Mappings into normed linear spaces

by

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We contribute a few new fragments to a still fragmentary theory—that of the topological structure of infinite-dimensional normed linear spaces. § 1 is concerned with a problem of Fréchet [6] and Banach [1]: Are all infinite-dimensional separable Banach spaces homeomorphic? Kadeč [7, 8] recently obtained an affirmative answer for the case of reflexive spaces. With the aid of a mapping theorem of Whyburn [29], we are able to extend the reasoning of [8] to cover all infinite-dimensional separable conjugate spaces. § 2 begins with some remarks on linear transformations of spaces $l\aleph$, extending a result of Banach and Mazur [2]. In conjunction with a theorem of Bartle and Graves [3], this leads to some interesting corollaries such as an embedding theorem of Dowker [5] and the fact that every metric space of cardinality $\leqslant \mathfrak{c}$ admits a biunique continuous map onto some totally bounded metric space (1). An example in § 2 substantiates a conjecture in Michael's selection theory [24]. A few other results are obtained and some unsolved problems are stated.

§ 1. The theorem of Kadeč.

A subset X of a metric space will be called a *Tchebycheff set* provided each point of the space admits a unique nearest point in X. An *admissible norm* for a normed linear space is one which generates the same topology as the given norm.

Kadeč first proved [7] that all infinite-dimensional separable uniformly convex Banach spaces are homeomorphic, then later observed [8] that the relevant consequences of uniform convexity can be obtained in more general spaces. By careful analysis of his reasoning, one arrives at the following conclusion.

1.1. Theorem (Kadeč). *Two infinite-dimensional normed linear spaces E_1 and E_2 are homeomorphic if (for $i=1,2$) there exist an admissible norm $\| \ \|$ for E_i, a linear subspace F_i of the conjugate space E_i^*, and a linearly independent sequence f_a in F_i such that the following three conditions are satisfied:*

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(1) Added in proof: A simpler proof of this fact has been communicated to the author by Professor H. H. Corson.

1° *the unit cell* $\{x \in E_i\colon \|x\| \leqslant 1\}$ *is compact in the topology* $\sigma(E_i, F_i)$;

2° *with* $L_n = \{x \in E_i\colon 0 = f_1x = \ldots = f_nx\}$, $\bigcap_1^\infty L_n = \{0\}$ *and each* L_n *is a Tchebycheff set in* E_i;

3° *if* x_a *is a sequence in* E_i, $x \in E_i$, $\|x_a\| \to \|x\|$, *and* $fx_a \to fx$ *for all* $f \in F_i$, *then* $\|x_a - x\| \to 0$.

The following corollary of 1.1 may be of interest in connection with the problem of Fréchet and Banach. Notice that *a priori* it does not seem obvious even that E is separable.

1.2. Corollary. *Suppose that* E *is an infinite-dimensional linear subspace of* $R^{\aleph_0}$, *topologized by means of a norm which satisfies the following two conditions*:

1° *the unit cell* $\{x = (x^1, x^2, \ldots) \in E\colon \|x\| \leqslant 1\}$ *is compact in the topology of pointwise convergence*;

2° *if* x_a *is a sequence in* E, $x \in E$, $\|x_a\| \to \|x\|$, *and* $x_a^n \to x^n$ *for each* n, *then* $\|x_a - x\| \to 0$.

Then the space E *is homeomorphic with Hilbert space.*

Proof. It is evident that Hilbert space satisfies the conditions of 1.1. We wish to prove the same for the space E. For $n = 1, 2, \ldots$, let the linear functional g_n on E be defined as follows: $g_n(x^1, x^2, \ldots) = x^n$. From 1° it follows that $\sup_{\|x\| \leqslant 1} |g_n x| = B_n < \infty$, whence g_n is continuous for the norm topology. Let F denote the linear extension of $\{g_n\}_1^\infty$, so that $F \subset E^*$. For each $x \in E$, let

$$(x) = \Big[\sum_1^\infty (2^{-n/2} x_n / B_n)^2\Big]^{1/2}.$$

Then $(\ )$ is a norm for E and $(\ ) \leqslant \|\ \|$, so the function $((\ ))$, with

$$((x)) = \|x\| + (x),$$

is an admissible norm for E. It can be verified that the unit cell $\{x \in E\colon (x) \leqslant 1\}$ is compact in the topology of pointwise convergence and that if x_a is a sequence in E, $x \in E$, $\|x_a\|$ is bounded, and $x_a^n \to x^n$ for each n, then $(x_a - x) \to 0$.

Since both $\|\ \|$ and $(\ )$ satisfy condition 1°, it follows that both are lower semicontinuous for the topology of pointwise convergence, whence $((\ ))$ is lower semicontinuous and satisfies condition 1°. Consequently, the unit cell $\{x \in E\colon ((x)) \leqslant 1\}$ is compact in the topology $\sigma(E, F)$. Now consider a sequence x_a in E and a point $x \in E$ such that $((x_a)) \to ((x))$ and $fx_a \to fx$ for all $f \in F$. Then of course $\|x_a\| + (x_a) \to \|x\| + (x)$, and $x_a^n \to x^n$ for all n. Clearly $\|x_a\|$ is bounded and consequently $(x_a - x) \to 0$, whence $(x_a) \to (x)$ and $\|x_a\| \to \|x\|$. Thus from 2° it follows that $\|x_a - x\| \to 0$ and hence $((x_a - x)) \to 0$. We have proved that the triple $(E, ((\ )), F)$ satisfies conditions 1° and 3° of 1.1.

Now it is easy to produce a subsequence f_a of g_a such that f_a is linearly independent over E and such that if $L_n = \{x \in E\colon 0 = f_1x = \ldots = f_nx\}$, then $\bigcap_1^\infty L_n = \{0\}$. To complete the proof it remains only to show that each L_n is a Tchebycheff set in E. Consider an arbitrary point $y \in E$ and let $U = \{x \in E\colon ((x)) \leqslant 1\}$. From compactness of U (in the topology of pointwise convergence) and the special form of L_n, it follows that the set $(y + tU) \cap L_n$ is also compact for every $t \geqslant 0$. Thus there exists a smallest value t_0 of t among those for which the intersection is nonempty. But it is easy to verify that the norm $(\ )$ is strictly convex, whence $((\ ))$ is also strictly convex and the set $\{x \in E\colon ((x)) = 1\}$ cannot contain any line segment. Consequently the set $(y + t_0 U) \cap L_n$ consists of a single point of L_n, and the set L_n must be a Tchebycheff set. It now follows from 1.1 that E is homeomorphic with Hilbert space.

Although Kadeč considers only pointwise convergence over the entire interval $[0, 1]$, his reasoning in [8] actually establishes the following result:

1.3. Theorem (Kadeč). *In the space* $C[0, 1]$, *let the norm* $\|\ \|$ *be defined as follows*:

$$\|\varphi\| = \max_{t \in [0,1]} |\varphi t| + \Big[\int_0^1 \varphi^2\Big]^{1/2} + \sum_{k=1}^\infty 2^{-k} \max_{|r-s| \leqslant 1/k} |\varphi r - \varphi s|.$$

Then $\|\ \|$ *is a strictly convex admissible norm for* $C[0, 1]$ *which satisfies the following two conditions*:

K^1 —*for each set* M *of measure* 1 *on* $[0, 1]$, *the unit cell* $\{\varphi \in C[0, 1]\colon \|\varphi\| \leqslant 1\}$ *is closed* (*in* $C[0, 1]$) *in the topology of pointwise convergence on* M;

K^2 —*if* M *is a set of measure* 1 *on* $[0, 1]$, φ_a *is a sequence in* $C[0, 1]$, $\varphi \in C[0, 1]$, $\|\varphi_a\| \to \|\varphi\|$, *and* $\varphi_a t \to \varphi t$ *for each* $t \in M$, *then* $\|\varphi_a - \varphi\| \to 0$.

Since ([1]) every separable Banach space is linearly homeomorphic with a linear subspace of $C[0, 1]$, there results

1.4. Corollary (Kadeč). *Every separable Banach space admits a strictly convex norm* $\|\ \|$ *such that whenever* x_a *is a sequence in* E, $x \in E$, $\|x_a\| \to \|x\|$, *and* x_a *is weakly convergent to* x, *then* $\|x_a - x\| \to 0$.

From 1.1 and 1.4 it follows that all infinite-dimensional separable reflexive Banach spaces are homeomorphic. In order to sharpen that result, we derive the following consequence of 1.3.

1.5. Corollary. *If* E *is a Banach space whose conjugate space* E^* *is separable, then there is an admissible norm for* E *such that the corresponding norm in* E^* *is strictly convex and such that whenever* f_a *is a sequence in* E^*, $f \in E^*$, $\|f_a\| \to \|f\|$, *and* f_a *is* w^*-*convergent to* f, *then* $\|f_a - f\| \to 0$.

Proof. Let D be the set of all dyadic rationals in $[0,1]$. By a well-known type of construction, there exists a set M of measure 1 in $[0,1]$ and a continuous map ζ of $[0,1]$ onto itself such that $\zeta M = D$. (See, for example, [21], p. 48-50.) Let U denote the unit cell of E, U^{**} that of the second conjugate space E^{**}, and τ the canonical embedding of E into E^{**}. Then τU is a w^*-dense subset of the w^*-compact convex set U^{**}. Since E^* is separable, the space (U^{**}, w^*) must be homeomorphic with the Hilbert parallelotope ([10], p. 31). In particular, the space (U^{**}, w^*) is a Peano continuum, and by a mapping theorem of Whyburn ([29], p. 31) there exists a continuous map η of $[0,1]$ onto (U^{**}, w^*) such that $\eta D \subset \tau U$.

For each $f \in E^*$, let μf be the function in $C[0,1]$ defined as follows:

$$(\mu f)t = (\eta\zeta t)f,$$

where of course $t \in [0,1]$ and $\eta\zeta t \in U^{**}$. Then μ is a linear homeomorphism of E^* into $C[0,1]$ under which the w^* topology in E^* corresponds to the topology in $C[0,1]$ given by pointwise convergence on the set $\zeta^{-1}\eta^{-1}(\tau U) \supset M$. For each $f \in E^*$, define $\|f\| = \|\mu f\|$, where the second $\|\ \|$ is Kadeč's norm for $C[0,1]$. Property K^1 in 1.3 implies that the unit cell $\{f \in E^*: \|f\| \leqslant 1\}$ is w^*-closed, whence the norm for E^* is generated by an admissible norm for E. The other properties claimed in 1.5 follow at once from the corresponding properties of Kadeč's norm.

We may now establish the principal result of the present section.

1.6. Theorem. *Suppose that S is a normed linear space and Z is an infinite-dimensional separable linear subspace of S^* which is closed in the w^*-topology. Then Z is homeomorphic with Hilbert space.*

Proof. Let z_a be a sequence dense in Z, and for each j let w_a^j be a sequence in the unit cell of S for which $\lim_{i\to\infty} z_j(w_i^j) = \|z_j\|$. Let F denote the closed linear extension of the set of all points w_i^j and let E denote the subset of F^* consisting of all $x \in F^*$ such that x is the restriction to F of some member of Z. Then E is isometric with Z and is a w^*-closed subspace of F^*. Clearly there exists a sequence f_a in F which is linearly independent over E and such that $\bigcap_1^\infty \{x \in E: f_n x = 0\} = \{0\}$, where we make the usual identification of F with a subspace of E^*. Now let F be given the sort of norm described (for E) in 1.5 and let $(\!(\ )\!)$ be the corresponding norm for F^*. It is easy to verify that conditions 1°-3° of 1.1 are satisfied by the space E, the norm $(\!(\ )\!)$, the subspace F of E^*, and the sequence f_a in F. Thus the desired conclusion follows from 1.1.

1.7. Corollary. *All infinite dimensional separable conjugate spaces are homeomorphic.*

§ 2. Linear mappings of $l\aleph$.

The algebraic dimension and density character of a normed linear space E will be denoted by d_E and δ_E respectively. It is well-known that $\operatorname{card} E = d_E$ when $d_E \geqslant \mathfrak{c}$, $\operatorname{card} E = \mathfrak{c}$ when $1 \leqslant d_E \leqslant \mathfrak{c}$, $d_E \geqslant \mathfrak{c}$ when $d_E \geqslant \aleph_0$ and E is complete, and $\delta_E \leqslant d_E$ when $d_E \geqslant \aleph_0$. (See, for example, [17] and [18], p. 159.) For a set Z, we denote by lZ the linear space of all real-valued functions φ on Z for which $\|\varphi\| < \infty$, where $\|\varphi\| = \sum_{z \in Z} |\varphi z|$; $l_F Z$ is the linear subspace of lZ consisting of all $\varphi \in lZ$ such that $\varphi z = 0$ for all but finitely many $z \in Z$. For a cardinal number $\aleph$, $l\aleph$ will denote (somewhat ambiguously) a space lZ such that $\operatorname{card} Z = \aleph$, and $l_F\aleph$ the corresponding subspace. Spaces $l^2\aleph$ and $l_F^2\aleph$ are similarly defined.

Banach and Mazur observed that every separable Banach space is a continuous linear image of the space $l\aleph_0$ ([2], p. 111). We may extend this result as follows:

2.1. Proposition. *Suppose that E is a normed linear space of dimension $d_E \geqslant \aleph_0$ and density character δ_E. Then for the existence of a continuous linear transformation which—*

carries $l_F\aleph$ biuniquely onto E, it is necessary and sufficient that $\aleph = d_E$;

is open and carries $l_F\aleph$ onto E, it is sufficient that $\aleph \geqslant \operatorname{card} E$;

is open and carries $l\aleph$ onto E it is necessary that $\aleph \geqslant \delta_E$, (when E is complete, this is also sufficient);

carries $l\aleph$ biuniquely onto a dense subspace of E, it is sufficient that E is complete, $\delta_E = \aleph_0$, and $\aleph_0 \leqslant \aleph \leqslant \mathfrak{c}$.

Proof. Let Z be a set of cardinality $\aleph$. For all $z \in Z$ and $z' \in Z \sim \{z\}$, set $\varphi_z z = 1$ and $\varphi_z z' = 0$. Then every map g of the set $\{\varphi_z: z \in Z\}$ onto a bounded subset B of E generates a continuous linear transformation T_g of the space $l_F Z$ onto the linear extension of B. (Continuity of T_g follows from subadditivity and positive homogeneity of the norm in E, in conjunction with the special form of the norm in lZ.) For the first assertion above, let B be a bounded Hamel basis for E and let g be biunique. For the second, let B be the unit cell of E. For the third, let B be a dense subset of the unit cell in E, with $\operatorname{card} B \leqslant \aleph$, and observe that when E is complete the transformation T_g admits a continuous linear extension u to lZ. As in [2], it can be verified that $u(lZ) = E$. That u is open can be verified directly or deduced from the open mapping theorem [1].

Under the hypotheses for the fourth assertion, E is an infinite-dimensional separable Banach space. From [12], p. 193, it follows that E admits a bounded *quasi-basis*—that is, a bounded sequence x_a whose linear extension is dense in E and such that whenever a_a and b_a are sequences of real numbers with $\sum_1^\infty a_i x_i = \sum_1^\infty b_i x_i$, then $a_j = b_j$ for all j. Let U^* denote the unit cell of the conjugate space $(lZ)^*$. Then in the usual way, U may be identified with the set of all real-valued functions ξ on Z for which $\sup_{z\in Z}|\xi z|\leqslant 1$—that is, with the product space $[-1,1]^Z$. It is easily verified that the w^*-topology for U coincides with the product topology. Since $\aleph\leqslant\mathfrak{c}$, a theorem of Marczewski ([19], p. 139) implies the existence of a countable sequence f_a which is w^*-dense in U. Clearly $\bigcap_1^\infty f_i^{-1}0=\{0\}$, and f_a admits a linearly independent subsequence g_a such that $\bigcap_1^\infty g_i^{-1}0=\{0\}$. For each $\varphi\in lZ$, let

$$T\varphi=\sum_1^\infty 2^{-n}(g_n\varphi)x_n\in E.$$

If $\sup_n\|x_n\|=M<\infty$, it can be verified that $\|T\varphi\|\leqslant M\|\varphi\|$, and hence T is continuous. Since the sequence g_a is linearly independent, it follows that $T(lZ)$ contains the linear extension of $\{x_n\}_1^\infty$, hence is dense in E. And the strong linear independence of x_a (from the definition of quasi-basis) implies that T is biunique. The proof of 2.1 is complete.

We next state a consequence of a theorem of Bartle and Graves ([3], p. 404). It was proved also by Michael [22, 23].

2.2. THEOREM (Bartle-Graves). *If E and F are Banach spaces and u is a continuous linear transformation of F onto E, then there exists a continuous map v of E into F such that $uvx=x$ for every $x\in E$.*

2.3. COROLLARY. *Let G denote the kernel $u^{-1}0$ of u, and for each $y\in F$ let $hy=(uy, vuy-y)\in E\times G$. Then h is a homeomorphism of F onto $E\times G$.*

The proof of 2.2 is based on the fact that every metric space is paracompact (A. H. Stone [28]). From 2.1 and 2.3 we obtain a new proof of an embedding theorem of Dowker ([5], p. 939), whose proof used paracompactness directly. While the proof is more complicated than Dowker's, it may nevertheless be of interest.

2.4. COROLLARY (Dowker). *If a metric space is of density character $\leqslant\aleph$, then it can be topologically embedded in the space $l\aleph$.*

Proof. It suffices to consider the case $\aleph\geqslant\aleph_0$. Consider a metric space (M,ϱ) having a dense subset D of cardinality $\leqslant\aleph$. Choose $q_0\in M$, and for each $q\in M$ let the real-valued function φ_q on M be defined as follows:

$$\varphi_q p=\varrho(p,q)-\varrho(p,q_0)\quad (p\in M).$$

For each $q\in M$, let $\tau q=\varphi_q\in CM$ (the space of all bounded continuous real-valued functions on M, with $\|\xi\|=\sup_{x\in M}|\xi x|$). Then τ is an isometric embedding of (M,ϱ) into CM, due essentially to Kuratowski ([16], p. 543). Let L denote the rational linear extension of the set τD, and let E denote the closure of L. Then $\operatorname{card}L\leqslant\aleph$ and E is a complete linear subspace of CM, so it follows by 2.1 that the space $l\aleph$ admits a continuous linear transformation onto E. The desired conclusion now follows from 2.3.

Dowker actually embedded in $l^2\aleph$ rather than $l\aleph$, but the two spaces are homeomorphic under Mazur's mapping [20] which sends the function $\varphi\in l^2\aleph$ into the function $(\operatorname{sgn}\varphi)\varphi^2\in l\aleph$.

2.5. COROLLARY. *For a metric space X, the following three assertions are equivalent:*

1° *X is of cardinality $\leqslant\mathfrak{c}$;*

2° *X admits a biunique continuous map into a compact metric space (that is, onto a totally bounded metric space);*

3° *there is a sequence G_a of open subsets of X such that whenever p and q are distinct points of X, then for some i, $p\in G_i\subset X\sim\{q\}$.*

Proof. It is easily verified that 2° implies 1° and 2° implies 3°. To prove that 1° implies 2° it suffices, in view of 2.4, to show that the space $l\mathfrak{c}$ admits a biunique continuous map into a compact metric space. But by 2.1, $l\mathfrak{c}$ must admit a biunique continuous map into Hilbert space $l^2\aleph_0$, and of course Hilbert space is homeomorphic with a subset of the (compact) Hilbert parallelotope. It remains only to show that 3° implies 2°.

Let ϱ be a bounded metric for X and let G_a be a sequence of open sets as described in 3°. For each $x\in X$ and each positive integer n, let

$$f_n x=\frac{1}{n}\inf_{y\in X\sim G_n}\varrho(x,y).$$

Then set $\varphi x=(f_1x, f_2x, \ldots)\in l^2\aleph_0$. Clearly φ is biunique. It is easy to see that φ is continuous and φX lies in a compact subset of $l^2\aleph_0$. The proof of 2.5 is complete.

Parhomenko [25] has treated some related problems, characterizing topological spaces which admits biunique continuous maps onto certain types of spaces. Still open is Banach's problem of characterizing those metric spaces, and especially those Banach spaces, which admit a biunique continuous map *onto* some compact metric space. Aspects of his problem have been treated in [9], [11], [13], [27]. It is known that the space (c_0), every separable conjugate space, and the space (m) all admit biunique continuous maps onto the Hilbert parallelotope. Does the space $l\mathfrak{c}$ admit such a map? Does $l\mathfrak{c}$ admit a homeomorphism into (m) or a biunique continuous map onto $l\aleph_0$? It can be seen that $l_F\mathfrak{c}$ admits a biunique continuous map onto a compact metric space, while $l_F\aleph_0$ does not.

Now consider a set S and the corresponding product space S^n, where n is a positive integer. In [12] we defined a *cross-section* of S^n to be a subset C_n of S^n such that if $(x_1,\ldots,x_n)\in C_n$, then the x_i's are all distinct, and such that whenever $y_1, \ldots, y_n$ are n distinct points of S, there is exactly one permutation $z_1, \ldots, z_n$ of the y_i's for which $(z_1, \ldots, z_n) \in C_n$.

2.6. COROLLARY. *If S is a metric space of cardinality $\leqslant \mathfrak{c}$, then S^n admits a cross-section which is the union of a countable family of closed sets.*

Proof. Let ξ be a biunique continuous map of S into a compact metric space K, and for $(x_1, \ldots, x_n) \in S^n$ define $\eta(x_1, \ldots, x_n) = (\xi x_1, \ldots, \xi x_n) \in K^n$. By [12], p. 191, there is an F_σ cross-section C_n for K^n, and then the set $\eta^{-1}(C_n \cap \eta S^n)$ is an F_σ cross-section for S^n.

Now from 2.6 and [12], p. 196, we deduce

2.7. COROLLARY. *Suppose that E is a normed linear space of cardinality $\leqslant \mathfrak{c}$ (or, equivalently, of dimension $\leqslant \mathfrak{c}$). Then in the unit sphere $\{x \in E: \|x\| = 1\}$ there is a countable sequence of closed sets whose union includes exactly one point from each antipodal pair.*

Corollary 2.7 should be compared with the antipodal point theorems of Lyusternik and Schnirelmann, and Borsuk [4]. I have no satisfactory results concerning extension of 2.6 and 2.7 to spaces of higher cardinality. With the aid of 2.6, some of the results of [12] can be extended. In particular, „separable metric spaces" may be replaced by „metric spaces of cardinality $\leqslant \mathfrak{c}$" in 2.1 of [12].

We wish, finally, to apply 2.1 to prove a conjecture of Michael [24]. For this purpose, we introduce two additional definitions. A separable metric space is *strongly infinite-dimensional* provided each of its nonempty open subsets is infinite-dimensional. A normed linear space E is *accessible* provided it admits a Hamel basis B such that whenever $J \subset B$ and card $J <$ card B, then the linear extension of J is closed in E. Clearly every $\aleph_0$-dimensional normed linear space is accessible, as is every space $l_F\aleph$.

2.8. PROPOSITION. *Suppose that X is a separable metric space which is strongly infinite-dimensional and Y is a closed subset of X which can be topologically embedded in an accessible normed linear space. Then Y is of the first category in X.*

Proof. The statement is obvious for all finite-dimensional normed linear spaces. Suppose it is known for all spaces of dimension $< \aleph$, and consider an $\aleph$-dimensional normed linear space E with Hamel basis B. Let B be well-ordered by a reflexive relation $\preccurlyeq$ such that for each $b \in B$, the set $P_b = \{d \in B: d \preccurlyeq b\}$ is of cardinality $< \aleph$. Consider a strongly infinite-dimensional separable metric space X, a closed subset Y of X, and a homeomorphism h of Y into E. For each $b \in B$, let L_b denote the linear extension of P_b; then L_b is a closed subspace of E and is itself an accessible normed linear space. For each b, let S_b denote the set $h^{-1}(L_b \cap hY)$. Then $Y = \bigcup_{b \in B} S_b$, each set S_b is closed, and $b \preccurlyeq b'$ implies $S_b \subset S_{b'}$. By a well-known property of separable metric spaces ([15], p. 146) there must exist a countable sequence $b_1 \preccurlyeq b_2 \preccurlyeq \ldots$ in B such that $\bigcup_{b \in B} S_b = \bigcup_{i=1}^{\infty} S_{b_i}$. But each set S_{b_i} is of the first category in X by the inductive hypothesis, so the same must be true of their union, the set Y. The proof is complete.

Since Hilbert space $l^2\aleph_0$ is of the second category in itself, 2.8 may be regarded as an extension of Kunugui's result [14] that $l^2\aleph_0$ cannot be topologically embedded in $l^2_F\aleph_0$.

Michael considered transformations φ (called *carriers*) which map a topological space X into the class of nonempty subsets of a topological space Y. A *selection* for φ is a continuous map f of X into Y such that $fx \in \varphi x$ for all $x \in X$. Michael proved ([22], [23]):

M⁰. *If X is a paracompact T_1-space and φ is a lower semicontinuous carrier mapping X into the class of nonempty closed convex subsets of a Banach space Y, then φ admits a selection.*

From M⁰ he deduced the result 2.2 above. He showed by example ([23], p. 374) that M⁰ may fail when Y is not complete. In [24] he considered continuous carriers, and conjectured ([24], p. 389) that even when φ is continuous, M⁰ may fail if Y is not complete. He also conjectured that 2.2 may fail in the absence of completeness. Now of course 2.2 can fail in a trivial way—use 2.1 to produce a *biunique* continuous linear map u of the space $l_F\mathfrak{c}$ onto a separable Banach space and observe that u^{-1} is discontinuous. But then u^{-1} is also not a continuous carrier in Michael's sense. Now consider an arbitrary infinite-dimensional separable Banach space E. By the second part of 2.1, there exists a continuous open linear transformation u of $l_F\mathfrak{c}$ onto E, and the carrier u^{-1} is continuous in the sense of [24], p. 377. The carrier u^{-1} does not admit a continuous selection (and the transformation u does not admit a continuous inverse in the sense of 2.2) because such a selection would be a homeomorphism of E into $l_F\mathfrak{c}$, and by 2.8 no such homeomorphism exists.

Note that in 2.2, the transformation u must be open by the open mapping theorem [1]. If it be assumed, on the other hand, that F is a Banach space and u is a continuous open linear transformation of F onto the normed linear space E, then completeness of E follows from a theorem of Ptak ([26], p. 70).

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Arithmetization of metamathematics in a general setting

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1. Introduction

The method of arithmetization, as developed by Gödel [10], exploits the possibility of defining within a formal theory $\mathcal{T}$, or in arithmetical theories closely related to $\mathcal{T}$, various syntactical and logical notions concerning $\mathcal{T}$. In broad terms, the applications of the method can be classified as being *extensional* if essentially only numerically correct definitions are needed, or *intensional* if the definitions must more fully *express* the notions involved, so that various of the general properties of these notions can be formally derived.

The following are some results of extensional type: incompleteness theorems (Gödel's first underivability theorem [10] Satz VI, Rosser [29] Theorem II); non-definability of predicates in formal theories (Tarski [31], Kleene [15] Theorem XIII); undecidability of various theories (Rosser [29] Theorem III, Tarski, Mostowski and Robinson [32]); and degrees of unsolvability of various theories (Myhill [25], our [7]). Among the intensional results we have the following: unprovability of consistency statements (Gödel's second underivability theorem [10] Satz XI), comparison of theories by relative consistency proofs (Novak [26], Wang [36], [37], Shoenfield [30]); and ordinal logics (Turing [33], our [8]). A result of mixed character is the arithmetization of Gödel's completeness theorem

* The results reported in this paper were obtained while the author was a student of Professor Alfred Tarski at the University of California, Berkeley. A more complete presentation of them has been given in the author's thesis [4]; announcement of the results has also been made in [5] and [6].

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