1, ω^{ω^ω} by a_2, $\omega^{\omega^{\omega^\omega}}$ by a_3. A. Tarski conjectures that x is definable in T_i if and only if $x < a_i$, $i = 1, 2, 3$.

It is not difficult to prove that if

(•) $x < a_i$

then x is definable in T_i; we will therefore consider converse implications only.

Let us denote by $D(\mathfrak{M}, \mathcal{T}_i)$ (or $D(\mathfrak{M}, T_i)$) the set of numbers definable in a model $\mathfrak{M}$ for $\mathcal{T}_i$ (T_i).

Obviously

(••) $D(\mathfrak{M}, T_i) \subset D(\mathfrak{M}, \mathcal{T}_i)$.

LEMMA. *If* 0 *is an initial segment of* $\mathfrak{M}$, $0 \subset D(\mathfrak{M}, \mathcal{T}_i)$ *and* 0 *is indiscernible from* $\mathfrak{M}$, *then* $0 = D(\mathfrak{M}, \mathcal{T}_i)$.

Proof. It is enough to consider the least number x which is in $D(\mathfrak{M}, \mathcal{T}_i) - 0$ and as Ψ a formula which defines x in $\mathfrak{M}$.

THEOREM 18. *The following conditions are equivalent* (*for* $i = 1, 2, 2$)

(a) *x is definable in* $\mathcal{T}_i$;

(b) *x is definable in* T_i;

(c) $x < a_i$.

Proof. $\mathfrak{M}_{a_i}$ and the class of all ordinals $\mathfrak{M}$ are indiscernible by means of finite sets, as models for $\mathcal{T}_i$, by Theorems 13, 15, 17; $\hat{x}(x < a_i)$ is an initial segment of $\mathfrak{M}$, and by (•) (••) $\hat{x}(x < a_i) \subset D(\mathfrak{M}, T_i) \subset D(\mathfrak{M}, \mathcal{T}_i)$.

Then by lemma $\hat{x}(x < a_i) = D(\mathfrak{M}, \mathcal{T}_i)$, q. e. d.

References

[1] A. Ehrenfeucht, *Application of games to some problems of mathematical logic*, Bull. Acad. Pol., Cl. III, 5 (1957), pp. 35-37.

[2] R. Fraïssé, *Sur les rapports entre la theorie des relations et la semantique au sens de A. Tarski*, Colloque de logique mathematique, Paris 1952.

[3] — *Sur les classifications des systems de relations*, Publications Sc. de l'Université D'Alger I, No I, juin 1954.

[4] A. Mostowski and A. Tarski, Bull. Amer. Math. Soc. 55 (1949-1950), p. 65.

[5] J. C. C. Mc Kinsey, *Introduction to the theory of games*, New York, Toronto, London 1952.

[6] A. Tarski, *Some model-theoretical results concerning weak second order logic*, Notices, Vol. 5, No. 6, Issue 34, pp. 550-556.

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