COROLLARY (Vaught). The sentence $\sigma_n$ is equivalent with the axiom of choice.

It is an open problem whether or not each $\sigma_n$, with $n \geq 3$, is equivalent with the axiom of choice.

References


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Measures in homogenous spaces

by

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1. Notation. Generally our notation will follow that of Weil [W] and Halmos [H]. Let $G$ be a locally compact topological group, $H$ a closed subgroup. Let $G/H$ be the homogeneous space of cosets $xH$ with the usual topology so that $G$ acts, by left translation, as a transitive group of homeomorphisms of $G/H$. The natural mapping $G \to G/H$ will be denoted by $\varphi$ but sometimes we shall use the shorter notation $\bar{x}$ instead of $\varphi(x)$ for the projection $xH$ of $x$ in $G/H$. We shall also use $\bar{x}$ to denote a generic element of $G/H$. We use $dx, d\bar{x}$ to denote integration with respect to the Haar measures in $G, H$, and $\lambda(x), \theta(\bar{x})$ to denote the modular functions in $G, H$ ([W], p. 39).

For any topological space $X$, $L(X)$ denotes the class of continuous real-valued functions with compact support and $L_+(X)$ denotes the subclass consisting of non-negative functions. Similarly $B(X)$ denotes the class consisting of all extended real-valued Baire functions on $X$, $B_+(X)$ the non-negative ones. (Extended real numbers include the values $\pm \infty$ as well as the ordinary real numbers.)

A set $Q \subseteq X$ will be called an LB-set (locally Baire) if $Q \cap E$ is a Baire set whenever $E$ is a Baire set. A function which is measurable with respect to the ring of LB-sets will be called an LB-function. It is convenient to extend the notion of a set of measure zero to LB-sets as follows. If $Q$ is an LB-set and $\mu$ is a Baire measure we say that $\mu(Q) = 0$ provided that $\mu(Q \cap E) = 0$ for each Baire set $E$. If $\mu(Q) = 0$ then we say that almost every $x$ in $X$ belongs to $X - Q$. If $f, g$ are LB-functions, $N$ is the set $\{x : f(x) \neq g(x)\}$, we say that $f = g[N]$ if $\mu(N) = 0$. These definitions do not introduce anything new if $X$ is a $\sigma$-compact space.

All measures we consider are non-negative Baire measures in the sense that they are defined on the ring of all Baire sets; our usage of the term “Baire measure” differs thus from that of Halmos [H], where a Baire measure is assumed to be finite on compact sets.

2. Definitions and main results. A Baire measure $\mu$ on $G/H$ is called (following Weil) relatively invariant with factor $h(x)$ if $\mu(xE) = h(x)\mu(E)$ for each Baire set $E$ and $x \in G$. Then $h(xy) = h(x)h(y)$
and Weil ([W], p. 42-45) showed that such a measure can exist only if
\[ h(\xi) \cdot A(\xi) = \delta(\xi) \quad \text{for each} \quad \xi \in H. \]

If \( G \) admits no non-trivial homomorphism into the multiplicative group of positive reals, and \( H \) is not unimodular, there can be no relatively invariant measure in \( G/H \). This situation occurs when \( G \) is the group of 3 by 3 real matrices with determinant 1 and \( H \) is the group of matrices of the form
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & a & b \\
0 & 0 & 1
\end{bmatrix}.
\]

Thus if we wish to have a class of measure which exists for every homogeneous space \( G/H \), we must weaken our demands about invariance. In this paper we define a class of measure with an invariance property which is weak enough to guarantee that such measures exist but which turns to be strong enough to imply a connection with the Haar measure.

**Definition 1.** A non-vanishing Baire measure \( \mu \) in \( G/H \) which is finite on compact sets is called pseudo-invariant if, for each pair of compact sets \( C_1, C_2 \subseteq G/H \), there is a finite real number \( k \) such that \( k \mu(C_1 \cap C_2) = \mu(C_1) \cdot \mu(C_2) \). In particular, \( \mu(E) = 0 \) if and only if \( \mu(E) = 0 \).

A pseudo-invariant measure is positive on every non-empty open set. For if \( U \) is open and \( \mu(U) = 0 \), then \( \mu(U) = 0 \). Each compact set \( C \) can be covered by a finite union of sets \( U \), so \( \mu(C) = 0 \) and the measure vanishes contrary to definition.

If \( f \in L_1(G) \), then the expression \( \int f(\xi) d\mu \), regarded as a function of \( \xi \) is constant on cosets \( xH \) and is therefore really a function \( \mathcal{f}(x) \). It is well known that \( \mathcal{f} \in L_1(G/H) \). The mapping of \( L_1(G) \) in \( L_1(G/H) \) defined by
\[ \mathcal{f}(x) = \int f(\xi) d\mu \]

is linear and monotone. Since (2) is invariant under taking limits of monotone sequences of non-negative functions, we derive that (2) defines also a mapping of \( B_1(G) \) in \( B_1(G/H) \). In particular, if \( E \) is a Baire set in \( G \) and \( \mathcal{f} \) denotes the characteristic function of \( E \), then
\[ \mathcal{f}(x) = \int \chi_E(x) d\mu(x). \]

Let \( \mu \) be a Baire measure in \( G/H \). For every Baire set \( E \subseteq G \) define
\[ \mathcal{f}(E) = \int \chi_E(x) d\mu(x). \]

It is obvious that \( \mathcal{f} \) is a Baire measure in \( G \).

**Definition 2.** The measure \( \mu \) will be called inherited if \( \mathcal{f} \) is absolutely continuous with respect to the Haar measure.

If \( \mu \) is inherited, then, by the generalized Radon-Nikodym theorem (proved in § 3) there is a non-negative LB-function \( h(x) \) such that
\[ \int h(x) \chi_E(x) d\mu = \int \mathcal{f}(E) d\mu(x). \]

The function \( h \) is called the factor function for the inherited measure \( \mu \). It is obvious that the above equality implies
\[ \int h(x) f(x) d\mu = \int \mathcal{f}(E) d\mu(x) \]
for every \( f \in B_1(G) \).

Our main results are as follows. We follow Halmos in calling two measures equivalent if each is absolutely continuous with respect to the other.

**Theorem 1.** For any \( G, H \) there exists at least one pseudo-invariant measure in \( G/H \). Any two pseudo-invariant measures in \( G/H \) are equivalent.

**Theorem 2.** An LB-function \( h(x) \) is the factor function for an inherited measure if and only if, for each \( \xi \in H \), and for almost all \( x \) (in the Haar measure)
\[ h(x) A(\xi) = h(\xi) A(\xi). \]
(This theorem is a generalization of Weil's formula (1).)

**Theorem 3.** A measure \( \mu \) in \( G/H \) is pseudo-invariant if and only if \( \mu \) is inherited and the factor function \( h(x) \) is essentially bounded away from 0 and \( \infty \), on each compact set \( C \), i.e., for each compact set \( C \) there are real positive numbers \( k_1, k_2 \) such that \( k_1 \leq h(x) \leq k_2 \) holds for almost all \( x \in C \).

3. The Radon-Nikodym theorem for \( G \). In this section we justify the use made of the Radon-Nikodym theorem in the last section to obtain the factor function \( h(x) \), even when the usual condition of total \( \sigma \)-finiteness is not satisfied. Our proof is based on a condition of Oxtoby, as indicated by Halmos ([H], p. 132, Ex. 10, p. 256, Ex. 1).

**Theorem A.** If \( \nu_1 \) and \( \nu_2 \) are Baire measures in \( G \), \( \nu_1 \) is finite on compact sets and \( \nu_2 \) is essentially continuous with respect to \( \nu_1 \), then there exists a non-negative LB-function \( h(x) \) such that, for each \( f \in B_1(G) \),
\[ \int f(x) d\nu_1(x) = \int f(x) h(x) d\nu_2(x). \]

The function \( h \) is unique in the sense that if \( h^* \) also has the above property, then \( h = h^* \).

We first show that \( G \) satisfies Oxtoby's condition, i.e., that \( G \) is the union of a disjoint class \( D \) of Baire sets of finite \( \nu_1 \) measure with the property that every Baire set can be covered by a countable subclass of \( D \). To show this consider an open subgroup \( \Gamma \) of \( G \) which is \( \sigma \)-compact.
Every $I$-coset is a countable union of disjoint bounded sets, let $D$ be the family of all these bounded sets. Now every compact Baire set is contained in a finite union of $I$-cosets since these are open, and therefore also in a countable union of sets $D \in D$. Since every Baire set is the subring generated by countably many compact sets (Hilbert theorem, p. 24, Theorem D), the family $D$ does what is required.

To prove Theorem 4, apply the Radon-Nikodym theorem (in the form given in [Hilbert], p. 131, § 31, Ex. 7) to each space $D \in D$. On each space $D$ there is a function $h_D \in B_1(D)$ such that, for every function $f \in B(G)$,

$$
\int f(x) dx = \int h_D(x) f(x) dx.
$$

or, for every $f \in B(G)$,

$$
\int f(x) dx = \int h_D(x) f(x) dx.
$$

The function $h_D$ with this property is essentially unique. The function $h(x)$ on $G$ such that, for each $D \in D$, $h(x) = h_D(x)$ when $x \in D$, clearly satisfies (4), and any such $h$ is essentially unique.

4. Pseudo-invariant measures on $G$. In this section we prove Theorems 1, 3 for groups, i.e., we prove the following theorem.

**Theorem B.** Any pseudo-invariant measure on $G$ is equivalent to the Haar measure. More precisely, any pseudo-invariant measure $\nu$ is definable by an equation of the form

$$
\int f(x) d\nu = \int f(x) W(x) dx,
$$

where $W$ is an LB function essentially bounded away from zero and infinity on every compact set.

To prove that $\nu$ is equivalent to the Haar measure, we have to show that, if $E$ is a Baire subset of $G$, then $E$, $\nu$ and the Haar measure both vanish or are both positive. If $F$ is an $\sigma$-compact open subgroup of $G$, which contains $E$ (Hilbert § 67, Theorem A), the Haar measure carried over from $F$ will be a Haar measure in $F$, and $\nu$ will be pseudo-invariant in $F$. Thus it is enough to prove the equivalence part of Theorem B for $F$ instead of $G$.

Having proved the equivalence, the existence of an essentially unique Radon-Nikodym derivative $W(x)$ follows from Theorem A. The property of $W(x)$ that we have to establish consists its values on a compact set $C$, which is also contained in an open $\sigma$-compact subgroup $\Gamma$. Thus it is enough to prove Theorem B for a $\sigma$-compact group, and we shall assume, in this section only, that $G$ is $\sigma$-compact.

Let $G' = G \times G'$ be the group of ordered pairs $(x, y)$ with Baire measure $\nu' = \nu \times \nu$, and $G''$ the group of ordered triples $(x, y, z)$ with Baire measure $\nu''$. We denote by $\nu$ the Baire measure in $G$ which is determined by the condition

$$
\int f(x, y) d\nu = \int f(x, y) d\nu''(x, y)
$$

for $f \in B(G')$.

If $f(x, y) \in B(G)$, then for each fixed $x$, by definition 1, the functions $P(x) = \int f(x, y) dy$ and $Q(x) = \int f(x, y) dx$ are both zero or both positive. Thus the Radon-Nikodym theorems has a positive function $\nu(x, y)$ such that for each $f \in B(G)$

$$
\int f(x, y) d\nu = \int f(x, y) d\nu(x, y).
$$

An analogous argument, expressing out the integration with respect to $y$ first, and then with respect to $\nu''(x, z)$, will show that, if $f(x, y, z) \in B(G''(x, y))$ and $\gamma(x, y, z)$ is any continuous mapping $G'' \to G$, then the integrals

$$
\int f(x, y, z) d\nu''(x, y, z), \quad \int f(x, y, z) d\nu''(x, y, z),
$$

both vanish or are both positive. In particular, if $T$ denotes the transformation

$$
T(x, y, z) = (x, x^{-1}y, z),
$$

then $\nu''(E) = 0$ if and only if $\nu''(TE) = 0$.

**Lemma 2.1.** The function $\nu(x, y)$ may be chosen to be bounded away from zero and infinity on every compact set.

**Proof.** We can alter $\nu(x, y)$ on a set of measure zero, so it is enough to show that $\nu(x, y)$ is essentially bounded on every compact Baire rectangle. Let $M, N$ be compact Baire subsets of $G$. By definition 1, there are numbers $\lambda_1, \lambda_2 > 0$ such that, for any $f \in B(G)$ vanishing outside $M \times N$,

$$
\lambda_1 \int f(x, y) dy \leq \int f(x, y) d\nu \leq \lambda_2 \int f(x, y) dy.
$$

Integrate with respect to $\nu(x)$.

$$
\lambda_1 \int f(x, y) dy \leq \int f(x, y) d\nu \leq \lambda_2 \int f(x, y) dy.
$$

Thus $\lambda_1 \int f(x, y) dy \leq \lambda_2$ almost everywhere in $M \times N$, and the lemma follows. We shall assume from now on that $\nu$ is bounded away from zero and infinity on every compact set.
Let \( \Phi(x, y, z) \in L(\mathcal{G}) \). We shall obtain a functional equation for \( J \) by transforming the following integral in two ways:
\[
I(\Phi) = \int \Phi(x, y, x^2z) \, d\pi(x, y, z).
\]

Firstly, we have, by Fubini's theorem and (5),
\[
(7) \quad I(\Phi) = \int d\tau(z) \int \Phi(x, y, x^2z) \, d\pi(y, z)
- \int d\tau(x) \int J(x, z) \Phi(x, y, z) \, d\pi(y, z)
+ \int d\tau(y) \int J(x, z) J(y, x^2z) \Phi(x, y, z) \, d\pi(y, z)
= \int J(x, z) J(y, x^2z) \Phi(x, y, z) \, d\pi(y, z),
\]

On the other hand, if we write \( g(x, y, z) = J(y, z) \Phi(x, x^{-1}y, z) \), we have
\[
(8) \quad I(\Phi) = \int d\tau(x) \int J(x, y) \Phi(x, x^{-1}y, y) \, d\pi(y, x)
- \int d\tau(x) \int J(x, y) J(y, z) \Phi(x, x^{-1}y, z) \, d\pi(y, x)
+ \int d\tau(x) \int J(x, y) g(x, y, y) \, d\pi(x, y)
= \int J(x, y) g(x, y, y) \, d\pi(x, y).
\]

Comparing the two expressions (7), (8), equal for all \( \Phi \in L(\mathcal{G}) \), we deduce the equation
\[
J(x, y, z) = J(x, y, z) J(x, z) \| x \|^2.
\]

Applying the transformation \( T \) defined at (6), we have
\[
(9) \quad J(x, y, z) = J(x^{2-1}y, x^{2-1}z) J(x, z) \| x \|^2.
\]

If \( E \) denotes the subset of \( \mathcal{G} \) for which the equation (9) is false, and if, for each fixed \( z \), \( E_z \) denotes the set of \( (x, y) \) for which it is false, then
\[
\nu(E) = \int \nu(E_x) \, d\pi(y) = 0.
\]

Thus there is at least one value \( y = a \) such that \( \nu(E_a) = 0 \). If \( J(x, a, z) \) is denoted by \( V(z) \), then by (9),
\[
(10) \quad V(z) = V(x^{-1}a) J(x, z) \| x \|^2.
\]

Now let \( d\pi \) be the integral on \( \mathcal{G} \) defined by
\[
\int f(x) \, d\pi(x) = \int f(x) V(x) \, d\pi(x).
\]

Lemma 2.1 shows that \( V(x) \) is bounded on every compact set, so that every \( f \in L(\mathcal{G}) \) is \( \pi \)-integrable. Since \( V(x) > 0 \), the measures \( \pi \) and \( \pi \) are equivalent, and Theorem B will follow if we show that \( \pi \) is a Haar measure, i.e., that for each fixed \( f \in L(\mathcal{G}) \), the function
\[
\int I(f(t)) \, d\pi(x)
\]
is a constant. Since \( f \) is uniformly continuous, with compact support, the function \( I(f(t)) \) is continuous, and the set \( \mathcal{N} \) of \( t \) for which \( I(f(t)) \neq I(f) \) is open. We shall show that \( \pi(\mathcal{N}) = 0 \), and Theorem B will follow since the empty set is the only open set with \( \nu \)-measure zero, as shown in § 3. Let \( g \in L(\mathcal{G}) \). Apply the formula (5) with the variables \( (x, y) \) replaced by \( (t, x) \) and the function \( f(x, y) \) replaced by \( f(x) V(x) J(t^{-1}x) \). Using (10), we have
\[
\int I(t) g(t) \, d\pi(t) = \int f(t) V(x) g(t) \, d\pi(t, x)
= \int J(t, x) V(t^{-1}x) f(x) g(t) \, d\pi(t, x)
= \int V(x) f(x) g(t) \, d\pi(t, x)
= \int g(t) I(t) f(t) \, d\pi(t).
\]

Thus, for every \( g \in L(\mathcal{G}) \), we have \( \int (I(t) - I(e)) g(t) \, d\pi(t) = 0 \). From this it follows that \( \pi(\mathcal{N}) = 0 \), \( \mathcal{N} = \emptyset \), and Theorem B follows with \( W(x) = I(1) V(x) \).

5. On the existence of certain functions. Later it will be necessary to make use of a function \( F(x) \) with the properties given in the following theorem, where \( S(f) \) denotes the minimal support of \( f \), i.e., \( S(f) = \{ x : f(x) \neq 0 \} \).

**Theorem C.** There exists a continuous non-negative function \( F(x) \) defined on \( G \) with the following properties

(i) For each \( x \in G \), \( F(x) = 1 \);

(ii) If \( G \subseteq H \) is compact, then \( \varphi^{-1}(G) \cap S(F) \) is bounded.

Let \( V \) be any bounded neighbourhood of \( e \). A set \( Y \subseteq G \) is called \( (V, H) \)-separated if \( V \cap \nu \cap V \cap H = \emptyset \) whenever \( x, y \in Y, x \neq y \).

**Lemma 5.1.** If \( Y \) is \( (V, H) \)-separated, and \( C \subseteq G \) is compact, then the set \( \varphi^{-1}(C) \cap Y \) is finite. More generally, if \( D \subseteq G \) is compact, the set of \( y \in Y \) for which \( \varphi^{-1}(C) \cap D y \neq \emptyset \) is finite.

**Proof.** Let \( U \) be any neighbourhood of \( e \) such that \( U U^{-1} \subseteq V \).

Now \( \varphi^{-1}(C) \) can be covered by a finite union of sets of \( U \) (since \( C \) is contained in a finite union of sets \( \varphi(Ua) \)). \( \varphi^{-1}(C) \cap Y \) is thus finite, since
each set $UxH$ contains at most one point $y$ of $Y$ if, for, on the contrary, $y, z \in Y \cap UxH$, then $y \in U^{-1}xHH^{-1} \subset VzH$, and $Y$ is not $(V, H)$-separated.

The second part of the lemma follows from the first on replacing the set $C$ by the set $E = D^{-1}C$; since $\varphi^{-1}(C) \cap Dy \neq \emptyset$ if and only if $y \in \varphi^{-1}(E)$. This completes the proof of Lemma 5.1.

Since $\varphi$ is completely regular, there exists a non-negative continuous function $g(x)$ equal to unity within the bounded set $V^{-1}V$, but equal to zero outside some compact set $D$. Having chosen $g(x)$, we define another function $f(x)$ by the relation

$$f(x) = \sum_{y \in Y} g(\varphi(y^{-1})).$$

The series may contain infinitely (perhaps uncountably) many terms, but, as we shall now show, only a finite number of them differ from zero.

**Lemma 5.2.** Let $C$ be any compact subset of $G/H$. Then there is a finite subset $Y_0$ of $Y$ such that, if $x \in C$, then

$$f(x) = \sum_{y \in Y_0} g(\varphi(y^{-1})).$$

Further $S(f) \cap \varphi^{-1}(C)$ is bounded and the functions $f, j$ are continuous.

Proof. If $x \in C$, i.e., $x \in \varphi^{-1}(C)$, and $g(\varphi(y^{-1})) \neq \emptyset$, we must have $\varphi(y^{-1}) \in D$, $x \in Dy$. Thus $Dy \cap \varphi^{-1}(C)$ contains the point $x$ and so is non-empty. By Lemma 5.1, this can be true only if $y$ belongs to a certain finite subset $Y_0$ of $Y$, and the first part of the lemma follows.

Next,

$$S(f) \cap \varphi^{-1}(C) = \bigcup_{y \in Y_0} S(g(\varphi(y^{-1}))) \cap \varphi^{-1}(C) \subset \bigcup_{y \in Y_0} (Dy : y \in Y_0).$$

The last set is compact because $D$ is compact and $Y_0$ is finite.

For the continuity, it is enough to show that $f, j$ are continuous on compact subsets, since $G, G/H$ are locally compact. On $\varphi^{-1}(C), O$ compact, $f$ is a finite sum of functions $g(x) = g(\varphi^{-1})$. Since $g(x) \in L_1(G)$, $g(x)$ and $\varphi^{-1}$ are continuous, and the lemma follows.

Proof of Theorem C. Now let $Y$ be a maximal $(V, H)$-separated set, and define $g(x), f(x)$ as before. Then $f(x) > 0$ for every $x$ if $f(\varphi(x)) \varphi(x) - 0$, then $f(\varphi(x)) = 0$ for $\varphi(x) \in H$, and, for each $y \in Y$, $g(\varphi(y^{-1})) = 0$. Hence $\varphi(y^{-1}) \notin V^{-1}V$, $\forall x \in (V, H)$-separated, contradicting the hypothesis that $Y$ is maximal.

Set $F(x) = f(x)/f(\varphi(x))$. Then $F(\varphi(x)) = 1$, $F$ is the quotient of two continuous functions, and, since $S(f) = S(F)$, Theorem C follows.


Let $\mu$ be an inherited measure on $G/H$, with factor function $h(x)$. Thus, for each $f \in L_1(G)$,

$$I(f) = \int f(x) \mu(x) = \int f(x) h(x) dx.$$  

(11)

Let $\sigma \in H$. If $f(x) = f(\sigma^{-1}x)$, we have $f(x) = h(x) \delta(\sigma^{-1}x)$, so that

$$\delta(\sigma) I(f) = \int f(x) \mu(x) = \int f(\sigma^{-1}x) h(x) dx = A(\sigma) \int f(x) h(x) dx.$$  

Thus, for each $f \in L_1(G)$,

$$\int f(x) (h(x) - h(x) \delta(\sigma^{-1}x)) dx = 0,$$

proving that the equation (3) must be satisfied for almost all $x$.

Suppose conversely that $h(x)$ is any LB-function satisfying (3). Let $F(x)$ be any function which satisfies the conditions of Theorem C. For each $g(x) \in L_1(G)$, we have $g(x) F(x) \in L_1(G)$. Define a measure $\mu$ in $G/H$ by

$$I(g) = \int g(x) \mu(x) = \int g(x) F(x) h(x) dx.$$  

(12)

Then the measure $\mu$ is an inherited measure with factor function $h$ since by Fedorin's theorem, (3), Theorem C (i) and the properties of $A, \delta$ ([W], p. 39-40), we have, for $f \in L_1(G)$,

$$\int f(x) \mu(x) = \int f(x) F(x) h(x) dx = \int f(x) h(x) dx = \int f(x) A(\sigma^{-1}) h(x) dx$$

$$= \int f(x) h(x) dx \int \delta(\sigma^{-1}) F(x) h(x) dx = \int f(x) A(\sigma^{-1}) h(x) dx = \int f(x) h(x) dx.$$  

This completes the proof of Theorem 2. We note that the measure associated with a given factor function must necessarily be unique (if it exists at all) because of (11). Thus we obtain the same measure $\mu$ independently of our choice of function $F(x)$ in equation (12), provided that $F(x)$ satisfies the conditions of Theorem C.

### 7. Pseudo-Invariant measures. Let $\mu$ be a pseudo-invariant measure on $G/H$. Then $F$ is a pseudo-invariant measure on $G$, since, if $x \in C$, and $\varphi(t) \in C_1 \subset G, C \subset G$, then $S(f) \subset \varphi(S(f))$, and there is a number $k$ such that

$$\int f(x) \mu(x) dx = \int f(x) \mu(x) \leq k \int f(x) \mu(x) = k \int f(x) \mu(x).$$


It follows from Theorem B, § 4, that \( \mu \) is inherited, with a factor function which is essentially bounded away from zero and infinity on each compact set.

Conversely, let \( h(x) \) be any LB-function satisfying (3), and bounded away from zero and infinity on each compact set. We shall show that the measure defined by the equation (12) of § 6 is pseudo-invariant. We have to show that if \( \mathcal{C} \) is a compact subset of \( G \) and \( g \in L_1(G/H) \), then there exists a constant \( k \) such that, if \( t \in \mathcal{C} \), \( f \leq g \), \( f \in B_r(G/H) \), then

\[
\int f(\mathcal{E}) \, d\mu(\mathcal{E}) \leq k \int f(\mathcal{E}) \, d\mu(\mathcal{E}).
\]

The set of \( x \) such that, for some \( t \in \mathcal{C} \), \( g(\mathcal{E}) F(t^{-1}x) > 0 \) is bounded, as a result of Theorem C (ii). Thus there is a number \( k \) such that, if \( t \in \mathcal{C} \) and \( g(\mathcal{E}) F(t^{-1}x) > 0 \), then \( h(t^{-1}x) \leq kh(x) \). Using this inequality and the invariance of the Haar measure, we derive:

\[
\int f(\mathcal{E}) \, d\mu(\mathcal{E}) = \int f(\mathcal{E}) P(\mathcal{E}) h(x) \, dx = \int f(\mathcal{E}) P(t^{-1}x) h(t^{-1}x) \, dx \\
\leq k \int f(\mathcal{E}) P(t^{-1}x) h(x) \, dx = k \int f(\mathcal{E}) \, d\mu(\mathcal{E}).
\]

The last equation holds because \( F(t^{-1}x) \) as well as \( P(x) \) satisfy the conditions of Theorem C. Thus Theorem 3 is proved.

Finally we prove Theorem 1. Let \( \mu \) be a pseudo-invariant measure and \( F \) again a function satisfying the conditions of Theorem C. Then, if \( f \in B_r(G/H) \), we have \( \mu(f) = \int f(\mathcal{E}) P(\mathcal{E}) h(x) \, dx \). Since \( h(x) > 0 \), \( \mu(f) \) vanishes if and only if \( f(\mathcal{E}) P(\mathcal{E}) h(x) \) vanishes except for a set of Haar measure zero in \( G \). This condition is independent of the particular pseudo-invariant measure \( \mu \) chosen, so any two pseudo-invariant measures are equivalent.

To show that at least one pseudo-invariant measure exists, we consider the inherited measure defined by the factor function \( h(x) = F(x) d(\mathcal{E}) h(t^{-1}x) \), which is non-zero, continuous, and satisfies the equation (3) identically.

References


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**Added in proof:** We have since shown that in Theorems 1 and 3 pseudo-invariance can be replaced by the weaker condition that \( \mu(E) = 0 \) implies \( \mu(E) = 0 \), where \( E \) is an infinite-dimensional normed space.

**Added in proof:** A simpler proof of this fact has been communicated to the author by Professor H. H. Corson.

Mappings into normed linear spaces

by

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We contribute a few new fragments to a still fragmentary theory—that of the topological structure of infinite-dimensional normed linear spaces. § 1 is concerned with a problem of Fréchet [6] and Banach [1]: Are all infinite-dimensional separable Banach spaces homeomorphic? Kadeč [7, 8] recently obtained an affirmative answer for the case of reflexive spaces. With the aid of a mapping theorem of Whyburn [29], we are able to extend the reasoning of [8] to cover all infinite-dimensional separable conjugate spaces. § 2 begins with some remarks on linear transformations of spaces \( G \), extending a result of Banach and Mazur [5].

In conjunction with a theorem of Bartle and Graves [3], this leads to some interesting corollaries such as an embedding theorem of Dowker [5] and the fact that every metric space of cardinality \( \leq \) admits a biunique continuous map onto some totally bounded metric space \( (\cdot) \). An example in § 2 substantiates a conjecture in Michael's selection theory [24]. A few other results are obtained and some unsolved problems are stated.

§ 1. The theorem of Kadeč. A subset \( X \) of a metric space will be called a *Tečkevich set* provided each point of the space admits a unique nearest point in \( X \). An **admissible norm** for a normed linear space is one which generates the same topology as the given norm.

Kadeč first proved [7] that all infinite-dimensional separable uniformly convex Banach spaces are homeomorphic, then later observed [8] that the relevant consequences of uniform convexity can be obtained in more general spaces. By careful analysis of his reasoning, one arrives at the following conclusion.

1.1. **Theorem** (Kadeč). Two infinite-dimensional normed linear spaces \( E_1 \) and \( E_2 \) are homeomorphic if (for \( i = 1, 2 \)) there exist an admissible norm \( \| \cdot \| \) for \( E_1 \), a linear subspace \( F \), and a linearly independent sequence \( j_i \) in \( F \) such that the following three conditions are satisfied:

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(†) Added in proof: A simpler proof of this fact has been communicated to the author by Professor H. H. Corson.