Dependence of mappings and equivalence of sets

by

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The notion of dependence of maps of spaces ([4], [5] and [7]) permits the introduction of some relations of an algebraic character between maps even in cases in which the introduction, in a natural manner, of algebraic operations on maps is impossible. In the present note I give some remarks concerning those relations and some related notions.

1. By a *space* we understand here always a topological normal space. The term *compact* is used in the sense of bicompact. By *compactum* we understand a metric compact space. By ANR-*set* we understand a compactum which is a retract of some of its neighbourhoods in every metric space containing it.

By a map $f\colon X\to Y$ we understand a continuous function mapping a space X into another space Y. The set of all maps $f\colon X\to Y$ will be denoted by Y^X. Two maps $f, g\in Y^X$ are said to be *homotopic* if there exists a continuous function $h(x,t)$ of two arguments $x\in X$ and $t\in\langle 0,1\rangle$ with values on Y such that

$$h(x,0)=f(x) \quad\text{and}\quad h(x,1)=g(x) \quad\text{for every}\quad x\in X.$$

The set of all maps $g\in Y^X$ homotopic to a given map $f\in Y^X$ is said to be a *homotopy class* in Y^X; it will be denoted by $[f]$ or by $\mathbf{f}$ (the same letter in bold type). The set of all homotopy classes in Y^X will be denoted by $[Y^X]$. More generally, if A is a subset of Y^X, then we denote by $[A]$, or by $\mathbf{A}$, the set of all homotopy classes $[f_A]\subset Y^X$ with $f_A\in A$.

If $f\in Y^X$, $g\in Z^Y$, then $gf\in Z^X$ and we see at once that the homotopy class $[gf]\subset Z^X$ depends only on the homotopy classes $[f]$ and $[g]$. The homotopy class $[gf]$ will be said to be the *composition* of the homotopy classes $[f]$ and $[g]$ and it will be denoted by $[g][f]$. Hence

$$[gf]=[g][f].$$

If X or Y is a compactum, then the set Y^X may be considered as a metric space with the distance given by the formula

$$\varrho(f,g)=\operatorname*{Sup}_{x\in X}\varrho[f(x),g(x)] \quad\text{for every}\quad f,g\in Y^X.$$

In the case where Y is a compactum the space Y^X is complete. If Y is an ANR-set, then Y^X is locally connected and we infer that the homotopy classes in Y^X coincide with the components of Y^X.

If X is a subset of a space X', then every map $f' \in Y^{X'}$ such that the partial map f'/X coincides with $f \in Y^X$ is said to be an *extension* of f over X'. The set of all extensions of f over X' will be denoted by $\eta_{X'}(f)$. If $\eta_{X'}(f) \neq 0$, then we say that f is *extendable* over X'. We see at once that if X is a closed subset of X' and Y is an ANR-set, then the extendability of f over X' implies the extendability over X' of every map homotopic to f. It follows that the extendability is actually a property of the homotopy class and we shall say that the homotopy class $\boldsymbol{f}$ is *extendable* over X'. Moreover, if $f' \in Y^{X'}$ is an extension of $f \in Y^X$ over X', then the homotopy class $\boldsymbol{f}' \in [Y^{X'}]$ is said to be an *extension of the homotopy class* $\boldsymbol{f} \in [Y^X]$.

2. Now let us consider a space X, two spaces Y and Z and a collection $\mathfrak{X}$ of spaces. For every set $A \subset Z^X$, let us denote by $D_{\mathfrak{X}}(A)$ the set of all maps $f \in Y^X$ such that for every space $X' \in \mathfrak{X}$ containing X as a closed subset the extendability of all maps $f_A \in A$ over X' implies the extendability of f over X'. The maps f belonging to $D_{\mathfrak{X}}(A)$ are said to be *dependent* ([7]) *on A relatively to the class $\mathfrak{X}$*. Evidently, if $\mathfrak{X} \subset \mathfrak{X}'$ then the dependence on A relatively to $\mathfrak{X}'$ implies the dependence on A relatively to $\mathfrak{X}$.

In the case where Y and Z are ANR-sets let us denote by $\boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A})$ the set of all homotopy classes $[f]$ with $f \in D_{\mathfrak{X}}(A)$. Hence $\boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A}) = [D_{\mathfrak{X}}(A)]$ and we see at once that $\boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A})$ coincides with the set of all homotopy classes $\boldsymbol{f} \in [Y^X]$ such that for every space $X' \in \mathfrak{X}$ the extendability of all homotopy classes $[f_A] \in \boldsymbol{A}$ over X' implies the extendability of $\boldsymbol{f}$ over X'. The homotopy classes $\boldsymbol{f}$ belonging to $\boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A})$ are said to be *dependent on $\boldsymbol{A}$ relatively to the class $\mathfrak{X}$*.

By fixing the class $\mathfrak{X}$ in various manners, we get various kinds of dependence. The most important are the following three cases:

1. *Normal dependence*, where X is an arbitrary normal space, Y and Z are ANR-sets, and $\mathfrak{X}$ is the collection of all normal spaces.

2. *Compact dependence*, where X is a compactum, Y and Z are ANR-sets and $\mathfrak{X}$ is the collection of all compacta.

3. *n-dimensional dependence*, where X is a compactum, Y and Z are ANR-sets and $\mathfrak{X}$ is the collection of all compacta X' satisfying the condition $\dim(X'-X) \leqslant n$.

Evidently, if X is a compactum and Y, Z are ANR-sets, normal dependence implies compact dependence and compact dependence implies n-dimensional dependence, for $n = 0, 1, 2, \ldots$

3. Consider now a set $A \subset Z^X$ and a set T, called the *set of indices*. Let us assign to every index $\tau \in T$ the space $Z_\tau = Z$ and a map $a_\tau \in A$ in such a manner that for each map $a \in A$ there exists at least one index $\tau \in T$ such that $a_\tau = a$. Let Z_T denote the Tychonoff product $\underset{\tau \in T}{\boldsymbol{P}} Z_\tau$, its points being all systems $\{z_\tau\}_{\tau \in T}$ with $z_\tau \in Z_\tau = Z$ for every index $\tau \in T$; in the case $A = T = 0$ we understand by Z_T the space containing only one point. Setting

$$g_A(x) = \{a_\tau(x)\}_{\tau \in T} \quad \text{for every} \quad x \in X,$$

we get a map $g_A \in Z_T^X$, called the *natural map* of X into Z_T.

Now we have the following

THEOREM 1. *A map $f \in Y^X$, where Y is an* ANR*-set, is normally dependent on a set $A \subset Z^X$ if and only if there exists a map $\varphi \in Y^{Z_T}$ such that the map $\varphi g_A \in Y^X$ is homotopic to f.*

Evidently this theorem can also be formulated as follows:

THEOREM 1'. *A homotopy class $\boldsymbol{f} \in [Y^X]$, where Y is an* ANR*-set, is normally dependent on a set $\boldsymbol{A} \subset [Z^X]$ if and only if there exists a homotopy class $\boldsymbol{\varphi} \in [Y^{Z_T}]$ such that $\boldsymbol{f} = \boldsymbol{\varphi} \boldsymbol{g}_A$.*

In the case where $Y = Z$ and where A consists of only one map the proof of theorem 1 is given in [3], p. 82. As was pointed by Hilton [7], p. 360, by the same argument we get the theorem also for $Y \neq Z$ if A consists of only one map. By a remark due also to Hilton [7], p. 376, the general case may be reduced to this special case, since the extendability over X' of the map $g_A \in Z_T^X$ is equivalent to the extendability of each of the maps $a \in A$.

Passing to compact dependence, let us assign to every natural n the space $Z_n = Z$ and let Z_0 denote the product $\overset{\infty}{\underset{n=1}{\boldsymbol{P}}} Z_n$. Then Z_0 is a compactum and we have the following

THEOREM 2. *Let X be a compactum, Y and Z two* ANR*-sets and $\boldsymbol{A}$ a subset of $[Z^X]$. Then there exists a map $g_A^0 \in Z_0^X$ such that the compact dependence of $\boldsymbol{f} \in [Y^X]$ on $\boldsymbol{A}$ is equivalent to the existence of a homotopy class $\boldsymbol{\varphi}_0 \in [Y^{Z_0}]$ satisfying the relation $\boldsymbol{f} = \boldsymbol{\varphi}_0 \boldsymbol{g}_A^0$.*

Proof. By our hypotheses the space Z^X is separable and locally connected, and consequently the set $[Z^X]$ of all components of Z^X is at most countable. It follows that the set $\boldsymbol{A}$ is also at most countable. We can assume that $\boldsymbol{A}$ coincides with the collection of all homotopy classes $[f_\mu]$, where the index μ runs through a subset M of the set T of all naturals. Let N denote the set consisting of all naturals which do not belong to M and let a_0 be a fixed point of Z. If we identify every point

$\{z_n\} \in Z_0$ satisfying the condition $z_n = a_0$ for every $n \in N$, with the point $\{z_\mu\} \in \mathbf{P}_{\mu \in M} Z_\mu = Z_M$, then we can consider Z_M as a subset of Z_0. Setting

$$r_A(\{z_n\}) = \{z_\mu\} \quad \text{for every point} \quad \{z_n\} \in Z_0,$$

we see at once that r_A is a retraction of Z_0 to Z_M.

Using the same argument as in the proof of theorem 1′, we see that the compact dependence of a homotopy class $\boldsymbol{f} \in [Y^X]$ on A is equivalent to the existence of a homotopy class $\boldsymbol{\varphi} \in [Y^{Z_T}]$ such that $\boldsymbol{f} = \boldsymbol{\varphi g}_A$. Now let us denote by g_A^0 the mapping of X into Z_0 defined by the formula

$$g_A^0(x) = g_A(x) \quad \text{for every} \quad x \in X,$$

and by φ_0 the map of Z_0 into Y defined by the formula

$$\varphi_0(z) = \varphi r(z) \quad \text{for every } z \in Z_0.$$

Then $\boldsymbol{f} = \boldsymbol{\varphi}_0 \boldsymbol{g}_A^0$. Consequently the compact dependence of $\boldsymbol{f} \in [Y^X]$ on A implies that $\boldsymbol{f} = \boldsymbol{\varphi}_0 \boldsymbol{g}_A^0$.

On the other hand, if $\boldsymbol{f} = \boldsymbol{\varphi}_0 \boldsymbol{g}_A^0$, where $\varphi_0 \in Y^{Z_0}$, then let us consider the map $g_A \in Z_A^X$ given by the formula

$$g_A(x) = g_A^0(x) \quad \text{for every} \quad x \in X,$$

and let φ denote the partial map of φ_0 on the subset Z_M of Z_0. Clearly we have

$$\boldsymbol{f} = \boldsymbol{\varphi g}_A$$

and we infer, by theorem 1′, that $\boldsymbol{f}$ is normally, whence also compactly dependent on A.

4. Now let us consider an abstract set W and an operation λ, assigning to every subset A of W a subset $\lambda(A)$ of W, subject to the following conditions:

1° $A \subset \lambda(A) \subset W$ for every set $A \subset W$.

2° If $A \subset B \subset W$ then $\lambda(A) \subset \lambda(B)$.

3° $\lambda\lambda(A) = \lambda(A)$ for every set $A \subset W$.

The set W, together with the operation λ, will be said to be a *d-set* (*dependence set*) and will be denoted by $(W)_\lambda$, or shorter, by W_λ. The operation λ will be said to be a *d-operation* (*dependence operation*).

The conditions belong to the axiomatic of closure of Kuratowski. But they are far weaker than the whole axiomatic of closure, because the condition of additivity is not included and the set $\lambda(0)$ can be non-empty.

More important for us than the interpretation of closure is the interpretation by which W is the set of all elements of an Abelian group $\mathfrak{W}$ and $\lambda(A)$ denotes the set of elements of the subgroup of $\mathfrak{W}$ generated by A, i.e. the set consisting of the element 0 and of all linear combinations

$$m_1 a_1 + m_2 a_2 + \ldots + m_k a_k,$$

where the elements $a_1, a_2, \ldots, a_k$ belong to A and the coefficients $m_1, m_2, \ldots, m_k$ are integers. The d-set obtained in this manner from the group $\mathfrak{W}$ will be denoted by $W(\mathfrak{W})$.

With this interpretation in mind, let us call the elements of the set $\lambda(A)$—the elements *dependent on* A. In the special case where A consists of only one element a the elements of the set $\lambda(A) = \lambda(a)$ will be said to be *d-multiplies* of a.

Many notions belonging to the theory of groups can easily be transferred onto the theory of d-sets. Consider a subset A of a d-set W_λ. If $\lambda(A) = W$, then A is said to be a *system of generators* of W_λ. If $A \subset W$ and for every two subsets A_1 and A_2 of A we have

$$\lambda(A_1 \cap A_2) = \lambda(A_1) \cap \lambda(A_2),$$

then the set A is said to be *independent* in W_λ.

Let W and W' be two d-sets with d-operations λ and λ' respectively. A transformation φ of W into W' satisfying the condition

$$\varphi\lambda(A) = \lambda'\varphi(A) \quad \text{for every set} \quad A \subset W \tag{1}$$

is said to be a *homomorphism* of W_λ into $W'_{\lambda'}$. If, moreover, $\varphi(W_\lambda) = W'_{\lambda'}$, then φ is said to be an *epimorphism*. If φ maps W_λ onto a subset of $W'_{\lambda'}$ in a 1-1 manner then φ is said to be a *monomorphism*. Finally, if φ is both an epimorphism and a monomorphism, then it is said to be an *isomorphism* of W_λ onto $W'_{\lambda'}$. Evidently, if φ is an isomorphism of W_λ onto $W'_{\lambda'}$, then the inverse transformation φ^{-1} is an isomorphism of $W'_{\lambda'}$ onto W_λ. The composition of two homomorphisms, epimorphisms, monomorphisms or isomorphisms is a homomorphism, epimorphism, monomorphism or isomorphism, respectively. Two d-sets are said to be *isomorphic* provided there exists an isomorphism transforming one of them onto the other.

5. Though the notion of the d-set is rather poor in comparison with the notion of the group, there exist cases in which the structure of a d-set determines the structure of a group. Consider an Abelian group $\mathfrak{W}$ which is a weak product of a class $\{\mathfrak{A}_\varkappa\}$ of cyclic groups, where the index $\varkappa$ runs through a set of indices K. Hence the elements of $\mathfrak{W}$ are all systems $\{a_\varkappa\}$ with $\varkappa \in K$, where $a_\varkappa \in \mathfrak{A}_\varkappa$ and $a_\varkappa = 0$ for almost all indices $\varkappa$. Let us observe that for any such Abelian group $\mathfrak{W}$ the isomorphism of two d-sets $W(\mathfrak{W})$ and $W(\mathfrak{W}')$ implies the isomorphism of the groups $\mathfrak{W}$ and $\mathfrak{W}'$.

In fact, let φ denote the isomorphism of $W(\mathfrak{W})$ onto $W(\mathfrak{W}')$ and let $\gamma_\varkappa$ denote a generator of the cyclic group $\mathfrak{A}_\varkappa$. Then $\mathfrak{A}_\varkappa = \lambda(\gamma_\varkappa)$. We see at once that the set $G = \{\gamma_\varkappa\}$ is an independent system of generators of $W(\mathfrak{W})$ and consequently the set $\varphi(G) = \{\varphi(\gamma_\varkappa)\}$ is an independent system of generators of $W(\mathfrak{W}')$. Moreover, the set $\mathfrak{A}'_\varkappa = \varphi(\mathfrak{A}_\varkappa) = \varphi[\lambda(\gamma_\varkappa)] = \lambda'[\varphi(\gamma_\varkappa)]$ is a cyclic subgroup $\mathfrak{A}'_\varkappa$ of $\mathfrak{W}'$ generated by $\varphi(\gamma_\varkappa)$. Since φ is 1-1, we infer that $\mathfrak{A}_\varkappa$ and $\mathfrak{A}'_\varkappa$ are isomorphic. Since $\varphi(G)$ is an independent system of generators of $\mathfrak{W}'$, we infer that the group $\mathfrak{W}'$ is the weak product of the groups $\mathfrak{A}'_\varkappa$ and thus the isomorphism of $\mathfrak{W}$ and $\mathfrak{W}'$ is proved.

By a remark due to Wiliam R. Scott, the last statement does not hold if we omit the hypothesis that $\mathfrak{W}$ is a weak product of cyclic groups. Consider in the additive group of all rationals two sub-groups: $\mathfrak{W}_2$ consisting of all numbers of the form $2^k \cdot m$, and $\mathfrak{W}_3$ consisting of all numbers of the form $3^k \cdot m$, where k and m are integers. Evidently every element of $\mathfrak{W}_2$ is divisible by 2, and this property does not hold for $\mathfrak{W}_3$. Consequently $\mathfrak{W}_2$ and $\mathfrak{W}_3$ are not isomorphic.

Now let us observe that the numbers $x \in \mathfrak{W}_2$ and $y \in \mathfrak{W}_3$, distinct from 0, are given by the formulas

$$x = \pm 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot \ldots; \qquad y = \pm 3^{\beta_1} \cdot 2^{\beta_2} \cdot 5^{\beta_3} \cdot 7^{\beta_4} \cdot \ldots,$$

where the exponents α_i and β_j are integers uniquelly determined by x and y respectively. Moreover, almost all exponents α_i and β_j vanish and $\alpha_2, \alpha_3, \ldots$ and $\beta_2, \beta_3, \ldots$ are not negative. Setting

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(\pm 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots) = \pm 3^{\alpha_1} \cdot 2^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots,$$

we get a 1-1 correspondence between the elements of $\mathfrak{W}_2$ and $\mathfrak{W}_3$. Let us show that φ is an isomorphism of the d-set $W(\mathfrak{W}_2)$ onto the d-set $W(\mathfrak{W}_3)$.

Consider a set $A \subset \mathfrak{W}_2$. If A is empty or A consists only of the number 0, then $\varphi(A)$ is empty or consists only of the number 0 and we have $\lambda\varphi(A) = (0) = \varphi\lambda(A)$. If A contains at least one element of the form $\pm 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots$ and if the collection of exponents α_1, for $x \in A$, is not bounded on the left side, then we see at once that $\lambda(A) = \mathfrak{W}_2$, $\varphi\lambda(A) = \mathfrak{W}_3$ and $\lambda\varphi(A) = \mathfrak{W}_3$. Finally, if A contains at least one element of the form $x = \pm 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots$ and the collection of exponents α_1 for $x \in A$ is bounded on the left, then $\lambda(A)$ is a cyclic infinite group generated by an element $x = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots$, with the minimal exponent α_1. It follows that $\varphi(A)$ is a cyclic infinite group generated by the element $\varphi(x) = 3^{\alpha_1} \cdot 2^{\alpha_2} \cdot 5^{\alpha_3} \cdot \ldots$, and we infer that $\lambda\varphi(A) = \varphi\lambda(A)$. It follows that φ is an isomorphism of $W(\mathfrak{W})$ onto $W(\mathfrak{W}')$.

6. Now let us return to the operation $D_{\mathfrak{X}}$, as defined in No. 2. Manifestly, in case $Y = Z$, the operation $D_{\mathfrak{X}}$ satisfies the conditions 1°, 2°, 3° of No. 4, whence it is a d-operation in the set Y^X. The set Y^X with the operation $D_{\mathfrak{X}}$ will be denoted by $Y^X_{\mathfrak{X}}$.

Moreover, if $Y = Z$ is an ANR-set, then the operation $\boldsymbol{D}_{\mathfrak{X}}$ is a d-operation in the set $[Y^X]$. The set $[Y^X]$ with the operation $\boldsymbol{D}_{\mathfrak{X}}$ will be denoted by $[Y^X_{\mathfrak{X}}]$.

Now let us prove the following

THEOREM 3. *Let p be a continuous map of a compactum X' into a compactum X*

$$p\colon X' \to X$$

and let Y be an ANR-set. If we assign to each homotopy class $\boldsymbol{f} = [f] \in [Y^X]$ the homotopy class $\chi^p(\boldsymbol{f}) = [fp] \in [Y^{X'}]$, then we get a transformation

$$\chi^p\colon [Y^{X'}] \leftarrow [Y^X].$$

If $\mathfrak{X}$ is the collection of all normal spaces, or the collection of all compacta, then the operation χ^p is a homomorphism of $[Y^X_{\mathfrak{X}}]$ into $[Y^{X'}_{\mathfrak{X}}]$.

Proof. Let A be a subset of Y^X. Consider a set of indices T and a function assigning to every index $\tau \in T$ a map $\alpha_\tau \in A$. Let us assume that this function is onto, i.e. for every $\alpha \in A$ there exists at least one index $\tau \in T$ such that $\alpha_\tau = \alpha$. Setting

$$\alpha'_\tau = \alpha_\tau p \quad \text{for every} \quad \tau \in T,$$

we get a function assigning to every index $\tau \in T$ a map $\alpha'_\tau \in Y^{X'}$. In order to prove our theorem, it suffices to show that

$$(1) \qquad \chi^p \boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A}) = \boldsymbol{D}_{\mathfrak{X}}(\chi^p(\boldsymbol{A})).$$

First let us consider the case where $\mathfrak{X}$ is the collection of all normal spaces. Let X_T denote the Tychonoff product of spaces $X_\tau = X$ with τ running through the set T. Let g_A denote the natural map of X into X_T given by the formula

$$g_A(x) = \{\alpha_\tau(x)\}_{\tau \in T} \quad \text{for every} \quad x \in X,$$

and $g_{A'}$—the natural map of X' into X_T given by the formula

$$g_{A'}(x') = \{\alpha'_\tau(x')\} \quad \text{for every} \quad x' \in X'.$$

Since $\alpha'_\tau = \alpha_\tau p$, we have

$$g_{A'} = g_A p.$$

By theorem 1′, the set $\boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A})$ coincides with the collection of all homotopy classes $[\varphi g_A] \in [Y^X]$, where $\varphi \in Y^{X_T}$. It follows that the set $\chi^p \boldsymbol{D}_{\mathfrak{X}}(\boldsymbol{A})$ coincides with the collection of all homotopy classes $[\varphi g_A p] \in [Y^{X'}]$.

On the other hand, $\chi^p(\boldsymbol{A})$ coincides with the collection of all homotopy classes $[\alpha_\tau p] = [\alpha'_\tau]$. We infer, by theorem 1′ of No. 3, that $\boldsymbol{D}_{\mathfrak{X}}(\chi^p(\boldsymbol{A}))$ coincides with the collection of all homotopy classes $[\varphi g_{A'}] = [\varphi g_A p]$, where $\varphi \in Y^{X_T}$. Consequently (1) holds in the case of normal dependence.

The proof in the case of compact dependence is quite analogous. We apply only theorem 2 of No. 3, instead of theorem 1′.

PROBLEM 1. *Does theorem 3 remain true also for dependence in dimension n* (that is when $\mathfrak{X}$ is the collection of all metric spaces X' satisfying the condition $\dim(X'-X) \leqslant n$)?

7. The operation of dependence is intimately related to the problem of the classification of spaces from the point of view of the properties of their maps into a given space. This classification may be considered as a relativisation of the classification of spaces into homotopy types in the sense of Hurewicz [8].

Let X_0 be a closed subset of a space X. For every map $f_0 \in Y^{X_0}$ let us denote by $\eta(f_0)$ the subset of Y^X consisting of all extensions of f_0. Now we say that X_0 is a *lower reduction of X relatively to Y* provided that for every map $f_0 \in Y^{X_0}$ the set $\eta(f_0)$ is non-empty. We say that X_0 is an *upper reduction of X relatively to Y*, provided that for every map $f_0 \in Y^{X_0}$ the set $\eta(f_0)$ is connected.

A set X_0 which is both a lower and an upper reduction of X relatively to Y will be said to be an *exact reduction of X relatively to Y*.

Using homological notions, we can say that a closed subset X_0 is a lower reduction of X relatively to Y if the set $\eta(f_0)$ is acyclic in the dimension -1 for every map f_0 of X_0 into X. And X_0 is an upper reduction of X relatively to Y if the set $\eta(f_0)$ is acyclic in the dimension 0. The acyclicity of the sets $\eta(f_0)$ in a given dimension n is a condition generalizing those notions.

In order to illustrate the sense of the notions of the lower and of the upper reductions, let us consider the following simple examples:

1. Let X denote the set obtained from an $(n+1)$-dimensional Euclidean ball Q_0^{n+1} by removing the interiors of two disjoint $(n+1)$-dimensional balls $Q_1^{n+1}, Q_2^{n+1} \subset Q_0^{n+1}$. Let S_ν^n denote the n-dimensional sphere which is the boundary of Q_ν^{n+1}, for $\nu = 0, 1, 2$. Setting $Y = S_0^n$, we easily see that each of the spheres S_ν^n is a lower but not an upper reduction of X relatively to Y and that the union of all spheres S_0^n, S_1^n, S_2^n is an upper but not a lower reduction of X relatively to Y. However, the union of two of those spheres is an exact reduction of X relatively to Y.

2. If $X = Q^{n+1}$ denotes the $(n+1)$-dimensional Euclidean ball and $X_0 = S^n$ denotes its boundary, then one easily sees that S^n is a lower reduction of Q^{n+1} relatively to an ANR-set Y if and only if the n-th homotopy group $\pi_n(Y)$ of Y is trivial. And S^n is an upper reduction of Q^{n+1} relatively to an ANR-set Y if and only if the $(n+1)$-th homotopy group $\pi_{n+1}(Y)$ of Y is trivial.

Now let us prove the following

LEMMA. *If X_0 is an exact reduction of a compactum X relatively to an* ANR-*set Y, then, for every connected subset A of Y^{X_0}, the set $\eta(A) \subset Y^X$ of all extensions over X of maps belonging to A is connected.*

Proof. Consider a decomposition of the set $\eta(A)$ into two non-empty open subsets M_1 and M_2. Let N_i denote the subset of Y^{X_0} consisting of all partial maps f/X_0 with $f \in M_i$. Evidently the sets N_1 and N_2 are non-empty and

$$A = N_1 \cup N_2,$$

because X_0 is a lower reduction of X. Moreover, the sets N_1 and N_2 are open in A, because M_1 and M_2 are open in $\eta(A)$ and the hypothesis that Y is an ANR-set implies that the operation assigning to every map $g \in Y^X$ the partial map $g/X_0 \in Y^{X_0}$ is open.

Thus we have a decomposition of the connected set A into two open and non-empty sets N_1 and N_2. It follows that there exists a map $f_0 \in N_1 \cap N_2$. But X_0 is an exact reduction of X. Hence the set $\eta(f_0)$ included in $\eta(A) = M_1 \cup M_2$ is non-empty and connected. Moreover, $f_0 \in N_1 \cap N_2$ implies that

$$\eta(f_0) \cap M_1 \neq 0 \neq \eta(f_0) \cap M_2 .$$

Thus we have a decomposition of the connected set $\eta(f_0)$ into two non-empty and open subsets $\eta(f_0) \cap M_1$ and $\eta(f_0) \cap M_2$. It follows that $\eta(f_0) \cap M_1 \cap M_2 \neq 0$, and consequently also $M_1 \cap M_2 \neq 0$.

8. The notions of the lower and the upper reductions are intimately related to the notions of the theory of retracts. In fact, we have the following simple theorems:

THEOREM 4. *A set X_0 is a retract of a space X if and only if it is a lower reduction of X relatively to every space Y.*

Proof. If there exists a retraction r of X to X_0, then for every map $f_0 \in Y^{X_0}$ the formula $f = f_0 r$ gives an extension $f \in Y^X$ of f_0. On the other hand, if a subset X_0 of X is a lower reduction of X relatively to every space Y, then setting $Y = X_0$ we infer that the identical map defined on X_0 has an extension onto X with values belonging to X_0. This extension is a retraction of X to X_0.

A subset X_0 of X is said to be a deformation retract of X if there exists a retraction r of X to X_0 homotopic in X^X to the identity.

THEOREM 5. *An* ANR-*set X_0 is a deformation retract of an* ANR-*set X if and only if it is an exact reduction of X relatively to every* ANR-*set Y.*

Proof. If X_0 is a deformation retract of X, then there exists a retraction r of X to X_0 homotopic to the identity. Consider now a continuous map f_0 of X_0 into an ANR-set Y. Then $f_0 r \in Y^X$ is an extension of f_0 over X and every other extension $f \in Y^X$ of f_0 is homotopic in Y^X to $fr = f_0 r$. Consequently, for every ANR-set Y the set $\eta(f_0) \subset Y^X$ is non-empty and connected, i.e. X_0 is an exact reduction of X relatively to every ANR-set Y.

On the other hand, if X_0 is an exact reduction of X relatively to every ANR-set Y, then, setting $Y = X_0$, let us consider the identity map f_0 of X_0 into X. Then $\eta(f_0)$ is non-empty and consequently there exists an extension f of f_0 over X with values in X_0 that is a retraction r of X into X_0. Moreover, r and the identity map on X are both extensions of f_0 over X considered as a map of X_0 into X. But X_0 is an upper reduction of X relatively to X, whence r and the identity on X belong to the same component of X^X and consequently they are homotopic in X^X. Thus we have shown that X_0 is a deformation retract of X.

9. Using the notions of the lower, upper and exact reductions, we get, as a corollary to theorem 3, the following

THEOREM 6. *Let X_0 be a closed subset of a compactum X, Y—an ANR-set, $\mathfrak{X}$—the collection of all normal spaces, or the collection of all compacta, and*

$$\chi^i\colon [Y^{X_0}_{\mathfrak{X}}] \leftarrow [Y^X_{\mathfrak{X}}]$$

the homomorphism induced by the inclusion $i\colon X_0 \to X$.

Under those hypotheses:

(1) *If X_0 is a lower reduction of X rel. to Y, then χ^i is an epimorphism.*

(2) *If X_0 is an upper reduction of X rel. to Y, then χ^i is a monomorphism.*

(3) *If X_0 is an exact reduction of X rel. to Y, then χ^i is an isomorphism.*

Proof. In order to prove (1) it suffices to observe that if X_0 is a lower reduction of X rel. to Y, then all maps belonging to Y^{X_0} are partial maps for maps belonging to Y^X.

Now let us assume that X_0 is an upper reduction of X rel. to Y. Then all extensions of one map $f_0 \in Y^{X_0}$ over X are homotopic. It follows that χ^i maps at most one homotopy class $\in [Y^X]$ onto $[f_0] \in [Y^{X_0}]$, whence χ^i is a monomorphism, i.e. (2) is proved.

Finally (3) is a direct consequence of (1) and (2).

10. Two compacta X_1 and X_2 will be said to be *equivalent relatively to an* ANR-*set* Y, symbolically

(1) $$X_1 \equiv X_2 \text{ rel. } Y,$$

provided that there exists a compactum X containing two exact reductions X_1' and X_2' rel. to Y, homeomorphic to X_1 and X_2 respectively. The compactum X will be said to *realize* equivalence (1). If there exists a compactum X realizing equivalence (1) and satisfying the condition

$$\dim(X - X_1 - X_2) \leqslant m,$$

then we say that spaces X_1 and X_2 are *equivalent relatively to Y in the dimension m* and we write

(2) $$X_1 \equiv X_2 \text{ rel. } Y \text{ in dimension } m.$$

It is clear that relation (2) implies relation (1) and also the relation $X_1 \equiv X_2$ rel. Y in all dimensions $m' > m$.

Evidently the relation of equivalence relatively to Y (and, for spaces of dimension $\leqslant m$, also relation of equivalence rel. Y in dimension m) is reflexive and symmetric. It is also topological, i.e. we can always replace in it spaces X_1 and X_2 by any spaces homeomorphic to them respectively.

In order to prove that the relation of equivalence rel. to Y (and also the relation of equivalence rel. Y in dimension m) is transitive, consider three spaces X_1, X_2, X_3 such that

$$X_1 \equiv X_2 \text{ rel. } Y \quad \text{and} \quad X_2 \equiv X_3 \text{ rel. } Y.$$

We have to show that

$$X_1 \equiv X_3 \text{ rel. } Y.$$

We can assume that X_1 and X_2 are exact reductions of a compactum X, and X_2 and X_3 are exact reductions of a compactum X' (in the case of equivalence rel. Y in dimension m we assume that $\dim(X - X_1 - X_2) \leqslant m$ and $\dim(X' - X_2 - X_3) \leqslant m$). Moreover, if we apply to X' a suitably chosen homeomorphism h, by which all points of the set X_2 remain fixed, we can assume—without loss of generality—that $X \cap X' = X_2$.

Now let us show that each of the sets X_1 and X_3 is an exact reduction rel. Y of the space $X'' = X \cup X'$. By the symmetry of assumptions, it suffices to show that X_1 is an exact reduction of X'' rel. Y.

Consider a map $f \in Y^X$. Since X_2 is an exact reduction of X' rel. Y, the set of all extensions $f' \in Y^{X'}$ of f/X_2 is non-empty and connected. Setting

$$f''(x) = f(x) \quad \text{for every} \quad x \in X,$$
$$f''(x) = f'(x) \quad \text{for every} \quad x \in X',$$

we get an extension $f'' \in Y^{X''}$ of f and we see at once that the set of all such extensions f'' is connected and non-empty. Consequently

(1) *X is an exact reduction of* X'' rel. Y.

Now let us consider a map $f_1 \in Y^{X_1}$. Since X is a lower reduction of X'' rel. Y, the set $\eta_{X''}(f_1) \subset Y^{X''}$ consisting of all extensions of f_1 over X'', coincides with the set of all extensions over X'' of maps belonging to the set $\eta_X(f_1) \subset Y^X$ of all extensions of f_1 over X. But $\eta_X(f_1)$ is a connected and non-empty set, since X_1 is an exact reduction of X rel. Y. Applying (1) and lemma of No. 7 we infer that the set $\eta_{X''}(f_1)$ is connected and non-empty, i.e. X_1 is an exact reduction of X'' over Y.

Thus we have the following

THEOREM 7. *The relation of equivalence rel. to a given* ANR-*set* Y, *and, for spaces of dimension* $\leqslant m$, *also the relation of equivalence in a given dimension* m *rel. to a given* ANR-*set* Y *are both reflexive, symmetric and transitive.*

11. EXAMPLES: 1. Let Y be the set consisting of two numbers 0 and 1 and let X_1 be a finite set consisting of k points $a_1, a_2, \ldots, a_k$. Then spaces equivalent to X_1 rel. Y coincide with spaces consisting of k components. In order to prove this, consider a space X containing two exact reductions rel. Y: a set X_1' homeomorphic to X_1 and a set X_2' homeomorphic to X_2. Since X_1' and X_2' are upper reductions of X rel. Y, we easily infer that each component of X contains at least one point of X_1' and at least one point of X_2'. Since X_1' and X_2' are lower reductions of X rel. Y, we infer at once that every component of X contains at most one component of X_2'. It follows that X_2', whence also X_2, consists of just k components.

On the other hand, if X_2 has k components $C_1, C_2, \ldots, C_k$ then let us pick up a point $a_i' \in C_i$ for $i = 1, 2, \ldots, k$ and let us set

$$X_1' = \{a_1', a_2', \ldots, a_k'\}, \qquad X_2' = X_2 = X.$$

Then we see at once that the sets X_1' and X_2' are homeomorphic with X_1 and X_2 respectively and that they are exact reductions rel. Y of the space X. Hence X_1 and X_2 are equivalent rel. Y.

2. Let X_1 and X_2 be two compact subsets of the Euclidean n-dimensional space E^n and let us assume that each of them decomposes E^n into the same finite number k of regions. Let us prove that the sets X_1 and X_2 are equivalent in dimension n relatively to the $(n-1)$-dimensional Euclidean sphere S^{n-1}.

For $n = 1$ this follows by example 1. Hence we can assume that $n > 1$. Since the relation of equivalence is transitive, it suffices to give the proof of our statement in the case where X_2 coincides with the union of $k-1$ disjoint $(n-1)$-spheres $S_1, S_2, \ldots, S_{k-1}$ which are boundaries of $k-1$ disjoint n-dimensional balls $Q_1, Q_2, \ldots, Q_{k-1}$ lying in E^n.

$$X_2 = S_1 \cup S_2 \cup \ldots \cup S_{k-1} \quad \text{and} \quad Q_i \cap Q_j = 0 \quad \text{for} \quad i \neq j.$$

Let $G_1, G_2, \ldots, G_k$ be the components of $E^n - X_1$. Since $n > 1$, one of these components, say G_k, is unbounded and all the other are bounded. Without loss of generality we can assume that Q_i lies in G_i, for $i = 1, 2, \ldots, k-1$.

Let us denote by Q_k a ball in E^n containing the sets X_1 and X_2. It remains to show that each of the sets X_1, X_2 is an exact reduction of the set $X = Q_k - \bigcup_{i=1}^{k-1} (Q_i - S_i)$ rel. S^{n-1}. But this is an immediate consequence of a theorem proved in [2], p. 227, 1° and 5°.

3. Let X_1 denote the subset of the Euclidean 1-dimensional space E^1, consisting of 0 and of all numbers $1/n$, where $n = 1, 2, \ldots$, and let X_2 denote the subset of E^1 consisting of all numbers 0, $1/n$ and $1+1/n$, where $n = 1, 2, \ldots$ Let us show that X_1 and X_2 are not equivalent relatively to the set Y consisting of numbers 0 and 1.

Suppose, on the contrary, that $X_1 \equiv X_2$ rel. Y. Then there exists a space X containing two exact reductions rel. Y: a set X_1' which is the image of X_1 by a homeomorphism h_1, and a set X_2' which is the image of X_2 by a homeomorphism h_2. Let $f \in Y^X$. Then, for almost all points $x \in X_1'$, we have $f(x) = f(0)$. Consequently there exist only a finite number of indices $n_1, n_2, \ldots, n_k$ such that

$$f\left(h_1\left(\frac{1}{n_i}\right)\right) \neq f\left(h_1(0)\right) \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Since X_1' is an exact reduction of X rel. Y, we easily infer that $f(x) \neq f[h_1(0)]$ only if x belongs to a component of X including one of the points $h_1(1/n_1), h_1(1/n_2), \ldots, h_1(1/n_k)$. It follows that X has only a finite number of components in which f is different from $f[h_1(0)]$. On the other hand, since X_2' is an exact reduction of X rel. Y, at most one point of the set X_2' lies in every component of X. It follows that

$$f(x) = f\left(h_1(0)\right) \quad \text{for almost all points} \quad x \in X_2',$$

and consequently

$$f\left(h_2(0)\right) = f\left(h_2(1)\right) \quad \text{for every map} \quad f \in Y^X.$$

But this is not true, because the function $f_2 \in Y^{X_2}$, defined by the formula

$$f_2\left(h_2(0)\right) = f_2\left(h_2\left(\frac{1}{n}\right)\right) = 0 \quad \text{for} \quad n = 2, 3, \ldots,$$

$$f_2\left(h_2(1)\right) = f_2\left(h_2\left(1+\frac{1}{n}\right)\right) = 1 \quad \text{for} \quad n = 1, 2, \ldots$$

has an extension $f \in Y^X$.

Let us observe that this example shows that the hypothesis that the number k appearing in examples 1 and 2 is finite is essential.

12. Two ANR-sets X_1 and X_2 are said (after Hurewicz [8]) to be of the same homotopy type provided that there exist two continuous maps $f_1 \in X_1^{X_2}$ and $f_2 \in X_2^{X_1}$ such that $f_1 f_2$ is homotopic to the identity in $X_1^{X_1}$ and $f_2 f_1$ is homotopic to the identity in $X_2^{X_2}$. By a theorem of Fox [6], two ANR-sets X_1 and X_2 are of the same homotopy type if and only if there exists an ANR-set X such that both the set X_1 and the set X_2 are homeomorphic with some deformation retracts of X.

It follows by theorem 5 that X_1 and X_2 are exact reductions of X relatively to every space Z. Hence two ANR-sets of the same homotopy type are equivalent relatively to every ANR-set Y.

PROBLEM 2. *Let X_1 and X_2 be two ANR-sets such that*

$$X_1 \equiv X_2 \text{ rel. } Y,$$

for every ANR-set Y. Is it true that X_1 and X_2 are necessarily of the same homotopy type?

PROBLEM 3. *Is it true that the equivalence*

$$X_1 \equiv X_2 \text{ rel. } S^n \quad \textit{for every} \quad n = 0, 1, 2, \ldots$$

implies the equivalence

$$X_1 \equiv X_2 \text{ rel. } Y$$

for every ANR-set Y?

As a simple corollary we get from theorem 6 the following

THEOREM 9. *Let X_1, X_2 be compacta, let Y be an ANR-set and let $\mathfrak{X}$ denote the collection of all normal spaces, or the collection of all compacta. Then the relation*

$$X_1 \equiv X_2 \text{ rel. } Y$$

implies the isomorphy of d-sets $[Y_{\mathfrak{X}}^{X_1}]$ and $[Y_{\mathfrak{X}}^{X_2}]$.

13. It would be interesting to find the relations between topological invariants of two spaces X_1, X_2 which are equivalent relatively to a given ANR-set Y. As yet this problem is far from being solved. We can only prove the following, rather special,

THEOREM 10. *If X_1, X_2 are compacta of dimension $\leqslant m$, equivalent in the dimension $m+1$ to the Euclidean sphere S^n of the dimension $n \geqslant (m+2)/2$, then the n-dimensional cohomotopy groups $\pi^n(X_1)$ and $\pi^n(X_2)$ are isomorphic.*

Proof. We can assume that X_1 and X_2 are exact reductions relatively to S^n of a space X satisfying the condition

$$\dim X \leqslant m+1.$$

Since $m+1 \leqslant 2n-1$, we infer, by [1] and [9], that for every two maps $f, g \in S^{n^X}$ there exist: a decomposition of X into two closed sets M and N and two maps $f', g' \in S^{n^X}$ homotopic to f and g respectively, and a point $a \in S^n$ such that

$$f'(x) = a \quad \text{for every} \quad x \in M,$$
$$g'(x) = a \quad \text{for every} \quad x \in N.$$

Then setting

$$h(x) = g'(x) \quad \text{for every} \quad x \in M,$$
$$h(x) = f'(x) \quad \text{for every} \quad x \in N,$$

we get a map $h \in S^{n^X}$, called the *union* of the maps f and g. Consider the partial maps:

$$f_i = f/X_i; \quad g_i = g/X_i; \quad f_i' = f'/X_i; \quad g_i' = g'/X_i; \quad h_i = h/X_i$$

for $i = 1, 2$. Evidently f_i is homotopic to f_i', and g_i is homotopic to g_i' in $S^{n^{X_i}}$ for $i = 1, 2$, and h_i is the union of f_i and g_i.

Since $\dim X_i \leqslant m \leqslant 2n-2$, we infer ([1] and [9]) that the homotopy class $\alpha(h_i)$ depends only on the homotopy classes $\boldsymbol{f}_i' = \boldsymbol{f}_i$ and $\boldsymbol{g}_i' = \boldsymbol{g}_i$. By the definition of the n-th cohomotopy group $\pi^n(X_i)$, it coincides with the set $[S^{n^{X_i}}]$ in which the group operation (addition) is given by the formula

$$\boldsymbol{f}_i + \boldsymbol{g}_i = \boldsymbol{h}_i.$$

On the other hand, since X_1 and X_2 are exact reductions of X relatively to S^n, the operation η_i assigning to every homotopy class $\boldsymbol{f}_i \in [S^{n^{X_i}}]$ the homotopy class $\boldsymbol{f} \in [S^{n^X}]$ of an extension f of f_i is one-to-one and it transforms $[S^{n^{X_i}}]$ onto $[S^{n^X}]$. Setting

$$\vartheta = \eta_2^{-1}\eta_1$$

we get a one-to-one operation transforming $[S^{n^{X_1}}]$ onto $[S^{n^{X_2}}]$.

In order to complete the proof, it suffices to show that the operation ϑ is a homomorphism, i.e. that

$$\vartheta(\boldsymbol{h}_1) = \vartheta(\boldsymbol{f}_1) + \vartheta(\boldsymbol{g}_1).$$

But this is evident, because

$$\vartheta(\boldsymbol{h}_1) = \eta_2^{-1}\eta_1(\boldsymbol{h}_1) = \eta_2^{-1}(\boldsymbol{h}) = \boldsymbol{h}_2,$$
$$\vartheta(\boldsymbol{f}_1) = \eta_2^{-1}\eta_1(\boldsymbol{f}_1) = \eta_2^{-1}\eta_1(\boldsymbol{f}_1') = \eta_2^{-1}(\boldsymbol{f}') = \boldsymbol{f}_2' = \boldsymbol{f}_2,$$
$$\vartheta(\boldsymbol{g}_1) = \eta_2^{-1}\eta_1(\boldsymbol{f}_1) = \eta_2^{-1}\eta_1(\boldsymbol{g}_1') = \eta_2^{-1}(\boldsymbol{g}') = \boldsymbol{g}_2' = \boldsymbol{g}_2,$$

and

$$(\boldsymbol{h}_2) = \boldsymbol{f}_2' + \boldsymbol{g}_2' = \boldsymbol{f}_2 + \boldsymbol{g}_2.$$

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