Absolute-valued algebras with an involution

by

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An algebra $A$ over the real field $R$ is a vector space over $R$ which is closed with respect to a product $xy$ which is linear in both $x$ and $y$ and satisfies the condition $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for any $\lambda \in R$ and $x, y \in A$. The product is not necessarily associative. An algebra is called absolute-valued if it is a normed space under a multiplicative norm $| |$, i.e. a norm satisfying in addition to the usual requirements the condition $|xy| = |x||y|$ for any $x$ and $y$. It is known ([3]) that an absolute-valued algebra with a unit element is isomorphic to either the real field, the complex field, the quaternion algebra or the Cayley-Dickson algebra.

A. A. Albert ([11]) has previously established this result under the restriction that the algebra is algebraic, in the sense that every element generates a finite-dimensional subalgebra and F. B. Wright ([4]) has shown that an absolute-valued division algebra is algebraic. On the other hand, infinite-dimensional absolute-valued algebras are known ([3]).

An operation $*$ defined on an absolute-valued algebra $A$ is called an involution if it satisfies the following conditions:

(i) $(\lambda x + \mu y)^* = \lambda x^* + \mu y^*$,
(ii) $x^{**} = x$,
(iii) $x^* x = x x^*$,
(iv) $(xy)^* = y^* x^*$,
(v) $|x|^2 = |x|

for any $\lambda, \mu \in R$ and $x, y \in A$.

Any absolute-valued algebra which is complete and admits an involution is said to be an absolute-valued *-algebra. Using the well-known process of embedding linear normed spaces in Banach ones, we can prove that any absolute-valued algebra with an involution can be extended to an absolute-valued *-algebra. Therefore we shall be discussing only absolute-valued *-algebras.

As well-known examples of absolute-valued *-algebras we quote the real field and the complex field with the identity transformation as an involution and the complex field, the quaternion algebra and the Cayley-Dickson algebra with the natural involution $(a + b z + c \bar{z} + d)^* = a - b z + c \bar{z} - d$. 


$-\mu - \ldots \quad \{ t_1, \mu_1, \ldots \in \mathcal{E} \}$. Now we shall give simple examples of infinite dimensional absolute-valued $^*$-algebras.

Let $X$ be an infinite dimensional real Hilbert space with an ortho-normal basis $\{ e_i \}_{i \in \mathcal{I}}$. By Zermelo's axiom of choice the set $T$ of indices can be decomposed into three disjoint subsets $T_1, T_2$, and $T_3$ such that $1 \leq \mathcal{I}_1 \leq \mathcal{T}_1 = \mathcal{T}_2$, where $T$ denotes the power of the set $B$. Let $\varphi$ be a one-to-one correspondence of the class of all two-point subsets of $T$ (i.e., non-ordered pairs $\{ t_1, t_2 \}$, where $t_1 \neq t_2, t_1, t_2 \in T$) onto the set $T$. Further, let $\psi$ be a function defined on the set of all ordered pairs $\langle t_1, t_2 \rangle$ ($t_1 \neq t_2, t_1, t_2 \in T$) taking the values 1 and $-1$ such that $\psi(t_1, t_2) + \psi(t_2, t_1) = 0$ both if $t_1, t_2 \in T_1$ and if $t_1, t_2 \notin T_1$ and $\psi(t_1, t_2) = 1$ otherwise. For example, if the set $T$ is ordered by a relation $t_1 \preceq t_2$ we put $\psi(t_1, t_2) = 1$ or $-1$, whenever $t_1 < t_2$ or $t_1 > t_2$, respectively and $t_1, t_2 \in T_1$ or $t \in T_2$. Let $\epsilon(t) = 1$ or $-1$ according as $t \in T_1$ or $t \notin T_2$.

Let us fix an element $e \in T$. It is clear that to define a multiplication and an involution on $X$ we must define the basis $(e_i)_{i \in \mathcal{I}}$. We define the multiplication and the involution of elements of the basis by the formulas

$$e_i^2 = \epsilon(t) e_i, \quad e_i e_j = \psi(t_1, t_2) \epsilon(\epsilon(t_1, t_2), t_1, t_2) e_i e_j$$

if $t_1 \neq t_2$,

$$e_i^2 = \epsilon(t) e_i.$$

These operations together with the usual addition and scalar-multiplication make $X$ an absolute-valued $^*$-algebra, which will be denoted by $X(T, T_1, T_2, \varphi, \psi)$. Indeed, setting (1)

$$x = \sum_{t \in \mathcal{I}} \lambda_t e_t, \quad y = \sum_{t \in \mathcal{I}} \mu_t e_t,$$

we have the equality

$$(1) \quad xy = \left( \sum_{t \in \mathcal{I}} \lambda_t \mu_t \epsilon(t) e_t \right) + \sum_{t, t \in \mathcal{I}} \left( \psi(t, t_1) \lambda_t \mu_t \epsilon(t, t_1, t_2) e_{t_1, t_2} \right) \epsilon(\epsilon(t_1, t_2), t_1, t_2) e_{t_1, t_2},$$

where the sum is running over all non-ordered pairs $(t_1, t_2)$ satisfying the condition $t_1 \neq t_2$. Hence, taking into account the equality

$$e(t) \epsilon(t) = \psi(t, t_1) \epsilon(t_1, t_2) = 0 \quad (t_1 \neq t_2),$$

we get

$$|xy| = \left[ \left( \sum_{t \in \mathcal{I}} \lambda_t \mu_t \epsilon(t) \right)^2 + \sum_{t, t_1, t_2 \in \mathcal{I}} \left( \psi(t, t_1) \lambda_t \mu_t \epsilon(t_1, t_2) + \psi(t, t_1) \lambda_t \mu_t \epsilon(t_1, t_2) \right)^2 \right]^{1/2}$$

$$= \left( \sum_{t \in \mathcal{I}} |\lambda_t|^2 \right)^{1/2} \left( \sum_{t, t_1, t_2 \in \mathcal{I}} |\psi(t, t_1)|^2 \right)^{1/2} = |x||y|.$$

Properties (i), (ii) and (v) of the involution are evident. From (1) we obtain the formula

$$(xy)^* = \left( \sum_{t \in \mathcal{I}} \lambda_t \mu^*_t \epsilon(t) \right) e_t^* + \sum_{t, t_1, t_2 \in \mathcal{I}} \left( \psi(t, t_1) \lambda_t \mu^*_t \epsilon(t_1, t_2) \lambda^* e_{t_1, t_2} \right) \epsilon(\epsilon(t_1, t_2), t_1, t_2) e_{t_1, t_2},$$

$$y^* x^* = \left( \sum_{t \in \mathcal{I}} \lambda_t \mu^*_t \epsilon(t) \right) e_t^* + \sum_{t, t_1, t_2 \in \mathcal{I}} \left( \psi(t, t_1) \lambda_t \mu^*_t \epsilon(t_1, t_2) \lambda^* e_{t_1, t_2} \right) \epsilon(\epsilon(t_1, t_2), t_1, t_2) e_{t_1, t_2}.$$

Since, for any $t \neq t_2, \psi(t, t_1) \epsilon(t_1, t_2) \epsilon(t_1, t_2) = -\psi(t_1, t_2)$ and $\psi(t, t_2) \epsilon(t_1, t_2) = -\psi(t_1, t_2)$, we get the equality $(xy)^* = y^* x^*$. Further, according to the equality

$$\psi(t, t_1) \epsilon(t_1, t_2) \epsilon(t_1, t_2) = \psi(t_1, t_2) \epsilon(t_1, t_2) \epsilon(t_1, t_2) + \psi(t_1, t_2) \epsilon(t_1, t_2) \epsilon(t_1, t_2),$$

we have the formula

$$ax^* = \left( \sum_{t \in \mathcal{I}} \lambda_t \epsilon(t) \right) e_t + \sum_{t, t_1, t_2 \in \mathcal{I}} \left( \psi(t, t_1) \lambda_t \epsilon(t_1, t_2) \epsilon(t_1, t_2) + \psi(t_1, t_2) \epsilon(t_1, t_2) \epsilon(t_1, t_2) \right) \epsilon(\epsilon(t_1, t_2), t_1, t_2) e_{t_1, t_2}$$

which completes the proof.

We note that if $U$ is a linear isometry on $A$ which commutes with the involution, then $A$ remains an absolute-valued $^*$-algebra with respect to the new product $x \cdot y = U(xy)$. This fact suggests the following definition: two absolute-valued $^*$-algebras $A$ and $A'$ are said to be similar if they are isomorphic as normed spaces with an involution: $A \sim A'$ and the multiplication $xy$ in $A$ is defined in terms of the multiplication $x' \cdot y'$ in $A'$ by the relation $xy = U(x' \cdot y')$, whenever $x \sim x'$, $y \sim y'$, where $U$ is a fixed invertible linear isometry on $A'$ which commutes with the involution.

It is not difficult to prove that two algebras $X(T, T_1, T_2, T_3, \varphi, \psi)$, $X'(T', T_1', T_2', T_3', \varphi', \psi')$ are similar if and only if $T_1 = T_1'$, $T_2 = T_2'$ and $T_3 = T_3'$. Therefore we shall call an absolute-valued $^*$-algebra $X(T, T_1, T_2, T_3, \varphi, \psi)$ a $(m_1, m_2, m_3)$-algebra, where $m_1 = T_1$, $m_2 = T_2$ and $m_3 = T_3$.

For any pair $x, y \in A$ we set

$$\langle x, y \rangle = \frac{1}{2} (xy^* + yx^*).$$

Fundamental Mathematics. T. XLIX.
The operation $\{x, y\}$ will be called a *-product. It imitates an inner product. More precisely, we have the equalities
\[
\{x, x\} = 0 \quad \text{if and only if} \quad x = 0,
\]
\[
\{ax + \mu x, y\} = \lambda \{x, y\} + \mu \{x, y\},
\]
\[
\{x, y\} = \{y, x\}.
\]
Moreover, the *-product is invariant under the involution, i.e.,
\[
\{(x, y)^*\} = \{x, y\}.
\]
An absolute-valued *-algebra is said to be regular if
\[
\{xy, xu\} = \{yx, xu\}
\]
for any $x, y, z, a$, and $u$. It is very easy to verify that the real field and the complex field (with $x^* = x$ or $x^* = -x$) regarded as *-algebras are regular. Now we shall show that all $\langle m_1, m_2, m_3 \rangle$-algebras are also regular. To prove this it is sufficient to show that all elements of the orthonormal basis $\{e_i\}_{i \in T}$ satisfy equality (2). By the definition of the involution in $\langle m_1, m_2, m_3 \rangle$-algebras, equality (2) can be written in the following form:
\[
\{e_i e_j, e_k e_l\} = \varepsilon \{e_j e_i, e_k e_l\} \quad (i, j, k, l \in T).
\]
If $t = t_k$ then the last equality is obvious. Therefore we may suppose that $t \neq t_k$.

If $t_1 = t_2$ and $t_3 = t_4$, then we have the equalities
\[
\{e_i e_j, e_k e_l\} = \{e_j e_i, e_k e_l\} = \{e_j e_k, e_i e_l\} = \{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k},
\]
which imply formula (3).

If $t_1 = t_2$ and $t_3 \neq t_4$, then taking into account the relations $t_k \in T$, and $\varphi(t_k, t_k) \not\in T$, we get the equality
\[
\{e_i e_j, e_k e_l\} = \varepsilon \{e_j e_i, e_k e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

Further, if $t_1 = t_1$, we have the formula
\[
\{e_i e_j, e_k e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

If $t_2 \neq t_3$, then the relations $\varphi(t_1, t_1) \neq \varphi(t_2, t_2)$, $\varphi(t_2, t_2) \neq \varphi(t_3, t_3)$, $\varphi(t_3, t_3) \in T$, imply the equality
\[
\varepsilon \{e_i e_k, e_j e_l\} = -\varepsilon \{e_i e_k, e_j e_l\}.
\]

Hence we get the equality
\[
\{e_i e_k, e_j e_l\} = \{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

Consequently, equality (3) is proved in the case $t_1 = t_2$ and $t_3 \neq t_4$. The case $t_1 = t_2$ and $t_3 = t_4$, in view of the commutativity of *-products, is reduced to the previous case. Now we assume that $t_1 = t_2$ and $t_3 \neq t_4$. Then we have
\[
\{e_i e_k, e_j e_l\} = \{e_i e_k, e_j e_l\} = \{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

and, consequently,
\[
\{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

If $t_1 \neq t_2$ and $t_3 \neq t_4$, then, in view of the assumption $t_1 = t_2$, we have the inequality $\varphi(t_2, t_3) \neq t_2$, $t_3$, $t_4$ and
\[
\{e_i e_k, e_j e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = \varepsilon \{e_j e_k, e_i e_l\} = e_{i k}.
\]

By the commutativity of the *-product, the case $t_2 = t_3$ is reduced to the previous case. Consequently, equality (3) holds for any system of indices $t_1, t_2, t_3, t_4$. In other words, we have proved that $\langle m_1, m_2, m_3 \rangle$-algebras are regular.

We remark that an absolute-valued *-algebra similar to a regular one is also regular.
In the present paper we shall represent regular absolute-valued \(*\)-algebras. Namely, we shall prove the following theorem, which is an answer to a problem raised by F. B. Wright.

**Theorem.** A regular absolute-valued \(*\)-algebra is similar to either the real field, the complex field (with \(x^* = x\) or \(x^* = \overline{x}\)) or a \((m_1, m_2, m_3)\)-algebra, where \(m_1, m_2, m_3\) are cardinals satisfying the inequalities
\[
1 \leq m_1 \leq m_2 \leq m_3 \leq m_n \geq n.
\]

Before proving the theorem we shall prove some lemmas. In the sequel \(A\) will denote an absolute-valued \(*\)-algebra. By \(A_0\) we shall denote the set of all self-adjoint elements of \(A\), i.e. the set of all elements \(x\) satisfying the equality \(x^* = x\). By \(A_1\) we shall denote the set of all skew elements of \(A\), i.e. the set of all elements \(x\) satisfying the equality \(x^* = -x\). Obviously, both \(A_0\) and \(A_1\) are linear subspaces of \(A\) and \(A_0 \cap A_1 = \{0\}\).

**Lemma 1.** Self-adjoint elements commute with skew elements.

**Proof.** Let \(x \in A_0\) and \(y \in A_1\). By property (iii), we have the equality
\[
0 = (x + y^*)(x + y) - (x + y^*)(x + y)^* = (x + y)(x + y) - (x + y)(x + y) = 2(xy - yx),
\]
which implies the assertion of the lemma.

**Lemma 2.** For any \(x \in A_0\) and \(y \in A_1\) we have the equality
\[
|x + y|^2 = |x|^2 + |y|^2.
\]

**Proof.** If either \(x \) or \(y \) is equal to \(0\), then our statement is obvious. Therefore we may suppose that \(x \neq 0\) and \(y \neq 0\). Let \(B\) be the linear span formed by \(x\) and \(y\). Since \(x^* = x\) and \(y^* = -y\), \(B\) is invariant under the involution. By Lemma 1, the elements of \(B\) commute with one another. Therefore, for every pair \(x_1, x_2\) of elements of \(B\) we have \([x_1, x_2]^* = -[x_1, x_2]\). Consequently, for \(|x| = |y| = 1\), we get the inequality
\[
|x_1 + x_2|^2 = |x_1|^2 + |x_2|^2 + |x_1 - x_2|^2 \geq |x_1 - x_2|^2 = 4|x_1||x_2| = 4.
\]

Hence, according to Schoenberg's Theorem ([2]), \(B\) is an inner product space over \(\mathbb{R}\). There are then a number \(\lambda\) and an element \(y_B \in B\) orthogonal to \(x\) such that \(y = \lambda x + y_B\). Since by (v) the involution is an isometry on \(B\), the element \(y_B^*\) is also orthogonal to \(x\). From the equality
\[
y = \frac{1}{2}(y - y^*) = \frac{1}{2}(\lambda x + y_B - \lambda x - y_B) = \frac{1}{2}(y_B - y_B)
\]
follows that \(y\) and \(x\) are orthogonal. The statement of the Lemma is a direct consequence of the orthogonality of \(x\) and \(y\).

**Lemma 3.** If \(A \neq \{0\}\), then there exists one and only one idempotent \(e \in A_0\) such that
\[
x^* = |x|^2 e, \quad y^* = -|y|^2 e
\]
for any \(x \in A_0\) and \(y \in A_1\).

**Proof.** Let \(x \in A_0\) and \(y \in A_1\). Since, by Lemma 1, \(x\) commutes with \(y\), we have the equality \((xy + yx)^* = (xy + yx)\). Consequently, \(xy + yx = (xy + yx)^* = (xy + yx)\). Further, we have the equality \((x^2 + y^2)^* = (x^2 + y^2)\), which implies the relation \(x^2 + y^2 \in A_0\). Hence, with the aid of the formula \((xy)^* = x^2 + y^2 + 2xy\) as well as Lemma 2, we obtain the following equality
\[
|x + y|^2 = |x + y|^2 = |x|^2 + |y|^2 + 4|x|^2|y|^2.
\]

But \(|x|^2 + |y|^2 = |x|^2 + |y|^2\) and, consequently, we have the equality
\[
(|x|^2 + |y|^2)^* = |x|^2 + |y|^2 = (x \in A_0, y \in A_1).
\]

By the assumption there exists a skew element \(y_0\) with \(|y_0| = 1\). Putting \(e = -y_0\), we have \(e^* = -y_0^* = -y_0 = e\), \(|e| = 1\) and, in view of (4),
\[
|x^2 - |x|^2 e| = |x^2 - |x|^2 y_0^2| = |x|^2 - |x|^2 |y_0|^2 = 0
\]
for any \(x \in A_0\).

Thus
\[
0 = |x| = |x|^2 e \quad (x \in A_0).
\]

In particular, \(e = \frac{1}{2}e\). The last equality and formula (4) imply the equation
\[
|y|^2 |x|^2 + |y|^2 |x|^2 = |y|^2 |x|^2 |y|^2 - |y|^2 |x|^2 = 0
\]
for any \(y \in A_1\). Thus \(y^* = -|y|^2 e\) for all skew elements \(y\). The uniqueness of the idempotent \(e\) follows from equality (5). The lemma is thus proved.

**Lemma 4.** Every absolute-valued \(*\)-algebra with \(A \neq \{0\}\) is a real Hilbert space.

**Proof.** Let \(x_1, x_2\) be a pair of elements of \(A\), with \(|x_1| = |x_2| = 1\). Writing \(x_1 = x_1 + y_1, x_2 = x_2 + y_2\) where \(x_1, x_2 \in A_0\) and \(y_1, y_2 \in A_1\), we have, according to Lemma 2,
\[
|x_1|^2 + |y_1|^2 = 1, \quad |x_1|^2 + |y_2|^2 = 1.
\]

Moreover, in view of Lemma 2, we have the equalities
\[
|x_1 - x_2|^2 = |x_1|^2 + |y_1|^2 - 2|x_1||y_1| = 2|x_1|^2 - 2|x_2|^2 - 2|x_1||y_1| = 2|x_1|^2 - 2|x_2|^2.
\]

Using Lemma 3 we obtain the inequalities
\[
|x_1 - x_2|^2 + |y_1 - y_2|^2 = 2|x_1|^2 + 2|x_2|^2 + 2|x_1|^2 + 2|x_2|^2 = 4|x_1|^2 + 4|x_2|^2.
\]

Therefore, \(x_1, x_2, y_1, y_2\) are orthogonal. The statement of the Lemma is a direct consequence of the orthogonality of \(x_1, x_2, y_1, y_2\).
Hence and from (9) and (7) we get the inequality
\[ |x_1-x_2|^2 + |y_1-y_2|^2 \geq 2(|x_3|^2 + |x_4|^2 + |y_5|^2 + |y_6|^2) = 4. \]
Thus, according to Schoenberg's Theorem (22), we know that \( \mathcal{A} \) is an inner product space over \( \mathbb{R} \) and, consequently, a real Hilbert space.

For any \( x \in \mathcal{A} \setminus \mathcal{A}_s \), \( x \neq 0 \), we set \( \delta(x) = 1 \) or \( -1 \) according as \( x \in \mathcal{A}_s \) or \( x \not\in \mathcal{A}_s \) and \( \delta(0) = 0 \).

**Lemma 5.** If \( \mathcal{A}_s \neq \{0\} \), then for any pair of orthogonal elements \( x, y \in \mathcal{A}_s \setminus \mathcal{A}_s \) we have the equality
\[
\delta(x) \delta(y) = 0.
\]

**Proof.** If \( \delta(x) \delta(y) = 0 \), then our assertion is obvious. Further, if \( \delta(x) \delta(y) = -1 \), then one element of the pair belongs to \( \mathcal{A}_s \) and the other belongs to \( \mathcal{A}_s \). Consequently, by Lemma 3, \( xy = 0 \). Now let us suppose that \( \delta(x) \delta(y) = 1 \), i.e. both elements \( x \) and \( y \) belong to either \( \mathcal{A}_s \) or \( \mathcal{A}_s \). From the orthogonality of \( x \) and \( y \) and from Lemma 5 we get the equality
\[
|\delta(x)|\left(|x|^2 + |y|^2\right) = |x^2 + y^2| = |x \pm y|^2 = |x \pm y|^2 = |\delta(x)|\left(|x|^2 + |y|^2\right).
\]
Hence, \( x = y \), which completes the proof of the Lemma.

By \( (x, y) \) we shall denote the inner product of two elements \( x \) and \( y \) \( (x, y) \in \mathcal{A} \).

**Lemma 6.** If \( \mathcal{A}_s \neq \{0\} \), then for any pair \( x, y \in \mathcal{A} \) we have the equality
\[
(x, y) = (x, x) + (y, y).
\]

**Proof.** Let \( x, y \in \mathcal{A} \). Since \( x, y \in \mathcal{A}_s \), the subspaces \( x, y \) and \( \mathcal{A}_s \) are orthogonal. Consequently, writing \( x = x + y, y = x + y \), where \( x, y \in \mathcal{A}_s \) and \( y, x \in \mathcal{A}_s \) we have the equality
\[
(x, y) = (x, x) + (y, y).
\]

Further, since by Lemma 1 self-adjoint elements commute with skew elements, we have the equalities
\[
(x, y) = \frac{1}{2} (x_1 y_2^* + y_1 x_2^*) = \frac{1}{2} (-x_1 y_2 + x_2 y_1) = 0,
\]
\[
(y, x) = \frac{1}{2} (y_1 x_2^* + x_1 y_2^*) = \frac{1}{2} (y_1 x_2 - y_2 x_1) = 0.
\]

Consequently,
\[
(x, y) = (x, x) + (y, y).
\]

**Absolute-valued algebras with an involution**

Let us represent the elements \( x \) and \( y \) in the form
\[
x = \lambda x_1 + x_2, \quad y = \mu y_1 + y_2,
\]
where \( \lambda \) and \( \mu \) are real numbers, \( x_1 \) is a self-adjoint element orthogonal to \( y \) and \( y_1 \) is a skew element orthogonal to \( y_2 \). Obviously, \( (x, x) = |x|^2 \), \( (y, y) = |y|^2 \) and, by Lemma 5, \( x_1 x_2 + x_2 x_1 = 0 \), \( y_1 y_2 + y_2 y_1 = 0 \). Hence, in view of Lemma 3, we get the equalities
\[
(x_1, x_2) = \frac{1}{2} (x_1 x_2 + x_2 x_1) = \frac{1}{2} (2\lambda^2 + x_1 x_2 + 2\lambda^2 + x_2 x_1) = \lambda |x_1|^2 = (x_1, x_2)^*,
\]
\[
(y_1, y_2) = \frac{1}{2} (y_1 y_2 + y_2 y_1) = -\frac{1}{2} (2\mu^2 + y_1 y_2 + 2\mu^2 + y_2 y_1) = \mu |y_1|^2 = (y_1, y_2)^*.
\]

Taking into account equalities (8) and (9) we get the assertion of the Lemma.

**Lemma 7.** Let \( \mathcal{A} \) be a regular absolute-valued \( * \)-algebra and \( \mathcal{A}_s \neq \{0\} \). For any system \( x_1, x_2, x_3, x_4 \) of orthogonal elements belonging to \( \mathcal{A}_s \) and \( x_5, x_6 \) are orthogonal, whenever the sets of indices \( (j_1, j_2), (j_3, j_4) \) are different and \( j_1 \neq j_3, j_2 \neq j_4 \), \( (j_1, j_2, j_3, j_4) = 1, 2, 3, 4 \).

**Proof.** First we suppose that \( (j_1, j_2) \cap (j_3, j_4) = \emptyset \). Using Lemma 5 we can write
\[
(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}) = (x_{j_1}, x_{j_2}) + (x_{j_3}, x_{j_4}),
\]
where \( s_{j_1} \neq s_{j_2} \). Hence and from the equality
\[
(x_{j_1}, x_{j_2} - x_{j_3} - x_{j_4}) = \frac{1}{2} \left( x_{j_1} x_{j_2} + x_{j_3} x_{j_4} - x_{j_1} x_{j_2} - x_{j_3} x_{j_4} \right) = 0,
\]
we get the orthogonality of \( x_{j_1}, x_{j_2} \) and \( x_{j_3}, x_{j_4} \).

Now let us assume that \( (j_1, j_2) \cap (j_3, j_4) = \emptyset \), i.e. the system \( j_1, j_2, j_3, j_4 \) is a permutation of \( 1, 2, 3, 4 \). Without loss of generality we may suppose that both \( x_1 \) and \( x_2 \) belong to either \( \mathcal{A}_s \) or \( \mathcal{A}_s \). Moreover, we may assume that
\[
\delta(s_1) 
eq 0 \quad \text{and} \quad \delta(s_2) = 1,
\]
because in the opposite case \( s_1 = 0, s_2 = 0 \) and the orthogonality of \( x_{j_1}, x_{j_2} \) and \( x_{j_3}, x_{j_4} \) is evident.

Since the algebra \( \mathcal{A} \) is regular, we have, according to Lemma 6,
\[
(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}) = (x_{j_1}, x_{j_2}) + (x_{j_3}, x_{j_4}).
\]

Hence and from Lemma 5 it follows that
\[
(x_{j_1}, x_{j_2}, x_{j_3}, x_{j_4}) = \pm (x_{j_1}, x_{j_2})
\]
for any permutation \( j_1, j_2, j_3, j_4 \) of integers \( 1, 2, 3, 4 \).

Put \( u = (x_{j_1} + x_{j_2})(x_{j_3} + x_{j_4}) \). By the orthogonality of \( x_1, x_2, x_3 \) and \( x_4 \) we have the relation
\[
|u|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2.
\]
Further, from Lemma 5 we infer that all the elements \( x_1, x_2, x_3, x_4 \) are skew. Hence we get the equality
\[
-\|x\|^2 = \left( (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + + 2 (x_1, x_2, x_3, x_4) + 2 (x_1, x_2, x_3, x_4) + 2 (x_1, x_2, x_3, x_4) + + 2 (x_1, x_2, x_3, x_4) + 2 (x_1, x_2, x_3, x_4) \right).
\]
Replacing in the last formula, in view of Lemma 6, the \(*\)-product by the inner product, we get the equality
\[
-\|x\|^2 = \left( \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 + \|x_4\|^2 \right).
\]

Since \( u = x_1 + x_2 + x_3 + x_4 \), and, consequently, is a skew element, we have, in virtue of Lemma 3, the equality \(-\|u\|^2 = \|u\|^2 e\). Hence and from (12) and (13) we get the relation
\[
(x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + + (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) + (x_1, x_2, x_3, x_4) = 0.
\]

By the first part of the proof we have the equality
\[
(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) = 0.
\]

Thus
\[
(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) = 0.
\]

By Lemma 3, we have the equalities \( a = -\delta(x) \overline{\delta(x)} a, a = -\delta(y) \overline{\delta(y)} a \). Hence we get the formulas
\[
(x_1, x_2, x_3, x_4) = -\delta(x) \overline{\delta(x)} a, a = (x_1, x_2, x_3, x_4) = -\delta(y) \overline{\delta(y)} a.
\]

Further, from the regularity of \( A \) it follows that \( (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \). Consequently, according to (10) and (14), \( (x_1, x_2, x_3, x_4) = 0 \). Thus, in view of (11), \( x_1, x_2, x_3, x_4 \) are orthogonal.

Proof of the Theorem. First let us suppose that \( A = \{0\} \), i.e. the involution is the identity transformation. Taking into account property (iv) of the involution we infer that \( A \) is a commutative absolute-valued algebra. Consequently, it is isomorphic to either the real field, the complex field or the algebra of complex numbers with the product of \( x \) and \( y \) defined as \( xy \) (see [3]). But the last algebra is similar to the complex field (as an isometry \( U \) we take \( x \rightarrow x \)).

Now let us suppose that \( A \neq \{0\} \). By Lemmas 3 and 4, \( A \) is at least a two-dimensional Hilbert space. Let \( A \) be finite-dimensional and let \( (e_1, e_2, \ldots, e_n) \) be an orthonormal basis for \( A \) consisting of elements belonging to \( A = \{0\} \). Since \( e \) is the idempotent defined by Lemma 3. If \( n \geq 2 \), then, by Lemma 7, \( e \) is orthogonal to \( e_1, e_2, \ldots, e_n \). Moreover, by Lemma 5, \( e_1, e_2, \ldots, e_n \) is a skew element and, consequently, it is orthogonal to \( e \). But the set \( (e, e_1, e_2, \ldots, e_n) \) is also a basis for \( A \). Thus \( e = 0 \), which is impossible. We have proved that any finite-dimensional absolute-valued \(*\)-algebra with a non-trivial involution is two-dimensional. Hence it follows that every element \( x \in A \) can be written in the form \( x = \lambda e + \mu x \), where \( \lambda, \mu, \in R \), and \( e_1, e_2, \ldots, e_n \) is a basis for \( A \). Thus \( U \) is isomorphic to the complex field if \( e_1 = e_2 \) and is similar to the complex field if \( e_1 = e_2 \) (as an isometry \( U \) we take the involution).

Finally let us suppose that \( A \) is infinite-dimensional. Let \( (e_1, e_2, \ldots) \) be an orthonormal basis of \( A \) consisting of elements belonging to \( A = \{0\} \) and containing the idempotent \( e \) defined by Lemma 3. By \( T_1 \) we denote the subset of indices such that \( (e_i, e_i) \) is a basis for \( A \). Evidently, \( (e_i, e_i) \) is a basis for \( A \). Let \( B \) be the linear subspace spanned by all products \( e_i e_j \) where \( i \neq j \) and \( i, j \in T \). Since, by Lemmas 1 and 2, all those products are skew, \( B \) is a subspace of \( A \). Using the axiom of choice, we can decompose the set of indices \( T \) into disjoint sets \( T_1 \) and \( T_2 \), where \( T_1 \) is the dimension of \( B \) and \( T_2 \) is the dimension of the orthogonal complement of \( B \) in \( A \). By definition, there exists an index \( t_i \in T \) such that \( e_i = e \). Further, let \( \psi \) be a pair of functions satisfying the requirements given in the definition of \( (e_i, m_i, m_i) \)-algebras. To prove our theorem it is sufficient to show that the \(*\)-algebra \( A \) is isomorphic to the algebra \( A(T_1, T_2, T_3, e, \psi) \).

It is very easy to see that the formula \( e_i e_2 \sim \psi((t_1, t_2), e_2, e_2) \) defines a one-to-one correspondence between the family \( (e_i, e_i) \) \( (i \neq j; i, j \in T) \), which, by Lemma 7, is an orthonormal basis of \( B \), and the orthonormal family \( (e_i, e_i) \). This correspondence and the identity transformation \( A \) can be extended to a unitary transformation \( U \) of the whole space \( A \). Obviously, \( A_1 \) and \( A_2 \) are invariant under the transformation \( U \). Hence, in particular, it follows that the transformation \( U \) commutes with the involution in the algebra \( A(T_1, T_2, T_3, e, \psi) \). Denoting by \( - \) the product in \( A(T_1, T_2, T_3, e, \psi) \) we have the equalities
\[
U(e_i e_j) = U(e_i e_j) = e_i e_j \;
( i, j \in T),
\]
\[
U(e_i e_2) = U(e_i e_2) = e_i e_2 \;
( i \neq j; i, j \in T) .
\]
Consequently, \( U(x, y) = xy \) for any \( x \) and \( y \) in \( A \). In other words, the algebras \( A \) and \( A(T_1, T_2, T_3, e, \psi) \) are similar.
References


Reçu par la Rédaction le 30. 3. 1960

Nilpotent free groups

by

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The following theorem, proved by A. I. Malcev in [3], will be denoted further by \( (M) \).

\( (M) \) Let \( G \) be a nilpotent free group of class \( c \), and \( X \) a subset of \( G \) of the cardinality \( |X| > 1 \). Then \( X \) is a nilpotent free generating system for some subgroup of \( G \), of the same class \( c \), if and only if \( X \) is linearly independent modulo the derived subgroup \( G' \).

This paper consists of two parts. The first is devoted to a group theoretical proof of \( (M) \). The proof is based on the following theorem.

\( (T) \) If \( x_1, \ldots, x_n \) is a free, or nilpotent free, generating system of a free, or nilpotent free, group \( G \), then a system \( x_1^{n_1} : x_1^{n_2} : \ldots : x_1^{n_k} \) is free, or nilpotent free, for any \( x_1 \in G' \), and for any positive integers \( n \).

The proof of \( (T) \) essentially needs M. Hall's theory of basic commutators exposed, for example, in [1].

The second part contains theorems that can be derived from \( (M) \):

**Theorem 1.** A subgroup \( H \) of a nilpotent free group \( G \) is a nilpotent free group if and only if it satisfies the condition \( H' = H \cap G' \) or is a cyclic group. (\( H' \) and \( G' \) are the derived subgroups of \( H \) and \( G \).)

**Theorem 2.** Every retract of a nilpotent free group \( G \), is a nilpotent free factor of \( G \) and a nilpotent free subgroup of \( G \).

The analogous statement fails for retracts of free or solvable free groups, see [4].

**Theorem 3.** An endomorphism of a nilpotent free group \( G \) is an automorphism of that group if and only if it induces an automorphism of \( G/G' \).

The terminology of the paper is the same as that in the book of M. Hall [1]. Some basic notions concerning varieties of nilpotent groups are listed in an introductory part. They can be found partly in papers [2], [5] and [6], and partly in the book quoted.

1. We define recursively the simple commutators as: \( [x] = x \), \( [x_1, x_2] = x_1^{-1} : x_2 : x_1 \), and \( [x_1, \ldots, x_{n+1}] = ([x_1, \ldots, x_n] : x_{n+1}] \). A group