Maximal n-disjointed sets and the axiom of choice

by

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This note contains a generalization of a result of R. L. Vaught [1] concerning the equivalence of the existence of maximal disjointed sets with the axiom of choice. Our generalization arises naturally when the notion of a disjointed set is considered as a special case (namely, when $n = 2$) of the notion of an n-disjointed set.

Let n be an integer greater or equal to 2. A set x is said to be n-*disjointed* if any n distinct elements of x has an empty intersection. An n-disjointed subset y of x is said to be a maximal n-disjointed subset of x if y is not properly contained in any n-disjointed subset of x. Notice that if y is an n-disjointed set then y is an m-disjointed set for each m greater or equal to n; also, if y is an n-disjointed set then every subset of y is an n-disjointed set.

Consider the following two sentences:

ξ_n: *Every n-disjointed subset of a set x can be extended to a maximal n-disjointed subset of x.*

ν_n: *Every set x contains a maximal n-disjointed subset.*

It is quite clear that for each n the sentence ξ_n implies the sentence ν_n. We shall now show that the sentence ξ_2 is equivalent with the sentence ν_2. Let y be a 2-disjointed subset of x, and let z be the set of those elements t of x such that t does not intersect any member of y, i. e.,

$$z = \{t;\, t \in x \text{ and, for each } s \in y,\, t \cap s = 0\}.$$

By ν_2, there exists a maximal 2-disjointed subset w of z. We assert that $y \cup w$ is a maximal 2-disjointed subset of x containing y. Clearly, $y \cup w$ is 2-disjointed and $y \subseteq y \cup w \subseteq x$. Suppose that $t \in x$ and $y \cup w \cup \{t\}$ is also 2-disjointed, then $t \in z$, $w \cup \{t\} \subseteq z$ and $w \cup \{t\}$ is 2-disjointed. Since w is maximal in z, $t \in w$ and $t \in y \cup w$. This proves the maximality of $y \cup w$ in x. While the above argument for the case when $n = 2$ is quite simple, we do not know at present whether ν_n implies ξ_n for any $n \geqslant 3$.

Our generalization is contained in the following

THEOREM. *For each $n \geqslant 2$, the sentence ξ_n is equivalent with the axiom of choice.*

Proof. Suppose that some integer $n \geqslant 2$ is chosen. The fact that ξ_n follows from the axiom of choice is a simple exercise involving an application of Zorn's lemma. Let us therefore assume the statement ξ_n. Let F be a 2-disjointed family of non-empty sets $p, q, r, \ldots$, we shall prove the axiom of choice by showing the existence of a choice set Z for the family F.

Let $F_1 = \{\{p\}; p \in F\}$ and $F_2 = \{\{p, q\}; p, q \in F\}$. Clearly F_1 is an n-disjointed subset of F_2. By ξ_n, we extend F_1 to a maximal n-disjointed subset X of F_2. We first show that X satisfies the following condition:

(1) *All but at most $n-2$ elements p of F have the following property* (P): *there are exactly $n-2$ distinct elements q of F, q different from p, such that $\{p, q\} \in X$.*

In the case $n = 2$, (1) is obvious; therefore assume $n \geqslant 3$ and assume, to the contrary, that

(2) *there are at least $n-1$ distinct elements $p_1, \ldots, p_{n-1}$ of F not enjoying the property* (P).

Since X is an n-disjointed subset of F_2, for each p_i there can not be more than $n-2$ distinct elements q of F different from p_i such that $\{p_i, q\} \in X$. For otherwise, together with the set $\{p\}$, there will be at least n distinct sets of X which have a non-empty intersection. Thus, since each p_i does not enjoy property (P), we have

(3) *for each p_i, there are no more than $n-3$ distinct elements q of F different from p_i such that $\{p_i, q\} \in X$.*

From (2) and (3), we see that for the element p_1, for instance, there is at least one p_j, $j \neq 1$, such that $\{p_1, p_j\} \notin X$. Now consider the set $X \cup \{p_1, p_j\}$, which we shall show to be n-disjointed. Suppose there are $n-1$ distinct elements $x_1, \ldots, x_{n-1}$ of X different from $\{p_1, p_j\}$ such that $x_1 \cap x_2 \cap \ldots \cap x_{n-1} \cap \{p_1, p_j\} \neq 0$. Then either $x_1 \cap \ldots \cap x_{n-1} = \{p_1\}$ or $x_1 \cap \ldots \cap x_{n-1} = \{p_j\}$. In either case we see that condition (3) can not be satisfied. Thus $X \cup \{p_1, p_j\}$ is n-disjointed; since $\{p_1, p_j\} \notin X$, this contradicts the maximality of X. Hence (2) is disproved and (1) holds.

Let $F_3 = X - F_1$, and $F_4 = F_3 \cup \big\{\{p, \{t\}\}; t \in p \in F\big\}$. (Cf. Vaught's original argument in [1]; in case $n = 2$, $X = F_1$ and $F_3 = 0$.) Clearly F_3 is an n-disjointed subset of F_4, thus using ξ_n once more, we extend F_3 to a maximal n-disjointed subset Y of F_4. First of all, we see that F_3 is an $(n-1)$-disjointed subset of F_4. For, if $n-1$ distinct elements of F_3 yield a non-empty intersection, then that intersection must include some $\{p\}$ of F_1 which belongs to X. Next, we show that the set Y satisfies the following condition:

(4) *for each element p of F having the property* (P), *there exists a unique $t \in p$ such that $\{p, \{t\}\} \in Y$.*

Suppose that

(5) *there exists a p in F with the property* (P) *such that $\{p, \{t\}\} \notin Y$ for each $t \in p$.*

In this case, we pick a p in F and a t in p satisfying

(6) $$\{p, \{t\}\} \notin Y.$$

Consider the set $Y \cup \{p, \{t\}\}$. Let $y_1, \ldots, y_{n-1}$ be any $n-1$ distinct elements of Y different from $\{p, \{t\}\}$, and consider $y_1 \cap \ldots \cap y_{n-1} \cap \{p, \{t\}\}$. If this intersection is non-empty, then either

(7) $$y_1 \cap \ldots \cap y_{n-1} = \{p\}$$

or

(8) $$y_1 \cap \ldots \cap y_{n-1} = \{\{t\}\}.$$

Since p has property (P), at least one of the y's, say y_i, must be of the form $\{q, \{s\}\}$ with $s \in q \in F$. By (6), $q \neq p$. By the 2-disjointedness of F, $\{s\} \neq p$ and $\{s\} \neq \{t\}$. Therefore, both (7) and (8) fail. Hence (5) does not hold. Now suppose that

(9) *there exists a p in F with the property* (P) *such that there are $\{p, \{t\}\} \in Y$ and $\{p, \{s\}\} \in Y$ with $t \neq s$ and $t, s \in p$.*

In this case, there are exactly $n-2$ distinct elements $q_1, \ldots, q_{n-2}$ of F all different from p such that

$$\{p, q_1\}, \ldots, \{p, q_{n-2}\} \in Y.$$

By the 2-disjointedness of F, for each i, $\{p, q_i\} \neq \{p, \{t\}\}$ and $\{p, q_i\} \neq \{p, \{s\}\}$. Thus, there will exist at least n distinct elements of Y whose intersection is $\{p\}$, which is a contradiction. Hence, (9) also does not hold. Condition (4) is now proved.

To conclude the proof, the set Z is defined as follows. Let $Z_1 = \big\{t; \{p, \{t\}\} \in Y \text{ and } p \text{ having property (P)}\big\}$. Clearly, in view of condition (1), a set Z_2 can be defined which will contain exactly one element from each p of F not having property (P). The set $Z = Z_1 \cup Z_2$ is obviously a choice set for F. The proof is complete.

Since we have already shown that the sentence ξ_2 is equivalent with the sentence ν_2, we obtain the

COROLLARY (Vaught). *The sentence ν_2 is equivalent with the axiom of choice.*

It is an open problem whether or not each ν_n, with $n \geqslant 3$, is equivalent with the axiom of choice.

References

[1] R. L. Vaught, *On the equivalence of the axiom of choice and a maximal principle*, Bull. Amer. Math. Soc. 58 (1952), p. 66.

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Measures in homogenous spaces

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1. Notation. Generally our notation will follow that of Weil [W] and Halmos [H]. Let G be a locally compact topological group, H a closed subgroup. Let G/H be the homogeneous space of cosets xH with the usual topology so that G acts, by left translation, as a transitive group of homeomorphisms of G/H. The natural mapping $G \to G/H$ will be denoted by φ but sometimes we shall use the shorter notation $\bar{x}$ instead of $\varphi(x)$ for the projection xH of x in G/H. We shall also use $\bar{x}$ to denote a generic element of G/H. We use dx, $d\xi$ to denote integration with respect to the Haar measures in G, H, and $\Delta(x)$, $\delta(\xi)$ to denote the modular functions in G, H ([W], p. 39).

For any topological space X, $L(X)$ denotes the class of continuous real-valued functions with compact support and $L_+(X)$ denotes the subclass consisting of non-negative functions. Similarly $B(X)$ denotes the class consisting of all extended real-valued Baire functions on X, $B_+(X)$ the non-negative ones. (Extended real numbers include the values $\pm\infty$ as well as the ordinary real numbers.)

A set $Q \subset X$ will be called an LB-*set (locally Baire)* if $Q \cap E$ is a Baire set whenever E is a Baire set. A function which is measurable with respect to the ring of LB-sets will be called an LB-function. It is convenient to extend the notion of a set of measure zero to LB-sets as follows. If Q is an LB-set and μ is a Baire measure we say that $\mu(Q) = 0$ provided that $\mu(Q \cap E) = 0$ for each Baire set E. If $\mu(Q) = 0$ then we say that *almost every x in X belongs to $X - Q$*. If f, g are LB-functions, N is the set $\{x\colon f(x) \neq g(x)\}$, we say that $f = g[\mu]$ if $\mu(N) = 0$. These definitions do not introduce anything new if X is a σ-compact space.

All measures we consider are non-negative Baire measures in the sense that they are defined on the ring of all Baire sets; our usage of the term "Baire measure" differs thus from that of Halmos [H], where a Baire measure is assumed to be finite on compact sets.

2. Definitions and main results. A Baire measure μ on G/H is called (following Weil) *relatively invariant with factor $h(x)$* if $\mu(xE) = h(x)\mu(E)$ for each Baire set E and $x \in G$. Then $h(xy) = h(x)h(y)$