Absolute-valued algebras with an involution

by

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An algebra A over the real field R is a vector space over R which is closed with respect to a product xy which is linear in both x and y and satisfies the condition $\lambda(xy)=(\lambda x)y=x(\lambda y)$ for any $\lambda\in R$ and $x,y\in A$. The product is not necessarily associative. An algebra is called *absolute-valued* if it is a normed space under a multiplicative norm $|\ |$, i.e. a norm satisfying in addition to the usual requirements the condition $|xy|=|x||y|$ for any x and y. It is known ([3]) that an absolute-valued algebra with a unit element is isomorphic to either the real field, the complex field, the quaternion algebra or the Cayley-Dickson algebra. A. A. Albert ([1]) has previously established this result under the restriction that the algebra is algebraic, in the sense that every element generates a finite-dimensional subalgebra and F. B. Wright ([4]) has shown that an absolute-valued division algebra is algebraic. On the other hand, infinite-dimensional absolute-valued algebras are known ([3]).

An operation * defined on an absolute-valued algebra A is called an *involution* if it satisfies the following conditions:

(i) $(\lambda x+\mu y)^*=\lambda x^*+\mu y^*$,

(ii) $x^{**}=x$,

(iii) $xx^*=x^*x$,

(iv) $(xy)^*=y^*x^*$,

(v) $|x^*|=|x|$

for any $\lambda,\mu\in R$ and $x,y\in A$.

Any absolute-valued algebra which is complete and admits an involution is said to be an absolute-valued **-algebra*. Using the well-known process of embedding linear normed spaces in Banach ones, we can prove that any absolute-valued algebra with an involution can be extended to an absolute-valued *-algebra. Therefore we shall be discussing only absolute-valued *-algebras.

As well-known examples of absolute-valued *-algebras we quote the real field and the complex field with the identity transformation as an involution and the complex field, the quaternion algebra and the Cayley-Dickson algebra with the natural involution $(\lambda+\mu i+\ldots)^*=\lambda-$

$-\mu i-\ldots$ $(\lambda, \mu, \ldots \in R)$. Now we shall give simple examples of infinite dimensional absolute-valued *-algebras.

Let X be an infinite dimensional real Hilbert space with an orthonormal basis $\{e_t\}_{t\in T}$. By Zermelo's axiom of choice the set T of indices can be decomposed into three disjoint sets T_1, T_2, and T_3 such that $1 \leqslant \overline{\overline{T}}_1 \leqslant \overline{\overline{T}}_2 = \overline{\overline{T}}$, where $\overline{\overline{B}}$ denotes the power of the set B. Let φ be a one-to-one correspondence of the class of all two-point subsets of T (i.e. non-ordered pairs $\{t_1, t_2\}$, where $t_1 \neq t_2$, $t_1, t_2 \in T$) onto the set T_2. Further, let ψ be a function defined on the set of all ordered pairs $\langle t_1, t_2\rangle$ $(t_1 \neq t_2,\ t_1, t_2 \in T)$ assuming the values 1 and -1 such that $\psi(\langle t_1, t_2\rangle)+\psi(\langle t_2, t_1\rangle) = 0$ both if $t_1, t_2 \in T_1$ and if $t_1, t_2 \notin T_1$ and $\psi(\langle t_1, t_2\rangle) = 1$ otherwise. For example, if the set T is ordered by a relation $\prec$, we put $\psi(\langle t_1, t_2\rangle) = 1$ or -1, whenever $t_1 \prec t_2$ or $t_2 \prec t_1$ respectively and $t_1, t_2 \in T_1$ or $\notin T_1$. Let $\varepsilon(t) = 1$ or -1 according as $t \in T_1$ or $t \notin T_1$.

Let us fix an element $t_0 \in T_1$. It is clear that to define a multiplication and an involution on X it is sufficient to define them on the basis $\{e_t\}_{t\in T}$. We define the multiplication and the involution of elements of the basis by means of the formulas

$$e_t^2 = \varepsilon(t)\, e_{t_0}, \qquad e_{t_1}e_{t_2} = \psi(\langle t_1, t_2\rangle)\, e_{\varphi(\{t_1,t_2\})} \quad \text{if} \quad t_1 \neq t_2,$$
$$e_t^* = \varepsilon(t)\, e_t.$$

These operations together with the usual addition and scalar-multiplication make X an absolute-valued *-algebra, which will be denoted by $X(T_1, T_2, T_3, \varphi, \psi)$. Indeed, setting [1]

$$x = \sum_{t\in T} \lambda_t e_t, \qquad y = \sum_{t\in T} \mu_t e_t$$

we have the equality

$$(1)\qquad xy = \Big(\sum_{t\in T} \lambda_t \mu_t \varepsilon(t)\Big) e_{t_0} + \\ + \sum_{t_1\neq t_2} \big(\psi(\langle t_1, t_2\rangle)\lambda_{t_1}\mu_{t_2} + \psi(\langle t_2, t_1\rangle)\lambda_{t_2}\mu_{t_1}\big)\, e_{\varphi(\{t_1,t_2\})},$$

where the sum is running over all non-ordered pairs $\{t_1, t_2\}$ satisfying the condition $t_1 \neq t_2$. Hence, taking into account the equality

$$\varepsilon(t_1)\varepsilon(t_2) + \psi(\langle t_1, t_2\rangle)\psi(\langle t_2, t_1\rangle) = 0 \qquad (t_1 \neq t_2),$$

[1] We write $x = \sum_{t\in T} x_t$, if for every positive number ε there exists a finite set $J_0 \subset T$ of indices such that $|x - \sum_{t\in J} x_t| < \varepsilon$ whenever J is a finite set of indices containing J_0.

we get

$$|xy| = \Big[\Big(\sum_{t\in T} \lambda_t\mu_t\varepsilon(t)\Big)^2 + \sum_{t_1\neq t_2} \big(\psi(\langle t_1, t_2\rangle)\lambda_{t_1}\mu_{t_2} + \psi(\langle t_2, t_1\rangle)\lambda_{t_2}\mu_{t_1}\big)^2\Big]^{1/2}$$
$$= \Big(\sum_{t_1,t_2\in T} \lambda_{t_1}^2\mu_{t_2}^2\Big)^{1/2} = \Big(\sum_{t\in T}\lambda_t^2\Big)^{1/2}\cdot\Big(\sum_{t\in T}\mu_t^2\Big)^{1/2} = |x|\,|y|.$$

Properties (i), (ii) and (v) of the involution are evident. From (1) we obtain the formula

$$(xy)^* = \Big(\sum_{t\in T}\lambda_t\mu_t\varepsilon(t)\Big)e_{t_0} + \\ + \sum_{t_1\neq t_2} \varepsilon\big(\varphi(\{t_1, t_2\})\big)\big(\psi(\langle t_1, t_2\rangle)\lambda_{t_1}\mu_{t_2} + \psi(\langle t_2, t_1\rangle)\lambda_{t_2}\mu_{t_1}\big)\, e_{\varphi(\{t_1,t_2\})},$$

$$y^*x^* = \Big(\sum_{t\in T}\lambda_t\mu_t\varepsilon(t)\Big)e_{t_0} + \\ + \sum_{t_1\neq t_2}\big(\psi(\langle t_1, t_2\rangle)\varepsilon(t_1)\mu_{t_1}\varepsilon(t_2)\lambda_{t_2} + \psi(\langle t_2, t_1\rangle)\varepsilon(t_2)\mu_{t_2}\varepsilon(t_1)\lambda_{t_1}\big)\, e_{\varphi(\{t_1,t_2\})}.$$

Since, for any $t_1 \neq t_2$, $\varepsilon(t_1)\varepsilon(t_2)\psi(\langle t_1, t_2\rangle) = -\psi(\langle t_2, t_1\rangle)$ and $\varepsilon(t_1)\varepsilon(t_2)\psi(\langle t_2, t_1\rangle) = -\psi(\langle t_1, t_2\rangle)$, we get the equality $(xy)^* = y^*x^*$. Further, according to the equality

$$\psi(\langle t_1, t_2\rangle)\varepsilon(t_2) + \psi(\langle t_2, t_1\rangle)\varepsilon(t_1) = \psi(\langle t_1, t_2\rangle)\varepsilon(t_1) + \psi(\langle t_2, t_1\rangle)\varepsilon(t_2);$$

we have the formula

$$xx^* = \Big(\sum_{t\in T}\lambda_t^2\Big)e_{t_0} + \sum_{t_1\neq t_2}\lambda_{t_1}\lambda_{t_2}\big(\psi(\langle t_1, t_2\rangle)\varepsilon(t_2) + \\ + \psi(\langle t_2, t_1\rangle)\varepsilon(t_1)\big)\, e_{\varphi(\{t_1,t_2\})} = x^*x,$$

which completes the proof.

We note that if U is a linear isometry on A which commutes with the involution, then A remains an absolute-valued *-algebra with respect to the new product $x\circ y = U(xy)$. This fact suggests the following definition: two absolute-valued *-algebras A and A' are said to be *similar* if they are isomorphic as normed spaces with an involution: $A \sim A'$ and the multiplication xy in A is defined in terms of the multiplication $x'\circ y'$ in A' by the relation $xy \sim U(x'\circ y')$, whenever $x \sim x'$, $y \sim y'$, where U is a fixed invertible linear isometry on A' which commutes with the involution.

It is not difficult to prove that two algebras $X(T_1, T_2, T_3, \varphi, \psi)$, $X'(T_1', T_2', T_3', \varphi', \psi')$ are similar if and only if $\overline{\overline{T}}_1 = \overline{\overline{T}}{}_1'$, $\overline{\overline{T}}_2 = \overline{\overline{T}}{}_2'$ and $\overline{\overline{T}}_3 = \overline{\overline{T}}{}_3'$. Therefore we shall call an absolute-valued *-algebra $X(T_1, T_2, T_3, \varphi, \psi)$ a $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\rangle$-*algebra*, where $\mathfrak{m}_1 = \overline{\overline{T}}_1$, $\mathfrak{m}_2 = \overline{\overline{T}}_2$ and $\mathfrak{m}_3 = \overline{\overline{T}}_3$.

For any pair $x, y \in A$ we set

$$\big((x, y)\big) = \tfrac{1}{2}(xy^* + yx^*).$$

The operation $((x, y))$ will be called a **-product*. It imitates an inner product. More precisely, we have the equalities

$$((x, x)) = 0 \quad \text{if and only if} \quad x = 0,$$
$$((\lambda x_1 + \mu x_2, y)) = \lambda((x_1, y)) + \mu((x_2, y)),$$
$$((x, y)) = ((y, x)).$$

Moreover, the *-product is invariant under the involution, i.e.

$$((x, y))^* = ((x, y)).$$

An absolute-valued *-algebra is said to be *regular* if

$$(2) \qquad ((xy, zu)) = ((xz^*, y^*u))$$

for any x, y, z, and u. It is very easy to verify that the real field and the complex field (with $x^* = x$ or $x^* = \bar{x}$) regarded as *-algebras are regular. Now we shall show that all $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \rangle$-algebras are also regular. To prove this it is sufficient to show that all elements of the orthonormal basis $\{e_t\}_{t \in T}$ satisfy equality (2). By the definition of the involution in $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \rangle$-algebras, equality (2) can be written in the following form:

$$(3) \qquad ((e_{t_1} e_{t_2}, e_{t_3} e_{t_4})) = \varepsilon(t_2)\varepsilon(t_3)((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) \qquad (t_1, t_2, t_3, t_4 \in T).$$

If $t_2 = t_3$, then the last equality is obvious. Therefore we may suppose that $t_2 \neq t_3$.

If $t_1 = t_2$ and $t_3 = t_4$, then we have the equalities

$$((e_{t_1} e_{t_2}, e_{t_3} e_{t_4})) = ((e_{t_2}^2, e_{t_3}^2)) = \varepsilon(t_2)\varepsilon(t_3)((e_{t_0}, e_{t_0})) = \varepsilon(t_2)\varepsilon(t_3) e_{t_0},$$
$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = ((e_{t_2} e_{t_3}, e_{t_2} e_{t_3})) = ((e_{\varphi(\{t_2,t_3\})}, e_{\varphi(\{t_2,t_3\})}))$$
$$= e_{\varphi(\{t_2,t_3\})} e^*_{\varphi(\{t_2,t_3\})} = e_{t_0},$$

which imply formula (3).

If $t_1 = t_2$ and $t_3 \neq t_4$, then, taking into account the relations $t_0 \in T_1$ and $\varphi(\{t_3, t_4\}) \notin T_1$, we get the equality

$$((e_{t_1} e_{t_2}, e_{t_3} e_{t_4})) = \psi(\langle t_3, t_4 \rangle)((e_{t_2}^2, e_{\varphi(\{t_3,t_4\})}))$$
$$= \varepsilon(t_2)\psi(\langle t_3, t_4 \rangle)((e_{t_0}, e_{\varphi(\{t_3,t_4\})}))$$
$$= \tfrac{1}{2}\varepsilon(t_2)\psi(\langle t_3, t_4 \rangle)(-e_{t_0} e_{\varphi(\{t_3,t_4\})} + e_{\varphi(\{t_3,t_4\})} e_{t_0}) = 0.$$

Further, if $t_2 = t_4$, we have the formula

$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = ((e_{t_2} e_{t_3}, e_{t_2}^2))$$
$$= \varepsilon(t_2)\psi(\langle t_2, t_3 \rangle)((e_{\varphi(\{t_2,t_3\})}, e_{t_0}))$$
$$= \tfrac{1}{2}\varepsilon(t_2)\psi(\langle t_2, t_3 \rangle)(e_{\varphi(\{t_2,t_3\})} e_{t_0} - e_{t_0} e_{\varphi(\{t_2,t_3\})}) = 0.$$

If $t_2 \neq t_4$, then the relations $\varphi(\{t_2, t_3\}) \neq \varphi(\{t_2, t_4\})$, $\varphi(\{t_2, t_3\}), \varphi(\{t_2, t_4\}) \notin T_1$ imply the equality

$$e_{\varphi(\{t_2,t_3\})} e_{\varphi(\{t_2,t_4\})} = -e_{\varphi(\{t_2,t_4\})} e_{\varphi(\{t_2,t_3\})}.$$

Hence we get the equality

$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = ((e_{t_2} e_{t_3}, e_{t_2} e_{t_4}))$$
$$= \psi(\langle t_2, t_3 \rangle)\psi(\langle t_2, t_4 \rangle)((e_{\varphi(\{t_2,t_3\})}, e_{\varphi(\{t_2,t_4\})}))$$
$$= -\tfrac{1}{2}\psi(\langle t_2, t_3 \rangle)\psi(\langle t_2, t_4 \rangle)(e_{\varphi(\{t_2,t_3\})} e_{\varphi(\{t_2,t_4\})} + e_{\varphi(\{t_2,t_4\})} e_{\varphi(\{t_2,t_3\})})$$
$$= 0.$$

Consequently, equality (3) is proved in the case $t_1 = t_2$ and $t_3 \neq t_4$. The case $t_1 \neq t_2$ and $t_3 = t_4$, in view of the commutativity of *-products, is reduced to the previous case.

Now let us assume that $t_1 = t_3$ and $t_2 = t_4$. Then we have

$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = ((e_{t_3} e_{t_2}, e_{t_3} e_{t_2})) = ((e_{\varphi(\{t_2,t_3\})}, e_{\varphi(\{t_2,t_3\})}))$$
$$= e_{\varphi(\{t_2,t_3\})} e^*_{\varphi(\{t_2,t_3\})} = e_{t_0},$$
$$((e_{t_1} e_{t_2}, e_{t_3} e_{t_4})) = ((e_{t_3}^2, e_{t_2}^2)) = \varepsilon(t_2)\varepsilon(t_3)((e_{t_0}, e_{t_0})) = \varepsilon(t_2)\varepsilon(t_3) e_{t_0},$$

whence equality (3) follows.

Finally we suppose that $t_1 \neq t_2$, $t_3 \neq t_4$ and $\{t_1, t_2\} \neq \{t_3, t_4\}$. Then taking into account the relation $\varphi(\{t_1, t_2\}), \varphi(\{t_3, t_4\}) \notin T_1$, we have the equality

$$e_{\varphi(\{t_1,t_2\})} e_{\varphi(\{t_3,t_4\})} = -e_{\varphi(\{t_3,t_4\})} e_{\varphi(\{t_1,t_2\})}$$

and, consequently,

$$((e_{t_1} e_{t_2}, e_{t_3} e_{t_4})) = \psi(\langle t_1, t_2 \rangle)\psi(\langle t_3, t_4 \rangle)((e_{\varphi(\{t_1,t_2\})}, e_{\varphi(\{t_3,t_4\})}))$$
$$= \tfrac{1}{2}\psi(\langle t_1, t_2 \rangle)\psi(\langle t_3, t_4 \rangle)(-e_{\varphi(\{t_1,t_2\})} e_{\varphi(\{t_3,t_4\})} -$$
$$- e_{\varphi(\{t_3,t_4\})} e_{\varphi(\{t_1,t_2\})}) = 0.$$

If $t_1 \neq t_3$ and $t_2 \neq t_4$, then, in view of the assumption $t_2 \neq t_3$ and $\{t_1, t_2\} \neq \{t_3, t_4\}$, we have the inequality $\{t_1, t_3\} \neq \{t_2, t_4\}$ and

$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = \psi(\langle t_1, t_3 \rangle)\psi(\langle t_2, t_4 \rangle)((e_{\varphi(\{t_1,t_3\})}, e_{\varphi(\{t_2,t_4\})}))$$
$$= \tfrac{1}{2}\psi(\langle t_1, t_3 \rangle)\psi(\langle t_2, t_4 \rangle)(-e_{\varphi(\{t_1,t_3\})} e_{\varphi(\{t_2,t_4\})} -$$
$$- e_{\varphi(\{t_2,t_4\})} e_{\varphi(\{t_1,t_3\})}) = 0.$$

If $t_1 = t_3$, then from the assumption $\{t_1, t_2\} \neq \{t_3, t_4\}$ we get the inequality $t_2 \neq t_4$ and, consequently,

$$((e_{t_1} e_{t_3}, e_{t_2} e_{t_4})) = ((e_{t_3}^2, e_{t_2} e_{t_4})) = \varepsilon(t_3)\psi(\langle t_2, t_4 \rangle)((e_{t_0}, e_{\varphi(\{t_2,t_4\})}))$$
$$= \tfrac{1}{2}\varepsilon(t_3)\psi(\langle t_2, t_4 \rangle)(-e_{t_0} e_{\varphi(\{t_2,t_4\})} + e_{\varphi(\{t_2,t_4\})} e_{t_0}) = 0.$$

By the commutativity of the *-product, the case $t_2 = t_4$ is reduced to the previous case. Consequently, equality (3) holds for any system of indices $t_1, t_2, t_3, t_4 \in T$. In other words, we have proved that $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \rangle$-algebras are regular.

We remark that an absolute-valued *-algebra similar to a regular one is also regular.

In the present paper we shall represent regular absolute-valued *-algebras. Namely, we shall prove the following theorem, which is an answer to a problem raised by F. B. Wright.

THEOREM. *A regular absolute-valued *-algebra is similar to either the real field, the complex field (with $x^* = x$ or $x^* = \bar{x}$) or a $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \rangle$-algebra, where $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$ are cardinals satisfying the inequalities $1 \leqslant \mathfrak{m}_1 \leqslant \mathfrak{m}_2$, $\mathfrak{m}_3 \leqslant \mathfrak{m}_2$ and $\mathfrak{m}_2 \geqslant \aleph_0$.*

Before proving the theorem we shall prove some lemmas. In the sequel A will denote an absolute-valued *-algebra. By A_a we shall denote the set of all self-adjoint elements of A, i.e. the set of all elements x satisfying the equality $x^* = x$. By A_s we shall denote the set of all skew elements of A, i.e. the set of all elements x satisfying the equality $x^* = -x$. Obviously, both A_a and A_s are linear subspaces of A and $A_a \cap A_s = \{0\}$. It is very easy to prove that every element $x \in A$ may be represented in one and only one manner as the sum $x_1 + x_2$, x_1 and x_2 being self-adjoint and skew respectively. Moreover, the equalities $x_1 = \frac{1}{2}(x + x^*)$ and $x_2 = \frac{1}{2}(x - x^*)$ hold. In other words, A is the direct sum of the subspace A_a and A_s.

LEMMA 1. *Self-adjoint elements commute with skew elements.*

Proof. Let $x \in A_a$ and $y \in A_s$. By property (iii), we have the equality

$$0 = (x+y)^*(x+y) - (x+y)(x+y)^* = (x-y)(x+y) - (x+y)(x-y) = 2(xy - yx),$$

which implies the assertion of the Lemma.

LEMMA 2. *For any $x \in A_a$ and $y \in A_s$ we have the equality*

$$|x+y|^2 = |x|^2 + |y|^2.$$

Proof. If either x or y is equal to 0, then our statement is obvious. Therefore we may suppose that $x \neq 0$ and $y \neq 0$. Let B be the linear set spanned by x and y. Since $x^* = x$ and $y^* = -y$, B is invariant under the involution. By Lemma 1, the elements of B commute with one another. Therefore for every pair z_1, z_2 of elements of B we have $(z_1+z_2)^2 - (z_1 - z_2)^2 = 4z_1 z_2$. Consequently, for $|z_1| = |z_2| = 1$, we get the inequality

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = |(z_1+z_2)^2| + |(z_1 - z_2)^2| \geqslant 4|z_1||z_2| = 4.$$

Hence, according to Schoenberg's Theorem ([2]), B is an inner product space over R. There are then a number λ and an element $y_0 \in B$ orthogonal to x such that $y = \lambda x + y_0$. Since by (v) the involution is an isometry on B, the element y_0^* is also orthogonal to x. From the equality $y = \frac{1}{2}(y - y^*) = \frac{1}{2}(\lambda x + y_0 - \lambda x - y_0) = \frac{1}{2}(y_0 - y_0^*)$ it follows that y and x are orthogonal. The statement of the Lemma is a direct consequence of the orthogonality of x and y.

LEMMA 3. *If $A_s \neq \{0\}$, then there exists one and only one idempotent $e \in A_a$ such that*

$$x^2 = |x|^2 e, \quad y^2 = -|y|^2 e$$

for any $x \in A_a$ and $y \in A_s$.

Proof. Let $x \in A_a$ and $y \in A_s$. Since, by Lemma 1, x commutes with y, we have the equality $(xy)^* = y^* x^* = -yx = -xy$. Consequently, $xy \in A_s$. Further, we have the equality $(x^2 + y^2)^* = (x^*)^2 + (y^*)^2 = x^2 + y^2$, which implies the relation $x^2 + y^2 \in A_a$. Hence, with the aid of the formula $(x+y)^2 = x^2 + y^2 + 2xy$ as well as Lemma 2, we obtain the following equality

$$|x+y|^4 = |(x+y)^2|^2 = |x^2 + y^2|^2 + 4|x|^2|y|^2.$$

But $|x+y|^2 = |x|^2 + |y|^2$ and, consequently, we have the equality

$$(4) \qquad |x^2 + y^2| = \big((|x|^2 + |y|^2)^2 - 4|x|^2|y|^2\big)^{1/2} = \big||x|^2 - |y|^2\big| \qquad (x \in A_a,\ y \in A_s).$$

By the assumption there exists a skew element y_0, with $|y_0| = 1$. Putting $e = -y_0^2$, we have $e^* = (-y_0^2)^* = -y_0^2 = e$, $|e| = 1$ and, in view of (4),

$$\big|x^2 - |x|^2 e\big| = \big|x^2 + (|x| y_0)^2\big| = \big||x|^2 - |x|^2|y_0|^2\big| = 0$$

for any $x \in A_a$. Thus

$$(5) \qquad x^2 = |x|^2 e \qquad (x \in A_a).$$

In particular, $e^2 = e$. The last equality and formula (4) imply the equation

$$\big||y|^2 e + y^2\big| = \big|(|y| e)^2 + y^2\big| = \big||y|^2|e|^2 - |y|^2\big| = 0$$

for any $y \in A_s$. Thus $y^2 = -|y|^2 e$ for all skew elements y. The uniqueness of the idempotent e follows from equality (5). The lemma is thus proved.

LEMMA 4. *Every absolute-valued *-algebra with $A_s \neq \{0\}$ is a real Hilbert space.*

Proof. Let z_1, z_2 be a pair of elements of A, with $|z_1| = |z_2| = 1$. Writing $z_1 = x_1 + y_1$, $z_2 = x_2 + y_2$, where $x_1, x_2 \in A_a$ and $y_1, y_2 \in A_s$, we have, according to Lemma 2,

$$(6) \qquad |x_1|^2 + |y_1|^2 = 1, \quad |x_2|^2 + |y_2|^2 = 1.$$

Moreover, in view of Lemma 2, we have the equalities

$$(7) \qquad |z_1 - z_2|^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2, \quad |z_1 + z_2|^2 = |x_1 + x_2|^2 + |y_1 + y_2|^2.$$

Using Lemma 3 we obtain the inequalities

$$|x_1 - x_2|^2 + |x_1 + x_2|^2 = |(x_1 - x_2)^2| + |(x_1 + x_2)^2| \geqslant |(x_1 - x_2)^2 + (x_1 + x_2)^2| = 2|x_1^2 + x_2^2| = 2\big|(|x_1|^2 + |x_2|^2)e\big| = 2(|x_1|^2 + |x_2|^2),$$

$$|y_1 - y_2|^2 + |y_1 + y_2|^2 = |(y_1 - y_2)^2| + |(y_1 + y_2)^2| \geqslant |(y_1 - y_2)^2 + (y_1 + y_2)^2| = 2|y_1^2 + y_2^2| = 2\big|-(|y_1|^2 + |y_2|^2)e\big| = 2(|y_1|^2 + |y_2|^2).$$

Hence and from (6) and (7) we get the inequality

$$|z_1-z_2|^2+|z_1+z_2|^2 \geqslant 2(|x_1|^2+|x_2|^2+|y_1|^2+|y_2|^2) = 4 .$$

Thus, according to Schoenberg's Theorem ([2]), we know that A is an inner product space over R and, consequently, a real Hilbert space.

For any $x \in A_a \cup A_s$, $x \neq 0$ we set $\delta(x) = 1$ or -1 according as $x \in A_a$ or $x \in A_s$ and $\delta(0) = 0$.

LEMMA 5. *If $A_s \neq \{0\}$, then for any pair of orthogonal elements $x, y \in A_a \cup A_s$ we have the equality*

$$xy+\delta(x)\delta(y)yx = 0 .$$

Proof. If $\delta(x)\delta(y) = 0$, then our assertion is obvious. Further, if $\delta(x)\delta(y) = -1$, then one element of the pair belongs to A_a and the other belongs to A_s. Consequently, by Lemma 1, $xy-yx = 0$. Now let us suppose that $\delta(x)\delta(y) = 1$, i.e. both elements x and y belong to either A_a or A_s. From the orthogonality of x and y and from Lemma 3 we get the equality

$$\begin{aligned}\big|\delta(x)(|x|^2+|y|^2)e\big| &= |x|^2+|y|^2 = |x\pm y|^2 = |(x\pm y)^2| \\ &= |x^2+y^2\pm(xy+yx)| = \big|\delta(x)(|x|^2+|y|^2)e\pm(xy+yx)\big| .\end{aligned}$$

Hence, $xy+yx = 0$, which completes the proof of the Lemma.

By (x, y) we shall denote the inner product of two elements x and y $(x, y \in A)$.

LEMMA 6. *If $A_s \neq \{0\}$, then for any pair $z_1, z_2 \in A$ we have the equality*

$$\big((z_1, z_2)\big) = (z_1, z_2)e .$$

Proof. Let $z_1, z_2 \in A$. Since $A_a \cap A_s = \{0\}$, the subspaces A_a and A_s are orthogonal. Consequently, writing $z_1 = x_1+y_1$, $z_2 = x_2+y_2$, where $x_1, x_2 \in A_a$ and $y_1, y_2 \in A_s$ we have the equality

$$(z_1, z_2) = (x_1, x_2)+(y_1, y_2) . \tag{8}$$

Further, since by Lemma 1 self-adjoint elements commute with skew elements, we have the equalities

$$\begin{aligned}\big((x_1, y_2)\big) &= \tfrac{1}{2}(x_1y_2^*+y_2x_1^*) = \tfrac{1}{2}(-x_1y_2+y_2x_1) = 0 , \\ \big((y_1, x_2)\big) &= \tfrac{1}{2}(y_1x_2^*+x_2y_1^*) = \tfrac{1}{2}(y_1x_2-x_2y_1) = 0 .\end{aligned}$$

Consequently,

$$\big((z_1, z_2)\big) = \big((x_1, x_2)\big)+\big((y_1, y_2)\big) . \tag{9}$$

Let us represent the elements x_2 and y_2 in the form

$$x_2 = \lambda x_1+x_3, \qquad y_2 = \mu y_1+y_3 ,$$

where λ and μ are real numbers, x_3 is a self-adjoint element orthogonal to x_1 and y_3 is a skew element orthogonal to y_1. Obviously, $(x_1, x_2) = \lambda|x_1|^2$, $(y_1, y_2) = \mu|y_1|^2$ and, by Lemma 5, $x_1x_3+x_3x_1 = 0$, $y_1y_3+y_3y_1 = 0$. Hence, in view of Lemma 3, we get the equalities

$$\begin{aligned}\big((x_1, x_2)\big) &= \tfrac{1}{2}(x_1x_2+x_2x_1) = \tfrac{1}{2}(\lambda x_1^2+x_1x_3+\lambda x_1^2+x_3x_1) = \lambda|x_1|^2e = (x_1, x_2)e , \\ \big((y_1, y_2)\big) &= -\tfrac{1}{2}(y_1y_2+y_2y_1) = -\tfrac{1}{2}(\mu y_1^2+y_1y_3+\mu y_1^2+y_3y_1) = \mu|y_1|^2e = (y_1, y_2)e .\end{aligned}$$

Taking into account equalities (8) and (9) we get the assertion of the Lemma.

LEMMA 7. *Let A be a regular absolute-valued *-algebra and $A_s \neq \{0\}$. For any system z_1, z_2, z_3, z_4 of orthogonal elements belonging to $A_a \cup A_s$ the products $z_{j_1}z_{j_2}$ and $z_{j_3}z_{j_4}$ are orthogonal, whenever the sets of indices $\{j_1, j_2\}$, $\{j_3, j_4\}$ are different and $j_1 \neq j_2$, $j_3 \neq j_4$ $(j_1, j_2, j_3, j_4 = 1, 2, 3, 4)$.*

Proof. First we suppose that $\{j_1, j_2\} \cap \{j_3, j_4\} \neq 0$. Using Lemma 5 we can write

$$(z_{j_1}z_{j_2}, z_{j_3}z_{j_4}) = \pm(z_kz_{s_1}, z_kz_{s_2}) ,$$

where $s_1 \neq s_2$. Hence and from the equality

$$\begin{aligned}(z_kz_{s_1}, z_kz_{s_2}) &= \tfrac{1}{4}(|z_kz_{s_1}+z_kz_{s_2}|^2-|z_kz_{s_1}-z_kz_{s_2}|^2) \\ &= \tfrac{1}{4}|z_k|^2(|z_{s_1}+z_{s_2}|^2-|z_{s_1}-z_{s_2}|^2) = |z_k|^2(z_{s_1}, z_{s_2}) = 0 ,\end{aligned}$$

we get the orthogonality of $z_{j_1}z_{j_2}$ and $z_{j_3}z_{j_4}$.

Now let us assume that $\{j_1, j_2\} \cap \{j_3, j_4\} = 0$, i.e. the system j_1, j_2, j_3, j_4 is a permutation of $1, 2, 3, 4$. Without loss of generality we may suppose that both z_2 and z_3 belong to either A_a or A_s. Moreover, we may assume that

$$\delta(z_1) \neq 0 \quad \text{and} \quad \delta(z_2)\delta(z_3) = 1 , \tag{10}$$

because in the opposite case $z_1 = 0$, $z_2 = 0$ or $z_3 = 0$ and the orthogonality of $z_{j_1}z_{j_2}$ and $z_{j_3}z_{j_4}$ is evident.

Since the algebra A is regular, we have, according to Lemma 6,

$$(z_{j_1}z_{j_2}, z_{j_3}z_{j_4}) = (z_{j_1}z_{j_3}, z_{j_2}z_{j_4}) .$$

Hence and from Lemma 5 it follows that

$$(z_{j_1}z_{j_2}, z_{j_3}z_{j_4}) = \pm(z_1z_2, z_3z_4) \tag{11}$$

for any permutation j_1, j_2, j_3, j_4 of integers 1, 2, 3, 4.

Put $u = (z_2+z_3)(z_1+z_4)$. By the orthogonality of z_1, z_2, z_3 and z_4 we have the relation

$$|u|^2 = (|z_2|^2+|z_3|^2)(|z_1|^2+|z_4|^2) . \tag{12}$$

Further, from Lemma 5 we infer that all the elements z_2z_1, z_2z_4, z_3z_1 and z_3z_4 are skew. Hence we get the equality

$$-u^2 = ((z_2z_1, z_2z_1)) + ((z_2z_4, z_2z_4)) + ((z_3z_1, z_3z_1)) + ((z_3z_4, z_3z_4)) + \\ + 2((z_2z_1, z_2z_4)) + 2((z_2z_1, z_3z_1)) + 2((z_2z_1, z_3z_4)) + \\ + 2((z_2z_4, z_3z_1)) + 2((z_2z_4, z_3z_4)) + 2((z_3z_1, z_3z_4)).$$

Replacing in the last formula, in view of Lemma 6, the *-product by the inner product, we get the equality

$$(13) \qquad -u^2 = (|z_2|^2|z_1|^2 + |z_2|^2|z_4|^2 + |z_3|^2|z_1|^2 + |z_3|^2|z_4|^2 + \\ + 2(z_2z_1, z_2z_4) + 2(z_2z_1, z_3z_1) + 2(z_2z_1, z_3z_4) + \\ + 2(z_2z_4, z_3z_1) + 2(z_2z_4, z_3z_4) + 2(z_3z_1, z_3z_4))e.$$

Since $u = z_2z_1 + z_2z_4 + z_3z_1 + z_3z_4$ and, consequently, is a skew element, we have, in virtue of Lemma 3, the equality $-u^2 = |u|^2e$. Hence and from (12) and (13) we get the relation

$$(z_2z_1, z_2z_4) + (z_2z_1, z_3z_1) + (z_2z_1, z_3z_4) + \\ + (z_2z_4, z_3z_1) + (z_2z_4, z_3z_4) + (z_3z_1, z_3z_4) = 0.$$

By the first part of the proof we have the equality

$$(z_2z_1, z_2z_4) = (z_2z_1, z_3z_1) = (z_2z_4, z_3z_4) = (z_3z_1, z_3z_4) = 0.$$

Thus

$$(14) \qquad (z_2z_1, z_3z_4) + (z_2z_4, z_3z_1) = 0.$$

By Lemma 5, we have the equalities $z_2z_1 = -\delta(z_1)\delta(z_2)z_1z_2$, $z_3z_1 = -\delta(z_1)\delta(z_3)z_1z_3$. Hence we get the formulas

$$(z_2z_1, z_3z_4) = -\delta(z_1)\delta(z_2)(z_1z_2, z_3z_4),$$
$$(z_3z_1, z_2z_4) = -\delta(z_1)\delta(z_3)(z_1z_3, z_2z_4).$$

Further, from the regularity of A it follows that $(z_1z_3, z_2z_4) = (z_1z_2, z_3z_4)$. Consequently, according to (10) and (14), $(z_1z_2, z_3z_4) = 0$. Thus, in view of (11), $z_{j_1}z_{j_2}$ and $z_{j_3}z_{j_4}$ are orthogonal.

Proof of the Theorem. First let us suppose that $A_s = \{0\}$, i.e. the involution is the identity transformation. Taking into account property (iv) of the involution we infer that A is a commutative absolute-valued algebra. Consequently, it is isomorphic to either the real field, the complex field or the algebra of complex numbers with the product of x and y defined as $\overline{xy}$ (see [3]). But the last algebra is similar to the complex field (as an isometry U we take $x \to \bar{x}$).

Now let us suppose that $A_s \neq \{0\}$. By Lemmas 3 and 4, A is at least a two-dimensional Hilbert space. Let A be finite-dimensional and let $\{e, e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for A consisting of elements belonging to $A_a \cup A_s$, where e is the idempotent defined by Lemma 3. If $n \geq 2$, then, by Lemma 7, e_1e_2 is orthogonal to $ee_1, ee_2, \ldots, ee_n$. Moreover, by Lemma 5, e_1e_2 is a skew element and, consequently, it is orthogonal to e. But the set $\{e, ee_1, ee_2, \ldots, ee_n\}$ is also a basis for A. Thus $e_1e_2 = 0$, which is impossible. We have proved that any finite-dimensional absolute-valued *-algebra with a non-trivial involution is two-dimensional. Hence it follows that every element $x \in A$ can be written in the form $x = \lambda e + \mu e_1$, where $e_1 \in A_s$, $|e_1| = 1$ and $\lambda, \mu \in R$. Since, by Lemma 5, $ee_1 \in A_s$ and $|ee_1| = 1$, we have either $ee_1 = e_1$ or $ee_1 = -e_1$. Further, by Lemma 3, $e_1^2 = -e$. Thus A is isomorphic to the complex field if $ee_1 = e_1$ and is similar to the complex field if $ee_1 = -e_1$ (as an isometry U we take the involution).

Finally let us suppose that A is infinite-dimensional. Let $\{e_t\}_{t \in T}$ be an orthonormal basis of A consisting of elements belonging to $A_a \cup A_s$ and containing the idempotent e defined by Lemma 3. By T_1 we denote the subset of indices such that $\{e_t\}_{t \in T_1}$ is a basis or A_a. Evidently, $\{e_t\}_{t \in T \setminus T_1}$ is a basis of A_s. Let B be the linear subspace spanned by all products $e_{t_1}e_{t_2}$, where $t_1 \neq t_2$ and $t_1, t_2 \in T$. Since, by Lemmas 1 and 5, all those products are skew, B is a subspace of A_s. Using the axiom of choice, we can decompose the set of indices $T \setminus T_1$ into disjoint sets T_2 and T_3, where $\overline{\overline{T}}_2$ is the dimension of B and $\overline{\overline{T}}_3$ is the dimension of the orthogonal complement of B in A_s [2]. By definition, there exists an index $t_0 \in T$ such that $e_{t_0} = e$. Further, let φ, ψ be a pair of functions satisfying the requirements given in the definition of $\langle \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \rangle$-algebras. To prove our theorem it is sufficient to show that the *-algebra A is similar to the algebra $A(T_1, T_2, T_3, \varphi, \psi)$.

It is very easy to see that the formula $e_{t_1}e_{t_2} \sim \psi(\langle t_1, t_2 \rangle) e_{\varphi(\langle t_1, t_2 \rangle)}$ defines a one-to-one correspondence between the family $\{e_{t_1}e_{t_2}\}$ $(t_1 \neq t_2;\ t_1, t_2 \in T)$, which, by Lemma 7, is an orthonormal basis of B, and the orthonormal family $\{e_t\}_{t \in T_2}$. This correspondence and the identity transformation of A_a can be extended to a unitary transformation U of the whole space A. Obviously, A_a and A_s are invariant under the transformation U. Hence, in particular, it follows that the transformation U commutes with the involution in the algebra $A(T_1, T_2, T_3, \varphi, \psi)$. Denoting by $\circ$ the product in $A(T_1, T_2, T_3, \varphi, \psi)$ we have the equalities

$$U(e_t \circ e_t) = U(\varepsilon(t)e) = \varepsilon(t)e = e_t^2 \qquad (t \in T),$$
$$U(e_{t_1} \circ e_{t_2}) = U(\psi(\langle t_1, t_2 \rangle) e_{\varphi(\langle t_1, t_2 \rangle)}) = e_{t_1}e_{t_2} \qquad (t_1 \neq t_2;\ t_1, t_2 \in T).$$

Consequently, $U(x \circ y) = xy$ for any x and y in A. In other words, the algebras A and $A(T_1, T_2, T_3, \varphi, \psi)$ are similar.

[2] The dimension of a subspace B is the power of its basis.

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Nilpotent free groups

by

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The following theorem, proved by A. I. Malcev in [3], will be denoted further by (M).

(M) *Let G be a nilpotent free group of class c, and X a subset of G of the cardinality $|X| > 1$. Then X is a nilpotent free generating system for some subgroup of G, of the same class c, if and only if X is linearly independent modulo the derived subgroup G'.*

This paper consists of two parts. The first is devoted to a group theoretical proof of (M). The proof is based on the following theorem.

(T) *If $x_1, \ldots, x_i, \ldots$ is a free, or nilpotent free, generating system of a free, or nilpotent free, group G, then a system $x_1^{n_1}\cdot x_1', \ldots, x_i^{n_i}\cdot x_i', \ldots$ is free, or nilpotent free, for any $x_i' \in G'$, and for any positive integers n_i.*

The proof of (T) essentially needs M. Hall's theory of basic commutators exposed, for example, in [1].

The second part contains theorems that can be derived from (M):

THEOREM 1. *A subgroup H of a nilpotent free group G is a nilpotent free group if and only if it satisfies the condition $H' = H \cap G'$ or is a cyclic group.* (H' and G' are the derived subgroups of H and G.)

THEOREM 2. *Every retract of a nilpotent free group G, is a nilpotent free factor of G and a nilpotent free subgroup of G.*

The analogous statement fails for retracts of free or solvable free groups, see [4].

THEOREM 3. *An endomorphism of a nilpotent free group G is an automorphism of that group if and only if it induces an automorphism of G/G'.*

The terminology of the paper is the same as that in the book of M. Hall [1]. Some basic notions concerning varietes of nilpotent groups are listed in an introductory part. They can be found partly in papers [2], [5] and [6], and partly in the book quoted.

1. We define recursively the simple commutators as: $(x_1) = x_1$, $(x_1, x_2) = x_1^{-1}\cdot x_2^{-1}\cdot x_1\cdot x_2$, and $(x_1, \ldots, x_{n+1}) = \big((x_1, \ldots, x_n), x_{n+1}\big)$. A group