Some conditions for a mapping to be a covering

by

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Introduction. The following problem was the origin of this paper:

(P) *Let* $p \in S^n$ (*= the n-dimensional sphere*) *and let* $f\colon S^n \to S^n$ *be a continuous mapping such that* $f(S^n - \{p\}) \subset S^n - \{p\}$, $f(p) = p$ *and* $f|S^n - \{p\}$ *is a local homeomorphism. Must* f *be a homeomorphism?*

The answer is affirmative (Corollary 2 of this paper) and the proof is not difficult. Nevertheless, we did not find in literature any theorem or lemma adequate for reference when the statement was needed in the approximation theory (1). Then we found some conditions for a mapping which imply that this mapping is a covering (in the sense of Chevalley [4], p. 40). This can be used for solving the above question and presents some analogy to the results of Eilenberg [5]. All this is given in the present paper.

Main definitions. All topological spaces are supposed to be Hausdorff spaces. A mapping is called *open* if the images of the open sets are open sets. A mapping $f\colon X \to Y$ is called a *local homeomorphism* if every point $p \in X$ has a neighbourhood V such that the partial mapping $f|V$ is a homeomorphism.

The theory of covering mappings is given in [4], p. 40-60 (see also [6]). The main definitions used in this paper are the following:

A pair (X, f) is called a *covering* of Y if X is a connected and locally connected space, f is a mapping of X onto Y and every point $y \in Y$ has a neighbourhood U such that for every connected component C of $f^{-1}(U)$ the partial mapping $f|C$ is a homeomorphism of C onto U (2).

Two coverings (X_1, f_1), (X_2, f_2) of Y are called *equivalent* if there exists a homeomorphism h of X_1 onto X_2 such that $f_1 = f_2 h$.

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(1) In a problem of existence and uniqueness of some polynomials (see [8]).

Added in proof: We have found now a paper of Browder [3] which contains results near to ours. In particular his Theorems 5 and 6 are similar to our Theorems 2 and 1 respectively. However our theorems are concerning different class of topological spaces. Corollaries 1 and 2 can also be derived from Browder's results.

(2) A less restrictive definition, which does not require the local connectedness of the spaces X and Y, is sometimes used (for instance see [7]).

A space X is called *simply connected* if it is connected and locally connected and every covering of X is equivalent to the covering (X, i), where i is the identity mapping [3].

Main theorems. We shall give some conditions for a pair (X, f), implying that it is a covering.

THEOREM 1. *If*

(i) *X is connected and X or Y is locally connected,*

(ii) *$f\colon X \to Y$ is an open local homeomorphism onto Y,*

(iii) *every point $p \in Y$ is an interior point of a set $H \subset Y$ such that $f^{-1}(H)$ is compact,*

then (X, f) is a covering of Y.

Proof. For every $p \in Y$ the set $f^{-1}(p)$ is finite. In fact, by (iii) $f^{-1}(p)$ is compact. Suppose that x_0 is a limit point of $f^{-1}(p)$. Then for every open set $V \subset X$ containing x_0 the set $f^{-1}(p) \cap V$ is infinite and $f|V$ is not a 1-1 mapping, which contradicts (ii).

Putting

$$(*)\qquad f^{-1}(p) = \{x_1, \ldots, x_n\},$$

we infer from (ii) and (iii) that there exist open sets $V_1, \ldots, V_n$ such that $x_i \in V_i \subset f^{-1}(H)$, $f|V_i$ is a homeomorphism and $V_i \cap V_j = \emptyset$ for $i, j = 1, \ldots, n$; $i \neq j$.

Let

$$(**)\qquad W = f(V_1) \cap \ldots \cap f(V_n).$$

It follows from (ii) that W is an open set and $p \in W$.

By (i) and (ii), X and Y are connected and locally connected. We shall show that there exists a connected open set U such that $p \in U \subset H \cap W$ and $f^{-1}(U) \subset V_1 \cup \ldots \cup V_n$. Suppose on the contrary that for every connected open set U contained in $H \cap W$ and containing p we have

$$f^{-1}(U) \cap [X - (V_1 \cup \ldots \cup V_n)] \neq \emptyset.$$

By the compactness of $f^{-1}(H)$ it follows from $(*)$ that [4]

$$\emptyset \neq \bigcap_U f^{-1}(\overline{U}) \cap [X - (V_1 \cup \ldots \cup V_n)] = \{x_1, \ldots, x_n\} \cap [X - (V_1 \cup \ldots \cup V_n)],$$

contrary to the definition of V_i. Hence such a U exists.

(3) A remark in [4], p. 44, concerning the word "every" used in this definition is not exact. In fact, it is completely correct to use the word "every" in each axiomatic set theory, and this does not imply the necessity of treating as a set the universum to which "every" is applied. On the other hand the construction performed, pp. 44-45, is not superfluous since it is used in the proof of Theorem 4, p. 54.

(4) $\overline{U}$ denotes the closure of U.

Let C be a component of $f^{-1}(U)$. Then $C \subset V_1 \cup \ldots \cup V_n$, and thus $C \subset V_j$ for some j. Therefore $f|C$ is a homeomorphism.

To finish the proof, it is enough to show that $f(C) = U$. In fact, since $U \subset W \subset f(V_j)$ according to $(**)$, and $f|V_j$ is a homeomorphism, the set $(f|V_j)^{-1}(U)$ is connected. But we obviously have $C \subset (f|V_j)^{-1}(U) \subset f^{-1}(U)$, whence $C = (f|V_j)^{-1}(U)$. It follows that

$$f(C) = (f|V_j)(C) = (f|V_j)(f|V_j)^{-1}(U) = U.$$

Theorem 1 clearly implies the following

COROLLARY 1. *If f is an open local homeomorphism of a connected locally connected and compact space X onto a space Y, then (X, f) is a covering of Y.*

Remark 1. It is easy to construct a mapping f of a compact disk D onto itself which is a local homeomorphism and therefore satisfies (iii) but is such that (D, f) is not a covering of D. This shows that the supposition "open" in Theorem 1 is essential.

Remark 2. It is easy to construct a mapping f of the plane R^2 onto itself which is a local homeomorphism and therefore is open but such that (R^2, f) is not a covering of R^2. This shows that condition (iii) in Theorem 1 is essential. On the other hand (iii) is very restrictive, for instance it is not satisfied by the covering (R, e^{it}) of the circle $|z| = 1$. This example suggests perhaps that the words "$f^{-1}(H)$ is compact" in (iii) can be replaced by "every component of $f^{-1}(H)$ is compact". This, however, is false—an example is described in the Appendix at the end of this paper. The theorem which follows applies to the most important examples of coverings, e.g. to (R, e^{it}).

THEOREM 2. *Condition* (iii) *in Theorem* 1 *can be replaced by the following one*:

(iii') *every point $p \in Y$ is contained in the interior of a set $H \subset Y$ such that*

1° *H is simply connected,*

2° *every component of $f^{-1}(H)$ is compact.*

Proof. Let C be a component of $f^{-1}(H)$. Since, by 1°, H is locally connected and f is a local homeomorphism, $f^{-1}(H)$ is locally connected. Thus C is open in $f^{-1}(H)$. The mapping f being open, the partial mapping $f|f^{-1}(H)$ is also open [5]. Therefore $f(C)$ is open in H and compact by 2°. This gives $f(C) = H$, because H is connected by 1°. Hence by Theorem 1 the pair $(C, f|C)$ is a covering of H. By 1° the mapping $f|C$ is a homeomorphism and Theorem 2 follows.

(5) This general implication is an immediate consequence of the trivial formula: $f[G \cap f^{-1}(H)] = f(G) \cap H$.

Remark 3. A simply connected space X is not necessarily locally simply connected, i.e. we can have $p \in V \subset X$ with V open and such that no neighbourhood of p contained in V is simply connected. Such is the continuum K constructed by Borsuk [1] (the above properties of Borsuk's continuum are consequences of the results proved in [1], [2] and [6]).

Remark 4. It is clear that if Y is supposed to be simply connected in Theorem 1 or 2 or Corollary 1, then the mapping f must be a homeomorphism. An analogous Theorem of Eilenberg [5] is the following: *if f is an open local homeomorphism of a compact arcwise connected space X onto a space Y such that $\pi_1(Y) = 0$, then f is a homeomorphism* (6). Note that this theorem is independent of our results, because 1° X is not supposed to be locally connected and 2° the condition $\pi_1(Y) = 0$ is independent of the simple connectedness of Y—of course it does not imply the local connectedness (and thus the simple connectedness) of Y and is not a consequence of the simple connectedness of Y, Borsuk's continuum K mentioned in Remark 3 being simply connected and such that $\pi_1(K) \neq 0$ (see [1]).

The original problem. We start with the following theorem:

THEOREM 3. *If*

(i) *X is a compact space,*

(ii) *$Q \subset X$ and Q is connected and locally connected,*

(iii) *f: $X \to Y$ is a continuous mapping and $f|Q$ is an open local homeomorphism,*

(iv) *Q or $f(Q)$ is locally compact,*

(v) *$f(Q) \cap f(X-Q) = \emptyset$,*

then $(Q, f|Q)$ is a covering of $f(Q)$.

Proof. By (iii) and (iv) every point $p \in f(Q)$ belongs to the interior relative to $f(Q)$ of a compact set H contained in $f(Q)$. Whence, by (v), we have $f^{-1}(H) \subset Q$. H being compact, and thus closed in $f(X)$, $f^{-1}(H)$ is closed in X by (iii). Therefore $f^{-1}(H)$ is compact by (i). This gives, according to (ii) and (iii), all hypotheses of Theorem 1 for the mapping $f|Q$: $Q \to f(Q)$, and Theorem 3 follows.

Now we return to problem (P) (see the Introduction). We shall use the fact that, owing to (i), Q is locally compact if it is open.

COROLLARY 2. *The answer to problem* (P) *is affirmative.*

Proof. We shall show first that (7)

$$(=) \qquad f(S^n - \{p\}) = S^n - \{p\}.$$

(6) $\pi_1(Y)$ denotes here the classical fundamental group, and not the fundamental group defined by Chevalley (see [4], p. 52). The vanishing of Chevalley's fundamental group is equivalent to the simple connectedness.

(7) This is known and can be derived from the results of Reichbach [9], for instance.

Indeed, let B be the boundary of $f(S^n)$ in S^n. If V is an open subset of S^n containing a point q such that $f(q) \in B$, then $f|V$ is not a homeomorphism by virtue of Brouwer's Theorem on the invariance of interior points. Therefore, $f|S^n - \{p\}$ being a local homeomorphism, we have $f^{-1}(B) \subset f^{-1}(p) = \{p\}$, i.e. $B = ff^{-1}(B) \subset \{f(p)\}$. But since $f(S^n)$ is compact and has interior points, its boundary cannot be a single point, whence $B = \emptyset$. It follows that $f(S^n) = S^n$ and $S^n - \{p\} = f(S^n) - f(\{p\}) \subset f(S^n - \{p\})$, which gives (=) according to the hypothesis of (P).

Now we put $X = S^n$, $Q = S^n - \{p\}$ and observe that conditions (i)-(v) from Theorem 3 hold. Therefore $(Q, f|Q)$ is a covering of $f(Q)$. But we have $f(Q) = Q$ according to (=). Since Q is simply connected (see [4], p. 58, Corollary), we infer that $f|Q$ is a homeomorphism of Q onto Q. Thus f is a homeomorphism of S^n onto S^n.

Appendix. We describe here the example mentioned in Remark 2. It will be a mapping f: $X \to Y$ such that:

(i) *X is a connected and locally connected closed subset of the plane R^2,*

(ii) *Y is a locally connected continuum in R^2,*

(iii) *f is an open local homeomorphism of X onto Y,*

(iv) *every point $y \in Y$ is contained in the interior relative to Y of a set $H \subset Y$ such that each component C of $f^{-1}(H)$ is compact and $f(C) = H$,*

(v) *(X, f) is not a covering of Y.*

The definition of Y is the following:

Y is a union of circles $K_1, K_2, \ldots$ tangent at a point p, K_{i+1} is contained in the closed disk bounded by K_i and the radius of K_i is $1/i$ for $i = 1, 2, \ldots$ (see fig. 1).

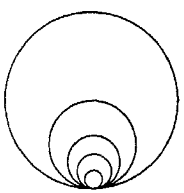

Fig. 1

The definition of X is the following:

Let L be a closed half-line and let $p_1, p_2, \ldots$ be a sequence of points of L such that p_1 is the extremity of L, $p_i \neq p_j$ for $i \neq j$, and the distance from p_i to p_{i+1} is 2 for $i = 1, 2, \ldots$

Let $Y_1, Y_2, \ldots$ be a sequence of sets such that Y_1 is isometrical to $Y - K_1 \cup \{p\}$, Y_{2i} and Y_{2i+1} are isometrical to $Y - K_1 - K_{i+1} \cup \{p\}$ and all circles in Y_i are tangent to L at the point p_i on the same side of L $(i = 1, 2, \ldots)$.

Let $I_1, I_2, \ldots$ be the sequence of closed intervals of L, the extremities of I_i being p_i and p_{i+1}, and let $U_1, U_2, \ldots$ be the sequence of closed half-circles, the extremities of U_i being p_i and p_{i+1} and U_i not lying on the same side of L as the sets Y_i $(i = 1, 2, \ldots)$.

X is the union $L \cup Y_1 \cup U_1 \cup Y_2 \cup U_2 \cup \ldots$ (see fig. 2).

f isometrically maps Y_i into Y, whence $f(p_i) = p$ for $i = 1, 2, \ldots$ f homeomorphically maps the interior of I_{2i-1} and of U_{2i-1} onto $K_1 - \{p\}$

and the interior of I_{2i} and of U_{2i} onto $K_{i+1}-\{p\}$, $i=1,2,\ldots$, in such a way that the mappings $I_{2i-1}\cup U_{2i-1}\to K_1$ and $I_{2i}\cup U_{2i}\to K_{i+1}$ are local homeomorphisms.

It is easy to verify properties (i)-(iv); let us show (v). If V is open in Y and contains p, then V contains a circle K_j, where $j>1$. Therefore

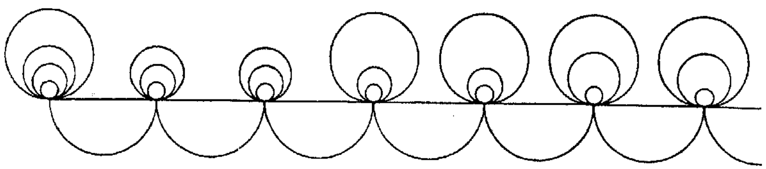

Fig. 2

$I_{2(j-1)}\subset f^{-1}(K_j)\subset f^{-1}(V)$. Hence there is a component C of $f^{-1}(V)$, which contains the interval $I_{2(j-1)}$, and thus also its extremities p_{2j-2} and p_{2j-1}. However, $p_{2j-2}\neq p_{2j-1}$ and $f(p_{2j-2})=f(p_{2j-1})=p$. It follows that $f|C$ is not a homeomorphism and (v) is proved.

Note. The space Y is not locally simply connected at the point p. This peculiarity of our example is necessary in view of Theorem 2.

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On plane dendroids and their end points in the classical sense

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§ 1. Dendroids. A continuum X (i.e. a compact connected metric space) is called a *dendroid* (¹) if it is arcwise connected and hereditarily unicoherent, i.e. if every two distinct points of it are joined by an arc contained in X and every subcontinuum of X (as well as whole X) is unicoherent (see [4], p. 104).

As has recently been proved by J. Charatonik (in a paper which is now being prepared for publication) the condition for X to be a dendroid is equivalent, among others, to each of the following ones:

(i) *every two distinct points of X are joined by exactly one irreducible continuum contained in X, namely by an arc*,

(ii) *X is an arcwise connected, homologically acyclic and 1-dimensional continuum* (*or a single point*).

By (ii) every non-degenerate dendroid is a unicoherent 1-dimensional continuum; therefore (see [4], p. 338) *every locally connected dendroid is a dendrite*. This means that dendroids constitute a generalization of dendrites. In this character they are found in literature. For instance in 1954 Borsuk [2] proved that dendroids have the fixed point property. This was generalized in 1958 by Ward [8], who was considering more abstract spaces, namely without the requirement that X be a metrizable space. Let us mention also paper [9] of Ward, where some additional references are given.

I was encouraged to study dendroids by Professor B. Knaster. I express my gratitude to him.

§ 2. End points in the classical sense. I say that a point x of an arcwise connected continuum X is an *end point of X in the classical sense* if x is an end point of every arc contained in X and containing x. It is well-known (see [4], p. 203) that if X is a locally connected con-

(¹) This term was proposed by B. Knaster and became usual in his Seminar in Wrocław.