Measures in homogenous spaces

by

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1. Notation. Generally our notation will follow that of Weil [W] and Halmos [H]. Let G be a locally compact topological group, H a closed subgroup. Let G/H be the homogeneous space of cosets xH with the usual topology so that G acts, by left translation, as a transitive group of homeomorphisms of G/H. The natural mapping $G \to G/H$ will be denoted by φ but sometimes we shall use the shorter notation $\bar{x}$ instead of $\varphi(x)$ for the projection xH of x in G/H. We shall also use $\bar{x}$ to denote a generic element of G/H. We use dx, $d\xi$ to denote integration with respect to the Haar measures in G, H, and $\Delta(x), \delta(\xi)$ to denote the modular functions in G, H ([W], p. 39).

For any topological space X, $L(X)$ denotes the class of continuous real-valued functions with compact support and $L_+(X)$ denotes the subclass consisting of non-negative functions. Similarly $B(X)$ denotes the class consisting of all extended real-valued Baire functions on X, $B_+(X)$ the non-negative ones. (Extended real numbers include the values $\pm\infty$ as well as the ordinary real numbers.)

A set $Q \subset X$ will be called an LB-*set (locally Baire)* if $Q \cap E$ is a Baire set whenever E is a Baire set. A function which is measurable with respect to the ring of LB-sets will be called an LB-function. It is convenient to extend the notion of a set of measure zero to LB-sets as follows. If Q is an LB-set and μ is a Baire measure we say that $\mu(Q) = 0$ provided that $\mu(Q \cap E) = 0$ for each Baire set E. If $\mu(Q) = 0$ then we say that *almost every x in X belongs to $X - Q$*. If f, g are LB-functions, N is the set $\{x: f(x) \neq g(x)\}$, we say that $f = g[\mu]$ if $\mu(N) = 0$. These definitions do not introduce anything new if X is a σ-compact space.

All measures we consider are non-negative Baire measures in the sense that they are defined on the ring of all Baire sets; our usage of the term "Baire measure" differs thus from that of Halmos [H], where a Baire measure is assumed to be finite on compact sets.

2. Definitions and main results. A Baire measure μ on G/H is called (following Weil) *relatively invariant with factor* $h(x)$ if $\mu(xE) = h(x)\mu(E)$ for each Baire set E and $x \in G$. Then $h(xy) = h(x)h(y)$

and Weil ([W], p. 42-45) showed that such a measure can exist only if

(1) $$h(\xi)\Delta(\xi) = \delta(\xi) \quad \text{for each} \quad \xi \in H.$$

If G admits no non-trivial homomorphism into the multiplicative group of positive reals, and H is not unimodular, there can be no relatively invariant measure in G/H. This situation occurs when G is the group of 3 by 3 real matrices with determinant 1 and H is the group of matrices of the form

$$\begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus if we wish to have a class of measure which exists for every homogeneous space G/H, we must weaken our demands about invariance. In this paper we define a class of measure with an invariance property which is weak enough to guarantee that such measures exist but which turns to be strong enough to imply a connection with the Haar measure.

DEFINITION 1. A non-vanishing Baire measure μ in G/H which is finite on compact sets is called *pseudo-invariant* if, for each pair of compact sets $C_1 \subset G$, $C_2 \subset G/H$, there is a finite real number k such that when $E \subset C_2$, $t \in C_1$, $\mu(tE) \leqslant k\mu(E)$. In particular $\mu(E) = 0$ if and only if $\mu(tE) = 0$.

A pseudo-invariant measure is positive on every non-empty open set. For if U is open and $\mu(U) = 0$, then $\mu(tU) = 0$. Each compact set C can be covered by a finite union of sets tU, so $\mu(C) = 0$ and the measure vanishes contrary to definition.

If $f \in L_+(G)$, then the expression $\int f(x\xi)d\xi$, regarded as a function of x is constant on cosets xH and is therefore really a function $\bar{f}(\bar{x})$. It is well known that $\bar{f} \in L_+(G/H)$. The mapping of $L_+(G)$ in $L_+(G/H)$ defined by

(2) $$\bar{f}(\bar{x}) = \int f(x\xi)d\xi$$

is linear and monotone. Since (2) is invariant under taking limits of monotone sequences of non-negative functions, we derive that (2) defines also a mapping of $B_+(G)$ in $B_+(G/H)$. In particular, if E is a Baire set in G and χ_E denotes the characteristic function of E, then $\bar{\chi}_E(\bar{x}) \in B_+(G/H)$.

Let μ be a Baire measure in G/H. For every Baire set $E \subset G$ define

$$\tilde{\mu}(E) = \int \bar{\chi}_E(\bar{x})\,d\mu(\bar{x}).$$

It is obvious that $\tilde{\mu}$ is a Baire measure in G.

DEFINITION 2. The measure μ will be called *inherited* if $\tilde{\mu}$ is absolutely continuous with respect to the Haar measure.

If μ is inherited, then, by the generalized Radon-Nikodym theorem (proved in § 3) there is a non-negative LB-function $h(x)$ such that

$$\int h(x)\chi_E(x)\,dx = \int \bar{\chi}_E(\bar{x})\,d\mu(\bar{x}).$$

The function h is called the *factor function* for the inherited measure μ. It is obvious that the above equality implies

$$\int h(x)f(x)\,dx = \int \bar{f}(\bar{x})\,d\mu(\bar{x})$$

for every $f \in B_+(G)$.

Our main results are as follows. We follow Halmos in calling two measures *equivalent* if each is absolutely continuous with respect to the other.

THEOREM 1. *For any G, H there exists at least one pseudo-invariant measure in G/H. Any two pseudo-invariant measures in G/H are equivalent.*

THEOREM 2. *An* LB*-function $h(x)$ is the factor function for an inherited measure if and only if, for each $\xi \in H$, and for almost all x (in the Haar measure)*

(3) $$h(x\xi)\Delta(\xi) = h(x)\delta(\xi).$$

(This theorem is a generalization of Weil's formula (1).)

THEOREM 3. *A measure μ in G/H is pseudo-invariant if and only if μ is inherited and the factor function $h(x)$ is essentially bounded away from 0 and ∞ on each compact set (i. e. for each compact set C there are real positive numbers k_1, k_2 such that $k_1 \leqslant h(x) \leqslant k_2$ holds for almost all $x \in C$).*

3. The Radon-Nikodym theorem for G. In this section we justify the use made of the Radon-Nikodym theorem in the last section to obtain the factor function $h(x)$, even when the usual condition of total σ-finiteness is not satisfied. Our proof is based on a condition of Oxtoby, as indicated by Halmos ([H], p. vii; p. 132, Ex. 10; p. 256, Ex. 7).

THEOREM A. *If ν_1 and ν_2 are Baire measures in G, ν_2 is finite on compact sets and ν_1 is absolutely continuous with respect to ν_2, then there exists a non-negative* LB*-function $h(x)$ such that, for each $f \in B(G)$,*

(4) $$\int f(x)\,d\nu_1(x) = \int f(x)h(x)\,d\nu_2(x).$$

The function h is unique in the sense that if h^0 also has the above property, then $h = h^0[\nu_2]$.

We first show that G satisfies Oxtoby's condition, i. e. that G is the union of a disjoint class $\boldsymbol{D}$ of Baire sets of finite ν_2 measure with the property that every Baire set can be covered by a countable subclass of $\boldsymbol{D}$. To show this consider an open subgroup Γ of G which is σ-compact.

Every Γ-coset is a countable union of disjoint bounded sets, let $\boldsymbol{D}$ be the family of all these bounded sets. Now every compact Baire set is contained in a finite union of Γ-cosets since these are open, and therefore also in a countable union of sets $D \in \boldsymbol{D}$. Since every Baire set belongs to a subring generated by countably many compact sets ([H], p. 24, Theorem D), the family $\boldsymbol{D}$ does what is required.

To prove Theorem 4, apply the Radon-Nikodym theorem (in the form given in [H], p. 131, § 31, Ex. 7) to each space $D \in \boldsymbol{D}$. On each space D there is a function $h_D \in B_+(D)$ such that, for every function $f_D \in B(D)$

$$\int f_D(x)\,d\nu_1(x) = \int h_D(x) f_D(x)\,d\nu_2(x)\,;$$

or, for every $f \in B(G)$,

$$\int_D f(x)\,d\nu_1(x) = \int_D h_D(x) f(x)\,d\nu_2(x)\,.$$

The function h_D with this property is essentially unique. The function $h(x)$ on G such that, for each $D \in \boldsymbol{D}$, $h(x) = h_D(x)$ when $x \in D$, clearly satisfies (4), and any such h is essentially unique.

4. Pseudo-invariant measures on G. In this section we prove Theorems 1, 3 for groups, i. e., we prove the following theorem.

THEOREM B. *Any pseudo-invariant measure on G is equivalent to the Haar measure. More precisely, any pseudo-invariant measure ν is definable by an equation of the form*

$$\int f(x)\,d\nu(x) = \int f(x) W(x)\,dx\,,$$

where W is an LB*-function essentially bounded away from zero and infinity on every compact set.*

To prove that ν is equivalent to the Haar measure, we have to show that, if E is a Baire subset of G, then, on E, ν and the Haar measure both vanish or are both positive. If Γ is a σ-compact open subgroup of G which contains E ([H], § 57, Theorem A), the Haar measure carried over from G will be a Haar measure in Γ, and ν will be pseudo-invariant in Γ. Thus it is enough to prove the equivalence part of Theorem B for Γ instead of G.

Having proved the equivalence, the existence of an essentially unique Radon-Nikodym derivative $W(x)$ follows from Theorem A. The property of $W(x)$ that we have to establish concerns its values on a compact set C, which is also contained in an open σ-compact subgroup Γ_1. Thus it is enough to prove Theorem B for a σ-compact group, and we shall assume, in this section only, that G is σ-compact.

Let $G^2 = G \times G$ be the group of ordered pairs (x, y) with Baire measure $\nu^2 = \nu \times \nu$, and G^3 the group of ordered triples (x, y, z) with

Baire measure ν^3. We denote by ϱ the Baire measure in G^2 which is determined by the condition

$$\int f(x, y)\,d\varrho(x, y) = \int f(x, xy)\,d\nu^2(x, y) \quad \text{for} \quad f \in B_+(G^2)\,.$$

If $f(x, y) \in B_+(G^2)$, then for each fixed x, by definition 1, the functions

$$P(x) = \int f(x, y)\,d\nu(y)\,, \qquad Q(x) = \int f(x, xy)\,d\nu(y)$$

are both zero or both positive. Therefore the integrals

$$\int f\,d\nu^2 = \int P(x)\,d\nu(x)\,, \qquad \int f\,d\varrho = \int Q(x)\,d\nu(x)$$

are both zero or both positive. Thus the measures ϱ and ν^2 are equivalent, and by the Radon-Nikodym theorem there is a positive function $J(x, y)$ such that for each $f \in B(G^2)$

$$\int f(x, xy)\,d\nu^2(x, y) = \int f(x, y) J(x, y)\,d\nu^2(x, y)\,. \tag{5}$$

An analogous argument, carrying out the integration with respect to y first, and then with respect to $\nu^2(x, z)$, will show that, if $f(x, y, z) \in B_+(G^3)$ and $\gamma(x, z)$ is any continuous mapping $G^2 \to G$, then the integrals

$$\int f(x, y, z)\,d\nu^3(x, y, z)\,, \qquad \int f\big(x, \gamma(x, z) y, z\big)\,d\nu^3(x, y, z)$$

both vanish or are both positive. In particular, if T denotes the transformation

$$T(x, y, z) = (x, z^{-1}xy, z)\,, \tag{6}$$

then $\nu^3(E) = 0$ if and ony if $\nu^3(TE) = 0$.

LEMMA 2.1. *The function $J(x, y)$ may be chosen to be bounded away from zero and infinity on every compact set.*

Proof. We can alter $J(x, y)$ on a set of measure zero, so it is enough to show that $J(x, y)$ is essentially bounded on every compact Baire rectangle. Let M, N be compact Baire subsets of G. By definition 1, there are numbers $k_1, k_2 > 0$ such that, for any $f \in B_+(G^2)$ vanishing outside $M \times N$,

$$k_1 \int f(x, y)\,d\nu(y) \leqslant \int f(x, xy)\,d\nu(y) \leqslant k_2 \int f(x, y)\,d\nu(y)\,.$$

Integrate with respect to $\nu(x)$:

$$k_1 \int f\,d\nu^2 \leqslant \int f(x, y) J(x, y)\,d\nu^2 \leqslant k_2 \int f\,d\nu^2\,.$$

Thus $k_1 \leqslant J(x, y) \leqslant k_2$ almost everywhere in $M \times N$, and the lemma follows. We shall assume from now on that J is bounded away from zero and infinity on every compact set.

Let $\Phi(x, y, z) \in L(G^3)$. We shall obtain a functional equation for J by transforming the following integral in two ways:

$$I(\Phi) = \int \Phi(x, y, xyz)\, d\nu^3(x, y, z).$$

Firstly, we have, by Fubini's theorem and (5),

$$(7)\qquad \begin{aligned} I(\Phi) &= \int d\nu(x) \int \Phi(x, y, xyz)\, d\nu^2(y, z) \\ &= \int d\nu(x) \int J(y, z)\Phi(x, y, xz)\, d\nu^2(y, z) \\ &= \int d\nu(y) \int J(x, z)J(y, x^{-1}z)\Phi(x, y, z)\, d\nu^2(x, z) \\ &= \int J(x, z)J(y, x^{-1}z)\Phi(x, y, z)\, d\nu^3(x, y, z). \end{aligned}$$

On the other hand, if we write $g(x, y, z) = J(y, z)\Phi(x, x^{-1}y, z)$, we have

$$(8)\qquad \begin{aligned} I(\Phi) &= \int d\nu(z) \int J(x, y)\Phi(x, x^{-1}y, yz)\, d\nu^2(x, y) \\ &= \int d\nu(x) \int J(x, y)J(y, z)\Phi(x, x^{-1}y, z)\, d\nu^2(y, z) \\ &= \int d\nu(z) \int J(x, y) g(x, y, z)\, d\nu^2(x, y) \\ &= \int d\nu(z) \int g(x, xy, z)\, d\nu^2(x, y) \\ &= \int J(xy, z)\Phi(x, y, z)\, d\nu^3(x, y, z). \end{aligned}$$

Comparing the two expressions (7), (8), equal for all $\Phi \in L(G^2)$, we deduce the equation

$$J(xy, z) = J(y, x^{-1}z)J(x, z)\,[\nu^3].$$

Applying the transformation T defined at (6), we have

$$(9)\qquad J(zy, z) = J(x^{-1}zy, x^{-1}z)J(x, z)\,[\nu^3].$$

If E denotes the subset of G^3 for which the equation (9) is false, and if, for each fixed y, E_y denotes the set of (x, z) for which it is false, then

$$\nu^3(E) = \int \nu^2(E_y)\, d\nu(y) = 0.$$

Thus there is at least one value $y = a$ such that $\nu^2(E_a) = 0$. If $J(za, z)$ is denoted by $V(z)$, then by (9),

$$(10)\qquad V(z) = V(x^{-1}z)J(x, z)\,[\nu^2].$$

Now let $d\pi$ be the integral on G defined by

$$\int f(x)\, d\pi(x) = \int f(x)V(x)\, d\nu(x).$$

Lemma 2.1 shows that $V(x)$ is bounded on every compact set, so that every $f \in L(G)$ is π-integrable. Since $V(x) > 0$, the measures ν and π are equivalent, and Theorem B will follow if we show that π is a Haar measure, i. e., that, for each fixed $f \in L(G)$, the function

$$I(t) = \int f(tx)\, d\pi(x)$$

is a constant. Since f is uniformly continuous, with compact support, the function $I(t)$ is continuous, and the set N of t for which $I(t) \neq I(e)$ is open. We shall show that $\nu(N) = 0$, and Theorem B will follow since the empty set is the only open set with ν-measure zero, as shown in § 2. Let $g \in L(G)$. Apply the formula (5) with the variables (x, y) replaced by (t, x) and the function $f(x, y)$ replaced by $f(x)V(t^{-1}x)g(t)$. Using (10), we have

$$\begin{aligned} \int I(t) g(t)\, d\nu(t) &= \int f(tx)V(x)g(t)\, d\nu^2(t, x) \\ &= \int J(t, x)V(t^{-1}x)f(x)g(t)\, d\nu^2(t, x) \\ &= \int V(x)f(x)g(t)\, d\nu^2(t, x) \\ &= \int g(t)I(e)\, d\nu(t). \end{aligned}$$

Thus, for every $g \in L(G)$, we have $\int [I(t) - I(e)]g(t)\, d\nu(t) = 0$. From this it follows that $\nu(N) = 0$, $N = \emptyset$, and Theorem B follows with $W(x) = 1/V(x)$.

5. On the existence of certain functions. Later it will be necessary to make use of a function $F(x)$ with the properties given in the following theorem, where $S(f)$ denotes the minimal support of f, i. e. $S(f) = \{x: f(x) \neq 0\}$.

THEOREM C. *There exists a continuous non negative function $F(x)$ defined on G with the following properties*

(i) *For each $x \in G$, $\overline{F}(\overline{x}) = 1$;*

(ii) *If $C \subset G/H$ is compact, then $\varphi^{-1}(C) \cap S(F)$ is bounded.*

Let V be any bounded neighbourhood of e. A set $Y \subset G$ is called *(V, H)-separated* if $VyH \cap VzH = \emptyset$ whenever $y, z \in Y$, $y \neq z$.

LEMMA 5.1. *If Y is (V, H)-separated, and $C \subset G/H$ is compact, then the set $\varphi^{-1}(C) \cap Y$ is finite. More generally, if $D \subset G$ is compact, the set of $y \in Y$ for which $\varphi^{-1}(C) \cap Dy \neq \emptyset$ is finite.*

Proof. Let U be any neighbourhood of e such that $UU^{-1} \subset V$. Now $\varphi^{-1}(C)$ can be covered by a finite union of sets UxH (since C is contained in a finite union of sets $\varphi(Ux)$). $\varphi^{-1}(C) \cap Y$ is thus finite, since each set UxH contains at most one point y of Y; for, if, on the contrary, $y, z \in Y \cap UxH$, then, $y \in UU^{-1}zHH^{-1} \subset VzH$, and Y is not (V, H)-separated.

The second part of the lemma follows from the first on replacing the set C by the set $E = D^{-1}C$; since $\varphi^{-1}(C) \cap Dy \neq \emptyset$ if and only if $y \in \varphi^{-1}(E)$. This completes the proof of Lemma 5.1.

Since G is completely regular, there exists a non-negative continuous function $g(x)$, equal to unity within the bounded set $V^{-1}V$, but equal to zero outside some compact set D. Having chosen $g(x)$, we define another function $f(x)$ by the relation

$$f(x) = \sum_{y \in Y} g(xy^{-1}) .$$

The series may contain infinitely (perhaps uncountably) many terms, but, as we shall now show, only a finite number of them differ from zero.

LEMMA 5.2. *Let C be any compact subset of G/H. Then there is a finite subset Y_0 of Y such that, if $\bar{x} \in C$, then,*

$$f(x) = \sum_{y \in Y_0} g(xy^{-1}) .$$

Further $S(f) \cap \varphi^{-1}(C)$ is bounded and the functions f, $\bar{f}$ are continuous.

Proof. If $\bar{x} \in C$, i. e., $x \in \varphi^{-1}(C)$, and $g(xy^{-1}) \neq \emptyset$, we must have $xy^{-1} \in D$, $x \in Dy$. Thus $Dy \cap \varphi^{-1}(C)$ contains the point x and so is non-empty. By Lemma 5.1, this can be true only it y belongs to a certain finite subset Y_0 of Y, and the first part of the lemma is proved.

Next,

$$S(f) \cap \varphi^{-1}(C) = \bigcup_{y \in Y_0} S\big(g(xy^{-1})\big) \cap \varphi^{-1}(C) \subset \bigcup \{Dy:\ y \in Y_0\} .$$

The last set is compact because D is compact and Y_0 is finite.

For the continuity, it is enough to show that f, $\bar{f}$ are continuous on compact subsets, since G, G/H are locally compact. On $\varphi^{-1}(C)$, C compact, f is a finite sum of functions $g_y(x) = g(xy^{-1})$. Since $g_y \in L_+(G)$, g_y and $\bar{g}_y$ are continuous, and the lemma follows.

Proof of Theorem C. Now let Y be a *maximal* (V, H)-separated set, and define $g(x)$, $f(x)$ as before. Then $\bar{f}(\bar{x}) > 0$ for every x; for if $\int f(x\xi)\,d\xi = 0$, then $f(x\xi) = 0$ for each $\xi \in H$, and, for each $y \in Y$, $g(x\xi y^{-1}) = 0$. Hence $x\xi y^{-1} \notin V^{-1}V$, so $Vx\xi \cap Vy = \emptyset$ for each $\xi \in H$. Thus $VxH \cap VyH = \emptyset$ and $Y \cup \{x\}$ is (V, H)-separated, contradicting the hypothesis that Y is maximal.

Set $F(x) = f(x)/\bar{f}(\bar{x})$. Then $\bar{F}(\bar{x}) = 1$, F is the quotient of two continuous functions, and, since $S(f) = S(F)$, Theorem C follows.

6. Inherited measures. Proof of Theorem 2. Let μ be an inherited measure on G/H, with factor function $h(x)$. Thus, for each $f \in B_+(G)$,

$$I(f) = \int \bar{f}(\bar{x})\,d\mu(\bar{x}) = \int f(x)\,h(x)\,dx . \tag{11}$$

Let $\sigma \in H$. If $f_\sigma(x) = f(x\sigma^{-1})$, we have $\bar{f}_\sigma(\bar{x}) = \delta(\sigma)\bar{f}(\bar{x})$, so that

$$\delta(\sigma)I(f) = \int \bar{f}_\sigma(\bar{x})\,d\mu(\bar{x}) = \int f(x\sigma^{-1})\,h(x)\,dx = \Delta(\sigma)\int f(x)\,h(x\sigma)\,dx .$$

Thus, for each $f \in L(G)$,

$$\int f(x)[h(x\sigma)\Delta(\sigma) - h(x)\delta(\sigma)]\,dx = 0 ,$$

proving that the equation (3) must be satisfied for almost all x.

Suppose conversely that $h(x)$ is any LB-function satisfying (3). Let $F(x)$ be any function which satisfies the conditions of Theorem C. For each $g(\bar{x}) \in B_+(G/H)$, we have $g(\bar{x})F(x) \in B_+(G)$. Define a measure μ in G/H by

$$I(g) = \int g(\bar{x})\,d\mu(\bar{x}) = \int g(\bar{x})F(x)\,h(x)\,dx . \tag{12}$$

Then the measure μ is an inherited measure with factor function h since by Fubini's theorem, (3), Theorem C (i) and the properties of Δ, δ ([W], p. 39-40), we have, for $f \in B_+(G)$,

$$\begin{aligned}\int \bar{f}(\bar{x})\,d\mu(\bar{x}) &= \int \bar{f}(\bar{x})F(x)\,h(x)\,dx = \int d\xi \int f(x\xi)F(x)\,h(x)\,dx \\ &= \int \Delta(\xi^{-1})\int f(x)F(x\xi^{-1})\,h(x\xi^{-1})\,dx\,d\xi \\ &= \int dx \int \delta(\xi^{-1})F(x\xi^{-1})f(x)\,h(x)\,d\xi \\ &= \int f(x)\,h(x)\,dx \int F(x\xi)\,d\xi = \int f(x)\,h(x)\,dx .\end{aligned}$$

This completes the proof of Theorem 2. We note that the measure associated with a given factor function must necessarily be unique (if it exists at all) because of (11). Thus we obtain the same measure μ independently of our choice of function $F(x)$ in equation (12), provided that $F(x)$ satisfies the conditions of Theorem C.

7. Pseudo-invariant measures. Let μ be a pseudo-invariant measure on G/H. Then $\tilde{\mu}$ is a pseudo-invariant measure on G, since, if $t \in C_1$ and $S(f) \subset C_2$, $(C_1, C_2 \subset G)$, then $S(\bar{f}) \subset \varphi(C_2)$, and there is a number k such that

$$\int f(tx)\,d\tilde{\mu}(x) = \int \bar{f}(t\bar{x})\,d\mu(\bar{x}) \leqslant k \int \bar{f}(\bar{x})\,d\mu(\bar{x}) = k \int f(x)\,d\tilde{\mu}(x) .$$

It follows from Theorem B, § 4, that μ is inherited, with a factor function which is essentially bounded away from zero and infinity on each compact set.

Conversely, let $h(x)$ be any LB-function satisfying (3), and bounded away from zero and infinity on each compact set. We shall show that the measure defined by the equation (12) of § 6 is pseudo-invariant. We have to show that if C is a compact subset of G and $g \in L_+(G/H)$, then there is a constant k such that, if $t \in C$, $f \leqslant g$, $f \in B_+(G/H)$, then

$$\int f(t\bar{x})\,d\mu(\bar{x}) \leqslant k \int f(\bar{x})\,d\mu(\bar{x}) .$$

The set of x such that, for some $t \in C$, $g(\bar{x})F(t^{-1}x) > 0$ is bounded, as a result of Theorem C (ii). Thus there is a number k such that, if $t \in C$ and $g(\bar{x})F(t^{-1}x) > 0$, then $h(t^{-1}x) \leqslant kh(x)$. Using this inequality and the invariance of the Haar measure, we derive:

$$\int f(t\bar{x})\,d\mu(\bar{x}) = \int f(t\bar{x})F(x)h(x)\,dx = \int f(\bar{x})F(t^{-1}x)h(t^{-1}x)\,dx$$
$$\leqslant k\int f(\bar{x})F(t^{-1}x)h(x)\,dx = k\int f(\bar{x})\,d\mu(\bar{x}) .$$

The last equation holds because $F(t^{-1}x)$ as well as $F(x)$ satisfies the conditions of Theorem C. Thus Theorem 3 is proved.

Finally we prove Theorem 1. Let μ be a pseudo-invariant measure and F again a function satisfying the conditions of Theorem C. Then, if $f \in B_+(G/H)$, we have $I(f) = \int f(\bar{x})F(x)h(x)\,dx$. Since $h(x) > 0$, $I(f)$ vanishes if and only if $f(\bar{x})F(x)$ vanishes except for a set of Haar measure zero in G. This condition is independent of the particular pseudo-invariant measure μ chosen, so any two pseudo-invariant measures are equivalent.

To show that at least one pseudo-invariant measure exists, we consider the inherited measure defined by the factor function $h(x) = \int F(x\xi)\Delta(\xi)\delta(\xi^{-1})\,d\xi$, which is non-zero, continuous, and satisfies the equation (3) identically.

References

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Added in proof: We have since shown that in Theorems 1 and 3 pseudo-invariance can be replaced by the weaker condition that $\mu(E)=0$ implies $\mu(tE)=0$, provided that h's property of being essentially bounded is replaced by $h > 0$ (see our joint paper: *Inherited measures*; to appear in Proc. Edinburgh Math. Soc.).

Mappings into normed linear spaces

by

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We contribute a few new fragments to a still fragmentary theory— that of the topological structure of infinite-dimensional normed linear spaces. § 1 is concerned with a problèm of Fréchet [6] and Banach [1]: Are all infinite-dimensional separable Banach spaces homeomorphic? Kadeč [7, 8] recently obtained an affirmative answer for the case of reflexive spaces. With the aid of a mapping theorem of Whyburn [29], we are able to extend the reasoning of [8] to cover all infinite-dimensional separable conjugate spaces. § 2 begins with some remarks on linear transformations of spaces $l_\aleph$, extending a result of Banach and Mazur [2]. In conjunction with a theorem of Bartle and Graves [3], this leads to some interesting corollaries such as an embedding theorem of Dowker [5] and the fact that every metric space of cardinality $\leqslant \mathfrak{c}$ admits a biunique continuous map onto some totally bounded metric space (1). An example in § 2 substantiates a conjecture in Michael's selection theory [24]. A few other results are obtained and some unsolved problems are stated.

§ 1. The theorem of Kadeč. A subset X of a metric space will be called a *Tchebycheff set* provided each point of the space admits a unique nearest point in X. An *admissible norm* for a normed linear space is one which generates the same topology as the given norm.

Kadeč first proved [7] that all infinite-dimensional separable uniformly convex Banach spaces are homeomorphic, then later observed [8] that the relevant consequences of uniform convexity can be obtained in more general spaces. By careful analysis of his reasoning, one arrives at the following conclusion.

1.1. THEOREM (Kadeč). *Two infinite-dimensional normed linear spaces E_1 and E_2 are homeomorphic if (for $i = 1, 2$) there exist an admissible norm $\| \;\|$ for E_i, a linear subspace F_i of the conjugate space E_i^*, and a linearly independent sequence f_α in F_i such that the following three conditions are satisfied:*

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(1) Added in proof: A simpler proof of this fact has been communicated to the author by Professor H. H. Corson.