Nilpotent free groups

by

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The following theorem, proved by A. I. Malcev in [3], will be denoted further by (M).

(M) *Let G be a nilpotent free group of class c, and X a subset of G of the cardinality $|X| > 1$. Then X is a nilpotent free generating system for some subgroup of G, of the same class c, if and only if X is linearly independent modulo the derived subgroup G'.*

This paper consists of two parts. The first is devoted to a group theoretical proof of (M). The proof is based on the following theorem.

(T) *If $x_1, \ldots, x_i, \ldots$ is a free, or nilpotent free, generating system of a free, or nilpotent free, group G, then a system $x_1^{n_1} \cdot x_1', \ldots, x_i^{n_i} \cdot x_i', \ldots$ is free, or nilpotent free, for any $x_i' \in G'$, and for any positive integers n_i.*

The proof of (T) essentially needs M. Hall's theory of basic commutators exposed, for example, in [1].

The second part contains theorems that can be derived from (M):

THEOREM 1. *A subgroup H of a nilpotent free group G is a nilpotent free group if and only if it satisfies the condition $H' = H \cap G'$ or is a cyclic group.* (H' and G' are the derived subgroups of H and G.)

THEOREM 2. *Every retract of a nilpotent free group G, is a nilpotent free factor of G and a nilpotent free subgroup of G.*

The analogous statement fails for retracts of free or solvable free groups, see [4].

THEOREM 3. *An endomorphism of a nilpotent free group G is an automorphism of that group if and only if it induces an automorphism of G/G'.*

The terminology of the paper is the same as that in the book of M. Hall [1]. Some basic notions concerning varietes of nilpotent groups are listed in an introductory part. They can be found partly in papers [2], [5] and [6], and partly in the book quoted.

1. We define recursively the simple commutators as: $(x_1) = x_1$, $(x_1, x_2) = x_1^{-1} \cdot x_2^{-1} \cdot x_1 \cdot x_2$, and $(x_1, \ldots, x_{n+1}) = \big((x_1, \ldots, x_n), x_{n+1}\big)$. A group

G is nilpotent of the class c, $c = 1, 2, \ldots$, if $(g_1, \ldots, g_{c+1}) = 1$ for all $g_1, \ldots, g_{c+1} \in G$. In this case we say briefly that G has nil-c. All groups having nil-c form a variety of groups (i.e. an equationally definable class of groups); thus we can speak about nil-c-free groups and nil-c-free products for every $c = 1, 2, \ldots$ For $c = 1$ we deal with abelian free groups and direct products.

We denote by $\Gamma_n(G)$ the word subgroup of G defined by the word $(x_1, \ldots, x_n)$, i.e. the subgroup generated by all $(g_1, \ldots, g_n)$ for $g_1, \ldots, g_n \in G$. For $n = 2$, $\Gamma_2(G) = G'$ is the derived subgroup of G.

One can easily prove that $\Gamma_{n+1}(G) = (\Gamma_n(G), G)$ so that the sequence of fully invariant subgroups

$$G = \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \ldots \supseteq \Gamma_n(G) \supseteq \ldots$$

is the lower central series of G.

Let X be a generating system of a group G. We shall call X a *nil-c-free generating system of* G if and only if there exists a free (1) group F, freely generated by some system Y, such that G can be represented as the factor group $F/\Gamma_{c+1}(F)$, in such a manner that between Y and X the representation is one to one. A group which has a nil-c-free generating system will be called a *nil-c-free group*. A system of a group which is a nil-c-free generating system for a subgroup will be called in the sequel a *nil-c-free system*.

If X is a nil-c-free generating system of G, then the image of X under the natural homomorphism of G onto G/G' is an abelian free generating system of G/G'. It simply means that G/G' is a direct product of cyclic groups generated by the images of elements belonging to X. This gives us a base for the following definition.

DEFINITION. A system X of a group G *is linearly independent modulo the derived subgroup* G' if and only if the image of X into G/G' is an abelian free system.

Now we can formulate some statements about nil-c-free systems.

P1. *A system X of any group is a nil-c-free system if and only if:*

1. *The subgroup $\{X\}$ generated by X has nil-c.*

2. *Every mapping of X into any group having nil-c, can be extended to the homomorphism of $\{X\}$ into that group.*

P2. *A system is nil-c-free if and only if it is a locally nil-c-free system, i.e. if every finite subsystem is nil-c-free.*

(1) In the sequel "free" without any "prefixes" always means free in the variety of all groups.

P3. *If $x_1, \ldots, x_i, \ldots, x_j, \ldots$ is a nil-c-free generating system of G, then the following three systems:*

$$(*)\qquad \begin{array}{l} x_1, \ldots, x_j, \ldots, x_i, \ldots, \\ x_1, \ldots, x_i^{-1}, \ldots, x_j, \ldots, \\ x_1, \ldots, x_i \cdot x_j, \ldots, x_j, \ldots \quad (i \neq j), \end{array}$$

are nil-c-free generating systems of G.

Property P1 easily follows from the definition since $\Gamma_{c+1}(G)$ is a fully invariant subgroup of G (2). Properties P2 and P3 may easily be checked by P1.

Now we report some facts about nilpotent groups.

P4. *For any group G, if $f \in \Gamma_k(G)$, $g \in \Gamma_l(G)$, and $h \in \Gamma_m(G)$, then*

4.1. $(f, g) \equiv 1 \pmod{\Gamma_{k+l}(G)}$,

4.2. $(f \cdot g, h) \equiv (f, h) \cdot (g, h) \pmod{\Gamma_{k+l+m}(G)}$,

4.3. $(h, f \cdot g) \equiv (h, f) \cdot (h, g) \pmod{\Gamma_{k+l+m}(G)}$.

P5. *If F is a free group freely generated by r elements, then $\Gamma_n(F)/\Gamma_{n+1}(F)$ are for $n = 1, 2, \ldots$, abelian free groups. Their dimensions $M_r(n)$, are defined by Witt formulas*

$$(**)\qquad M_r(n) = 1/n \sum_{d|n} \mu(d) \cdot r^{n/d}$$

where $\mu(d)$ denotes the Möbius function of d.

For the proof of P4 and P5 we refer the reader to a paper of Witt [6] or to M. Hall's book [1] cited above. In this book the reader will find a proof of the following property of nilpotent groups quoted below as (P).

(P) *If H is a subgroup of G and $H \cdot G' = G$, then $H = G$.*

The meaning of this is the following: If, for a nilpotent group G, we take in G/G' a generating system, then every preimage in G of that system generates G.

The remaining of this part is devoted to the proof of (M). The definition of linear independence given in the introduction may be stated in a more operative form as follows:

A system $z_1, z_2, \ldots$ of a group G is *linearly independent* modulo G' if and only if from the relation $z_{i_1}^{m_1} \ldots z_{i_l}^{m_l} \in G'$, $i_1 < \ldots < i_l$, it follows that $z_{i_1}^{m_1} = 1, \ldots, z_{i_l}^{m_l} = 1$.

(2) Analogous to properties P1-P3 are generally valid for free systems in a variety. In this case it is convenient to take the property analogous to P1 as a definition.

The notion of independence is close to the following notion of regularity:

DEFINITION. A system $z_1, z_2, \ldots$ of a nil-c-free group G is *regular* if there exists a nil-c-free generating system of G, $x_1, x_2, \ldots$ say, such that for some positive integers n_j we have

$$z_1 = x_{i_1}^{n_1} \cdot x_1',\; z_2 = x_{i_2}^{n_2} \cdot x_2',\; \ldots, \qquad i_j \neq i_k \quad \text{for} \quad j \neq k,$$

for suitably chosen $x_1', x_2', \ldots \in G$ (3). A system such that each of its finite subsystems is regular will be called *locally regular*.

The following lemma is true.

LEMMA 1. *Let U be a system contained in a nil-c-free group G and linearly independent modulo G'; then every finite subsystem of U may be transformed by operations* (∗) *onto a regular system.*

Proof. Choose a nil-c-free generating system Y of G. Let $u_1, \ldots, u_r$ be any given finite subsystem of U. The elements of the subsystem depend on a finite number of elements belonging to Y, e.g. on $y_1, \ldots, y_s$. Therefore we may write $u_1, \ldots, u_r$ in the form

$$(**)\qquad \begin{aligned} u_1 &= y_1^{n_{11}} \ldots y_s^{n_{1s}} \cdot y_1', \\ &\ldots\ldots\ldots\ldots \\ u_r &= y_1^{n_{r1}} \ldots y_s^{n_{rs}} \cdot y_r', \quad \text{where} \quad y_1', \ldots, y_r' \in G'. \end{aligned}$$

Note that we do not need here the linear independence of U. The representation (∗∗) may be obtained by successive application of the formula $a \cdot b = b \cdot a \cdot (a, b)$. But it may easily be seen that from the linear independence $\operatorname{mod} G'$ of the whole system Y follows the linear independence of the rows of the matrix $N = [n_{ij}]_{i=1,\ldots,r;\, j=1,\ldots,s}$ with integral coefficients.

By elementary operations on the rows and on the columns of the matrix

$$N = \begin{bmatrix} n_{11} \ldots n_{1s} \\ n_{r1} \ldots n_{rs} \end{bmatrix},$$

we can transform it into the diagonal form

$$N \sim \begin{bmatrix} n_1 & & 0 \ldots 0 \\ & \ddots & \\ 0 & & n_r \ldots 0 \end{bmatrix} \qquad (s \text{ columns, } r \text{ rows}).$$

(All $n_1, \ldots, n_r$ are positive because of the linear independence of rows.) Elementary operations on rows are permutations, multiplications by -1, and additions of rows. Each of those operations on the matrix may be obtained by a suitable operation (∗) on $u_1, \ldots, u_r$. E.g. the product

$$u_1 \cdot u_2 = y_1^{n_{11}} \ldots y_s^{n_{1s}} \cdot y_1' \cdot y_1^{n_{21}} \ldots y_s^{n_{2s}} \cdot y_2' = y_1^{n_{11}+n_{21}} \ldots y_s^{n_{1s}+n_{2s}} \cdot y_1'', \qquad y_1'' \in G',$$

gives the sum of the two rows. Of course it changes the element belonging to G', but this does not matter.

Similarly every elementary operation on the columns of the matrix can be obtained by a suitable operation (∗) on $y_1, \ldots, y_s$. According to P3 it is only a change of the initial nil-c-free generating system of G into another one.

Thus there exist a sequence of operations (∗) on $u_1, \ldots, u_r$ giving $z_1, \ldots, z_r$, and such $x_1, \ldots, x_s$ belonging to a nil-c-free generating system X of G that

$$z_1 = x_1^{n_1} \cdot x_1', \quad \ldots, \quad z_r = x_r^{n_r} \cdot x_r',$$

for some $x_1', \ldots, x_r' \in G'$. (It may easily be seen that $x_1', \ldots, x_r'$ can be taken from $H' \subset G'$ where $H = \{x_1, \ldots, x_s\}$.) This proves the lemma.

(Note that the proof is correct for any group which is free in some variety of groups (4).)

To prove theorem (T) we need an arithmetical lemma about commutators.

LEMMA 2. *If in a commutator $f(x_1, \ldots, x_r)$ of weight k, each argument x_i appears m_i times, $m_i \geq 0$ for $i = 1, \ldots, r$, then for every group G and every $g_1, \ldots, g_r \in G$ and every $h_1, \ldots, h_r \in G'$ we have an identity*

$$(1)\qquad f(g_1^{n_1} \cdot h_1, \ldots, g_r^{n_r} \cdot h_r) \equiv f(g_1, \ldots, g_r)^{n_1^{m_1} \ldots n_r^{m_r}} \;(\operatorname{mod} \Gamma_{k+1}(G)).$$

The lemma is true for any integers $n_1, \ldots, n_r$ (if we put $n_i^{m_i} = 1$ when $n_i = m_i = 0$); but we shall use them only for positive integers. Therefore we shall give the proof only in this case. It shall be proved by induction.

For $k = 1$ the lemma is obvious. Let

$$f(x_1, \ldots, x_r) = \big(f_1(x_1, \ldots, x_r), f_2(x_1, \ldots, x_r)\big)$$

be formed from commutators f_1 and f_2 of weights k_1 and k_2 where $k_1 + k_2 = k$. If an argument x_i appears t_i times in f_1 and s_i times in f_2, then it appears $m_i = t_i + s_i$ times in f (5).

(3) The notion of regularity may be defined in the same manner for system contained in any group that is free for some variety. See also note 4 on page 263.

(4) Some troubles arise for varieties with the non-zero exponent. (The exponent is defined in paper [2] for instance.) In this case all is well if we take statement on page 261 as a definition of linear independence, and if in the definition of regularity we add that all the $n_1, \ldots, n_r$ are prime with the exponent. Also the linear independence of rows of the matrix N ought to be understood over the ring of integers modulo the exponent.

(5) By the assumption that f is a non-empty word. For the empty word the lemma is trivial; then (1) is an identity $1 \equiv 1$.

Assume that the lemma is true for f_1 and f_2; from this assumption we have

$$f_1(g_1^{n_1}\cdot h_1, \ldots, g_r^{n_r}\cdot h_r) = f_1(g_1, \ldots, g_r)^{n_1^{t_1}\ldots n_r^{t_r}}\cdot h',$$
$$f_2(g_1^{n_1}\cdot h_1, \ldots, g_r^{n_r}\cdot h_r) = f_2(g_1, \ldots, g_r)^{n_1^{s_1}\ldots n_r^{s_r}}\cdot h'',$$

where $h' \in \Gamma_{k_1+1}(G)$, $h'' \in \Gamma_{k_2+1}(G)$. Then

$$f(g_1^{n_1}\cdot h_1, \ldots, g_r^{n_r}\cdot h_r) = \big(f_1(g_1, \ldots, g_r)^{n_1^{t_1}\ldots n_r^{t_r}}\cdot h',\, f_2(g_1, \ldots, g_r)^{n_1^{s_1}\ldots n_r^{s_r}}\cdot h''\big).$$

Applying P4.1-P4.3 to the right side of this formula we obtain congruence (1), as required.

Theorem (T), basic for this section, can now be formulated as follows:

Let a group F be free or nil-c-free. Then any system contained in F and locally regular in F is free or nil-c-free respectively.

Proof. According to P2 we need to prove the theorem only for finite systems. Let $z_1, \ldots, z_r$ be a regular system contained in F. Then there exists a generating system $x_1, x_2, \ldots$ of F, which is free or nil-c-free, such that $z_1 = x_1^{n_1}\cdot x_1', \ldots, z_r = x_r^{n_r}\cdot x_r'$ for some $x_1', \ldots, x_r' \in F'$ and some positive integers $n_1, \ldots, n_r$. The primed x-es are generated by a finite number of the unprimed ones. We may suppose that these elements and the $x_1, \ldots, x_r$ are contained among the first s elements $x_1, \ldots, x_s$. Denote by H the free, or nil-c-free, group generated by $x_1, \ldots, x_s$. Since $H' = H \cap F'$ (6), we need not keep in mind the group F (perhaps infinitely generated). The elements $x_1', \ldots, x_r'$ belong to a meet of H and F' which is H'. Everything is reduced to the proof of (T) in the case where F is finitely generated.

Now we give the proof in the case of $F = \{x_1, \ldots, x_s\}$.

To prove that the system $z_1, \ldots, z_r$ is free (or nil-c-free) we need (according to P1) to prove that there exists no word f such that

$$f(x_1, \ldots, x_s) \neq 1 \quad \text{and} \quad f(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s) = 1.$$

Let us assume the opposite, i.e. that the such word exists. Then for some k, $f(x_1, \ldots, x_s) \in \Gamma_k(F)$ and $f(x_1, \ldots, x_s) \notin \Gamma_{k+1}(F)$. The word f can be represented in a unique manner as a product

$$(2) \qquad f(x_1, \ldots, x_s) \equiv f_1(x_1, \ldots, x_s)^{l_1}\ldots f_d(x_1, \ldots, x_s)^{l_d} \big(\mathrm{mod}\ \Gamma_{k+1}(F)\big)$$

of basic commutators $f_1, \ldots, f_d$ of weights k. (See [1].) Since $f(x_1, \ldots, x_s) \notin \Gamma_{k+1}(F)$, not all integers $l_1, \ldots, l_d$ in (2) are equal to zero.

(6) This is weaker than lemma 6, and can be checked in the same manner.

Let us notice that in the uniqueness of (2) we have used the assumption that the system $x_1, \ldots, x_s$ a free (or nil-c-free) generating system of the group F. Now we use this assumption to show that there exists an endomorphism of F which maps x_1 onto $x_1^{n_1}\cdot x_1'$, ..., x_r onto $x_r^{n_r}\cdot x_r'$, and the other x-es onto themselves. Under this isomorphism congruence (2) is mapped onto the following one:

$$(3) \qquad 1 = f(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s)$$
$$= f_1(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s)^{l_1}\ldots f_d(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s)^{l_d} \big(\mathrm{mod}\ \Gamma_{k+1}(F)\big).$$

Now we use the assumption that $x_1', \ldots, x_r' \in F'$. By this assumption and by the last lemma we obtain for the commutators $f_1, \ldots, f_d$ the formulae:

$$\left.\begin{array}{l} f_1(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s) \equiv f_1(x_1, \ldots, x_s)^{u_1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_d(x_1^{n_1}\cdot x_1', \ldots, x_r^{n_r}\cdot x_r', x_{r+1}, \ldots, x_s) \equiv f_d(x_1, \ldots, x_s)^{u_d} \end{array}\right\} \big(\mathrm{mod}\ \Gamma_{k+1}(F)\big)$$

where every u_j $(j = 1, 2, \ldots, d)$ is > 0 and has the form $n_1^{m_1}\ldots n_r^{m_r}$ (the exponents m_i being, in general, different for different u_j's). From these formulas and from (3) we obtain the congruence

$$1 \equiv f_1(x_1, \ldots, x_s)^{l_1\cdot u_1}\ldots f_d(x_1, \ldots, x_s)^{l_d\cdot u_d} \big(\mathrm{mod}\ \Gamma_{k+1}(F)\big).$$

It follows from the uniqueness of representations of elements by basic commutators that the last congruence has all exponents $l_1\cdot u_1 = 0$, ..., $l_d\cdot u_d = 0$. This is in spite of the fact that not all $l_1, \ldots, l_d$ are equal to zero. The proof is finished.

The last step towards the proof of (M) is the following lemma.

LEMMA 3. *Let G be any group having nil-c (not neccessarily nil-c-free). Then any system X of G, of cardinality $|X| > 1$, and nil-c-free is linearly independent modulo G'.*

Proof. If the system X is not linearly independent then for some elements e.g. $x_1, \ldots, x_s \in X$, $s \geq 1$, we have a relation

$$(4) \qquad x_1^{n_1}\ldots x_s^{n_s} = g \in G',$$

where $n_1, \ldots, n_s$ are non-zero integers. Changing some of the x-es into x^{-1}, we can assume that the exponents in (4) are all positive. Then the elements

$$(5) \qquad z_1 = x_1^{n_1}, \ldots, z_s = x_s^{n_s}, \quad \text{and the other} \quad x \in X,$$

form a regular subsystem of the nil-c-free group $H = \{X\}$. Using (T) we prove that this system is nil-c-free. From this system we construct in the following manner a new nil-c-free system consisting of two elements: the element $g \in G'$ and a new element h: If the length s of relation (4) is (i) $s = 1$, then h is any element of X, $h \neq x_1$; (ii) if $s > 1$, then $h = x_2$. In the case (i) it is a subsystem of (5). In the case (ii) the situation is more complicated; then we replace $z_1, z_2, \ldots, z_s$ of (5) by $z_1 \cdot z_2 \ldots z_s = g, z_2, \ldots, z_s$. Using P3 we prove that the changed system (5) is nil-c-free. Evidently this system contains g, h as a subsystem. In both cases it is proved that the system g, h is nil-c-free; the group $K = \{g, h\}$ is nil-c-free. Then according to the Witt theorem given by the formula (**) the group $\Gamma_c(K)$ is abelian free of the dimension $M_2(c) \neq 0$.

Since $g \in G'$, we prove by induction on i that $\Gamma_i(K) \subset \Gamma_{i+1}(G)$, $i = 1, \ldots, c$. The group G has nil-c; then $\Gamma_c(K) \subset \Gamma_{c+1}(G) = \{1\}$. It is contrary to the fact that the dimension of $\Gamma_c(K)$ is different from zero.

This proves that no relation such as (4) takes place, and X is linearly independent modulo G', as required.

Lemma 3 is the first half of theorem (M). The second half can be deduced as follows:

Let X be a linearly independent system, modulo the derived subgroup of a nil-c-free group G. Let $u_1, \ldots, u_r$ be a finite subsystem of X. Then we use Lemma 1, and transform, by operations (*), $u_1, \ldots, u_r$ onto the regular system $z_1, \ldots, z_r$. By (T) this system is nil-c-free. Since the inverse to the transformation (*) is also a transformation (*), we prove by P3 that $u_1, \ldots, u_r$ is a nil-c-free system.

We have proved that every finite subsystem of X is nil-c-free; now P2 finishes the proof of (M).

In the same way we can deduce the following:

COROLLARY. *Every system of a free group linearly independent modulo the derived subgroup is a free system.*

2. In this part we give some results derived from (M). Let G be any nilpotent group such that G/G' is an abelian free group, and let a subgroup H satisfy the condition

$$H' = H \cap G', \tag{6}$$

where $H' = (H, H)$ is a derived subgroup of H, and $G' = (G, G)$ a derived subgroup of G.

The subgroup $H \cdot G'/G'$ of G/G' is an abelian free group. Denote an abelian free generating system of this subgroup by $\bar{y}_1, \bar{y}_2, \ldots$ Choose any elements $y_1, y_2, \ldots$ of H such that $h(y_1) = \bar{y}_1$, $h(y_2) = \bar{y}_2, \ldots$; there h is the natural homomorphism of G onto G/G'. By (6) and by the second isomorphism theorem, groups $H \cdot G'/G'$ and H/H' are isomorphic. The superposition of h and that isomorphism maps the system $y_1, y_2, \ldots$ onto an (abelian free) generating system of H/H'. Using property (P) we prove that $y_1, y_2, \ldots$ is a generating system of H. This system is linearly independent modulo G', because the images under h form an abelian free system. We formulate this fact as a lemma.

LEMMA 4. *Let G be any nilpotent group (not neccessarily nil-c-free) such that G/G' is abelian free. Then every subgroup H satisfying condition* (6) *is generated by a system linearly independent modulo G'.*

Now let G be any group (not neccessarily nilpotent), and H a subgroup of G, generated by a system linearly independent modulo G'. Our aim is to prove that H satisfies condition (6).

The inclusion $H' \subset H \cap G'$ is evident. We prove the inverse inclusion. Let $g \in H \cap G'$. Then g as an element of H may be represented in the following form: $g = y_1^{n_1} \ldots y_s^{n_s} \cdot k$, $k \in H'$, where $y_1, \ldots, y_s$ denote elements that are from the generating system of H which is linearly independent modulo G'. Since $g \in G'$, we have a relation $y_1^{n_1} \ldots y_s^{n_s} = g \cdot k^{-1} \in G'$ between the elements of the linearly independent system. This proves that $y_1^{n_1} = 1, \ldots, y_s^{n_s} = 1$, and therefore $g \cdot k^{-1} = 1$. It means that $g = k \in H'$ and proves the inclusion $H \cap G' \subset H'$.

This proves the lemma

LEMMA 5. *If a subgroup H of a group G (not neccessarily nilpotent) is generated by a system linearly independent modulo G', then H satisfies condition* (6).

Both lemma proved in connection with (M) give

THEOREM 1. *A subgroup H of a nil-c-free group G is a nil-c-free group if only if it satisfies condition* (6)

$$H' = H \cap G'$$

or is a cyclic group.

Now we give a definition of a retract of a group: A subgroup H is called a *retract of a group* G if there exists an endomorphism e of G onto H, such that $e(h) = h$ for $h \in H$. We prove the following lemma:

LEMMA 6. *If a subgroup H is a retract of a group G, then $H' = H \cap G'$.*

Proof. For any subgroup we have an inclusion $H' \subset H \cap G'$; thus we need only to prove opposite inclusion $H \cap G' \subset H'$. Suppose $h \in H \cap G'$ and suppose that e is the endomorphism retracting G onto H. Since for any endomorphism e of G onto H, $e(G') \subset H'$, it follows by $h \in G'$ that $e(h) \in H'$. But $h \in H$; then $e(h) = h$ for that endomorphism e. This proves that $h \in H'$, and completes the proof.

In the paper [2] P. Hall has proved that if H is a retract of G, then the subgroup $H \cdot G'/G'$ is a direct factor of G/G'. This fact connected with (M) is a basis for the following theorem.

THEOREM 2. *A retract H of a nil-c-free group G is a nil-c-free group and a nil-c-free factor of G.*

Proof. According to lemmas 6 and 4 there exists a generating system $y_1, y_2, \ldots$ of H, which is linearly independent modulo G'. Let $\bar{y}_1 = h(y_1)$, $\bar{y}_2 = h(y_2)$, ... be the image of this system under the natural homomorphism h of G onto G/G'. Basing ourselves on the result of P. Hall we can complete the $\bar{y}_1, \bar{y}_2, \ldots$ by some $\bar{z}_1, \bar{z}_2, \ldots$ to an abelian free generating system of G/G'. Using property (P) we infer that any system $z_1, z_2, \ldots$ such that $h(z_1) = \bar{z}_1$, $h(z_2) = \bar{z}_2$, ..., generates together with $y_1, y_2, \ldots$ the whole group G. We have thus found a generating system of G,

$$y_1, y_2, \ldots, z_1, z_2, \ldots, \tag{7}$$

linearly independent modulo G'.

Now we infer from (M) that (7) is a nil-c-free generating system of G. Its part $y_1, y_2, \ldots$ generates H. This completes the proof.

If h is the natural homomorphism of a group G onto G/G', and e any endomorphism of G, then there exists a unique endomorphism e' of G/G' such that following diagram is commuting (7).

$$\begin{array}{ccc} G & \xrightarrow{h} & G/G' \\ e\downarrow & & \downarrow e' \\ G & \xrightarrow{h} & G/G' \end{array}$$

We shall write that e' is "induced" by e.

Let the group G be nil-c-free, and let $y_1, y_2, \ldots$ be its nil-c-free generating system. For the mappings h and e we put $e(y_i) = z_i$, $h(y_i) = \bar{y}_i$ and $h(z_i) = \bar{z}_i$. Then for the induced endomorphism e', $e'(\bar{y}_i) = \bar{z}_i$. If e is an automorphism of G, then $z_1, z_2, \ldots$ is a nil-c-free generating system of G, and therefore $\bar{z}_1, \bar{z}_2, \ldots$ is an abelian free generating system of G/G'. This proves that the induced mapping e' is an automorphism of G/G'.

Conversely, if the induced mapping e' is an automorphism of G/G', then $\bar{z}_1, \bar{z}_2, \ldots$ is an abelian free generating system of G/G'. Then the system $z_1, z_2, \ldots$ is, by (P), a generating system of G, and it is linearly independent modulo G'. This proves that $z_1, z_2, \ldots$ is a nil-c-free generating system of G; therefore the endomorphism e is an automorphism of G. We have thus proved the following theorem.

THEOREM 3. *An endomorphism of a nil-c-free group G is an automorphism of G if and only if it induces an automorphism of G/G'.*

Note that this theorem has been proved for the group G finitely or infinitely generated.

(7) It means that $e'h = he$.

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