Note on arcs in semigroups *

by

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Koch ([1]) has shown that a compact connected continuous semigroup with unit element must contain an arc. If S is such a semigroup then, under certain conditions, S must be arcwise connected. This is true, for instance, if S is one dimensional and has a zero element (Theorem 3.2 of [6]).

This note is concerned with the arcwise connectedness of certain semigroups and the existence of subsemigroups which are arcs. We recall now some of the standard terminology. An element e is called idempotent if $e^2 = e$. The collection of idempotent elements is denoted by E. If S is a compact semigroup then E is compact. A non-empty set M is a left (right) ideal if $SM \subseteq M$, $(MS \subseteq M)$. The minimal (two-sided), closed ideal is denoted by K. We note that if S is compact, K exists and must be the minimal ideal of S, since $K = SxS$ for any $x \in K$. If a is any point of S we define L_a as $\{x|\ x \cup Sx = a \cup Sa\}$ and R_a as $\{x|\ x \cup xS = a \cup aS\}$. For each a the set H_a is defined as $L_a \cap R_a$. If $e \in E$ then H_e is the maximal subgroup containing e.

It is easy to show that if S is compact the sets L_a (as do R_a and H_a), form an upper semi-continuous decomposition.

We shall, for the sake of simplicity, assume that S is metric.

A semigroup is said to admit relative inverses ([8]) if for every element a there is an idempotent e such that $a = ae = ea$ and an element a' such that $aa' = a'a = e$.

THEOREM 1. *Let S be a compact connected semigroup which admits relative inverses. If S has a zero then E, and consequently S, is arcwise connected.*

Proof. We first form the upper semi-continuous decomposition of S whose elements are the sets H_a. It is easy to see that since S admits relative inverses, each set H_a contains at least one idempotent. We assert that it contains no more than one. To see this let $e, f \in E \cap H_a$ so that $L_e \cap R_e = L_f \cap R_f$. Since $e \in L_f$, we see that $ef = e$. But since $f \in R_e$, we see that $ef = f$. Hence we see that $e = f$.

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If we denote the hyperspace of S by S', we see that S' and E are homeomorphic. The homeomorphism from E onto S being obtained by considering the canonical mapping from S to S', cut down to E. In particular then, E is a continuum.

Now for e and f in E, define $e \leqslant f$ if and only if $ef = fe = e$.

For $g \in E$ let $L(g) = \{x|\ x \in E,\ x \leqslant g\}$. We assert now that $L(g)$ is a continuum. We note that $L(g) = E \cap gSg$. Since S admits relative inverses, gSg admits relative inverses. For if $a \in gSg$ there is an e and an a' such that $ea = ae = a$ and $aa' = a'a = e$. We see that $ge = g(aa') = (ga)a' = aa' = e$ and $ga' = g(ea') = (ge)a' = ea' = a'$. In the same way $eg = e$ and $a'g = a'$ so that e and a' are in gSg. Hence gSg admits relative inverses and, from the first part of the argument, $E \cap gSg = L(g)$ is a continuum. An easy argument shows that "$\leqslant$" is a partial order with closed graph, i.e. $\{(a, b)|\ a \leqslant b\}$ is closed in $S \times S$.

Koch (Corollary 1 of [1]) has shown that if $(X, \leqslant)$ is a compact partially ordered space with unique minimal element 0 such that the graph of $\leqslant$ is closed in $X \times X$ and $L(x)$ is connected for each $x \in X$, then X is arcwise connected.

It follows from this that E is arcwise connected, the minimal element being the zero of S.

Finally, to show that S is arcwise connected, let $x, y \in S$ and $e, f \in E$ be such that $xe = x$ and $yf = y$. If $[0, e]$ and $[0, f]$ denote arcs from 0 to e and 0 to f, the continuum $x[0, e] \cup y[0, f]$ is arcwise connected and contains x and y. Hence S is arcwise connected.

Let X be a continuum and N a closed subset of X. We denote by X/N the space formed by identifying the elements of N, i.e. shrinking N to a point.

Let S be a compact semigroup and M a closed ideal of S. We recall that the Rees quotient S/M is formed by identifying the elements of M. It follows that S/M is a compact (continuous) semigroup (with zero equal to M/M).

By an *I-semigroup* we shall mean a semigroup which is an arc having a zero at one endpoint and a unit at the other.

An inverse semigroup ([15]) is one having the following two properties:

(1) For every a there is an idempotent e such that $ea = a$ and the equation $ax = e$ is solvable.

(2) $ef = fe$ for arbitrary idempotents.

THEOREM 2. *If S is a compact connected inverse semigroup then:*

(i) *K is a group;*

(ii) *E is homeomorphic to the hyperspace of the upper semi-continuous decomposition of sets $\{R_x\}$;*

(iii) *if k is the unit of K and g any idempotent, then there is an I-semigroup from k to g;*

(iv) *S is arcwise connected if and only if K is arcwise connected.*

Proof. (i) It is known [9] that if S contains a minimal right (left) ideal, then K is the union of disjoint right (left) ideals. Let I and I' be minimal right (left) ideals of S. Then, again from [9], we know that $I = eS$ ($I = Se$) and $I' = fS$ ($I' = Sf$). It follows that $ef \in I$ and $ef = fe \in I'$, which implies (since I' and I are disjoint), that $I = I' = K$. Hence K is a group. We let k be the unit of K.

(ii) Since for each a in S there is an idempotent e such that $ea = a$ and $ax = e$ is solvable, we see that for each a there is an idempotent e such that $aS = eS$. Hence each set R_a contains an idempotent. Furthermore, R_a contains only one idempotent, for if $e, f \in R_a \cap E$ we see that $eS = fS$ and it follows that $f = ef = fe = e$. We now let $\mathfrak{S}$ denote the hyperspace of the upper semi-continuous decomposition of S whose elements are the sets R_a. Define $\{R_x\} \leqslant \{R_y\}$ if $xS \subseteq yS$. It follows that "$\leqslant$" is a partial order on $\mathfrak{S}$, with a closed graph, and with unique minimal element $\{R\ \}$. Furthermore, the set $L(R_x) = \{R_\tau|\ R_\tau \leqslant R_y\}$ is connected since $L(R_k) = \alpha(xS)$ where α: $S \to \mathfrak{S}$ is the natural mapping, $\alpha(p) = \{R_p\}$.

From Koch's theorem quoted in the proof of Theorem 1, we see that $\mathfrak{S}$ is arcwise connected.

Now the mapping α considered over E defines a continuous 1-1 mapping from E on to $\mathfrak{S}$. Hence E and $\mathfrak{S}$ are homeomorphic. Since $(ef)^2 = (ef)(ef) = e(fe)f = e(ef)f = (ee)(ff) = ef$, we see that E is a sub-semigroup. E is then a compact connected semigroup with zero element k.

(iii) Koch (Corollary, [2]) has shown that if S is a compact connected semigroup with zero and unit and is such that $aS = bS$ implies $a = b$, then S contains an I-semigroup from its zero to its unit.

Now the semigroup $eEe = E'$ where $e \in E$ is compact connected with unit e and zero k. We assert that if $f, g \in E'$ then $fE' = gE'$ implies $f = g$. For if $fE' = gE'$ then $gf = f$ since $f \in gE$ and $fg = g$ since $g \in fE$ so that $f = g$. Hence, by the above corollary of Koch, there is an I-semigroup $[k, e]$ from k to e.

(iv) If x, y are points of S let $ex = x$ and $fy = y$. If K is arcwise connected then $[k, e]x \cup [k, f]y \cup K$ contains an arc from x to y. The assertation that K is arcwise connected if S is, follows immediately from the fact that if $x \in K$ then $K = S \times S$.

An arc from x to y will be noted by $[x, y]$ and the half open arc $[x, y] - y$ will be denoted by $[x, y)$.

THEOREM 3. *Let $[0, e]$ be an I-semigroup contained in a compact semigroup S. Then $[0, e]$ is contained in an I-semigroup $[0, f]$, maximal with respect to having 0 as a zero.*

Proof. Let T be the collection of all I-semigroups of the form $[0, g]$ where $[0, e] \subseteq [0, g]$. Define on T the partial order $\leqslant$ as follows: for $[0, m]$, $[0, n]$ elements of T, let $[0, m] \leqslant [0, n]$ if and only if $[0, m] \subset [0, n]$. By the Hausdorff maximality principle, the partially ordered set $(T, \leqslant)$ contains a maximal linearly ordered subcollection (chain), T_0. Denote by T_1 the point set which is the union of the elements of T_0, i.e. $T_1 = \bigcup_a [0, e_a]$, $[0, e_a] \in T_0$.

We assert first that if $g \in T_1 \cap E$ then $\overline{T_1 - [0, g]}$ contains no points from $[0, g]$, i.e. $[0, g)$ is open in T_1. Suppose on the contrary that $x \in [0, g) \cap \overline{T_1 - [0, g]}$. Now $xg = gx = x$ since $[0, g]$ is an I-semigroup. If t is any point of $T_1 - [0, g]$ then $gt = tg = g$. To see this we note that since $t \in T_1 - [0, g]$ there is an I-semigroup $[0, f]$ such that $[0, g] \subset [0, t] \subseteq [0, f]$, and from 1.2 of [10], we see that $gt = g$.

Since the set of all points x such that $xg = gx = g$ is compact it is clear that $x \notin \overline{T_1 - [0, g]}$.

We now assert that the compact connected semigroup T_1 is irreducible between two points, one of which is 0. For let p be a point of the common part of the sets $\overline{T_1 - [0, e_a)}$ where $e_a \in T_1 \cap E$. (Such a point p exists since S is compact.) We assert that $\overline{T}_1$ is irreducible from 0 to p. To show this let $x \in \overline{T}_1$ such that $x \neq 0$ and $x \neq p$. From the above argument we know that $[0, x)$ is an open subset of $\overline{T}_1$ and it follows that x separates $\overline{T}_1$ into two sets, $[0, x)$ and $\overline{T}_1 - [0, x)$. It is clear then that no proper subcontinuum of $\overline{T}_1$ contains 0 and p.

Now certainly $T_1^2 = T_1$ since every element in T_1 has an identity in T_1. By continuity, it follows that $(\overline{T}_1)^2 = \overline{T}_1$.

From Theorem 2.1 of [6], we know that if M is any compact connected semigroup with zero, irreducible between two points and satisfying the condition $M^2 = M$ then M is an arc, one endpoint of which is a non zero idempotent.

Hence $T_1 = [0, f]$ where $f^2 = f$. Clearly $[0, f]$ is the desired I-semigroup. For the existence of an I-semigroup $[0, I]$ such that $[0, f] \subsetneq [0, I]$ would contradict the maximality of T_0.

Theorem 4. *Let S be a compact semigroup which contains a unique arc between the points 0 and 1. If 0 is a zero, and 1 is a unit for $[0,1]$ then $[0,1]$ is an I-semigroup.*

Proof. Let x and y be points of $[0,1]$ and assume $xy \notin [0,1]$. The continuum $x[0, y]$ contains an arc $[0, xy]$. Let z be the last point of $[0,1] \cap [0, xy]$ in the order from 0 to 1. Now $x[y, 1]$ contains xy and x. A straightforward argument using the fact that $[0,1]$ is the only arc from 0 to 1 and the fact that $x[y,1]$ is arcwise connected shows that $z \in x[y, 1]$. Let $z = xt$ where $t \in [y, 1]$. Now tS contains an arc from 0 to t so that we must have $y = ts$ for some $s \in S$. We further note that $[0, z]s$ contains an arc from 0 to $zs = (xt)s = x(ts) = xy$. From the uniqueness of $[0, 1]$. It follows that $[0, z] \subset [0, z]s$. But since both $[0, z]$ and $\overline{\{s, s^2, s^3, \ldots\}}$ are compact sets, we can apply Corollary 1 of [3] to get $[0, z] = [0, z]s$. This implies that $xy = zs \in [0, z] \subset [0, 1]$.

This includes the known result that a dendion contains an I-semigroup from its zero to its unit ([10]).

For the convenience of the reader we now state a topological extension of a theorem of Rees-Suschkewitsch (Theorem 1, [4]).

Theorem. *Let S have a compact minimal ideal K, and let $e \in K \cap E$. Form $eSe \times (eS \cap E) \times (Se \cap E)$ with the multiplication $(t, q, f)(t', q', f') = (tqf't', q', f)$ and define φ: $eSe \times (eS \cap E) \times (Se \cap E) \to K$ by $\varphi(t, q, f) = fqt$. Then $eSe \times (eS \cap E) \times (Se \cap E)$ is a compact semigroup and φ is a topological isomorphism. Further, if η: $K \to K$ is defined by "$\eta(t)$ is the idempotent in the group containing t", then r: $S \to K$ defined by $r(x) = x\eta(qx)$ is a retraction of S onto K.*

Lemma 1. *If $eS \cap E$ or $Se \cap E$ is degenerate then eSe is a group.*

Indeed: Suppose $eSe \cap E = e$ or $Se \cap E = e$ then since $eSe \subset eS$ and $eSe \subset Se$, we have $eSe \cap E = e$. But since S is compact, any left (right) ideal I of eSe contains an idempotent. Thus $e \in I$ and consequently $eSe = I$, which proves that eSe is a group.

We also state the following corollary to the above which is proved in [4].

By multiplication of type (i) we mean the multiplication defined by $xy = x$ for all x, y. By type (ii) we mean that defined by $xy = y$ for all x, y.

Corollary. *Let K be a continuum and suppose K is not the cartesian product of two non-degenerate continua. Then either K is a group or multiplication in K is of type* (i) *or* (ii).

We now consider further consequences of the Rees-Suschkewitsch Theorem.

By a *simple semigroup* S, we mean one which satisfies the conditions $SxS = S$ for each x in S. When K exists we note that S is simple if and only if $S = K$.

Theorem 5. *Let S be a compact connected simple semigroup which is a subset of the plane. Then*

(a) *multiplication in S is of type* (i) *or* (ii) or

(b) *S is the cartesian product of two arcs, multiplication in the first being* (i) *and in the second type* (ii) or

(c) *S is the usual circle group or is a cartesian product of a simple closed curve C and an arc A, where multiplication in C is* (i), (ii) *or the usual circle group and in A the multiplication is of type* (i) *or* (ii).

For our proof we shall require the following:

LEMMA 2. *Let X and Y be two non-degenerate continua such that $X\times Y$ is embeddable in the plane. Then either X or Y is an arc and the other is an arc or a simple closed curve.*

Proof. We assert first that both X and Y are locally connected. Suppose on the contrary that say X is not locally connected. Then X contains a non-degenerate continuum of convergence M (see p. 18 of [19]). Now $M\times Y$ since it is two dimensional must contain a disc (Corollary 1, p. 46, [21]), say D. Now D clearly contains an open subset of $X\times Y$. It easily follows that this is impossible since M was a continuum of convergence. (One may, for instance, use the fact that the projection mapping is an open map.) Hence it follows that both X and Y are locally connected. We now assert that neither X nor Y can contain a triod. Suppose that X contains T, a triod. Then $X\times Y$ contains uncountably many triods in violation of the classical theorem of Moore. Now a continuum which is locally connected and contains no triods must be either a simple closed curve or an arc (Theorem 71, Chapter 4, [12]). Clearly we cannot have both X and Y as simple closed curves. Our proof of the Lemma is complete.

Proof of Theorem 5. We recall the description of S as

$$eSe\times(eS\cap E)\times(Se\cap E)\qquad (e\in E).$$

Clearly only two of these three terms can be non-degenerate. If, in fact, only one term is non-degenerate then it follows that multiplication in S is either (i), (ii) or S is a group using Lemma 1. If S is a group then, since S is in the plane, S must be the circle group. Hence we suppose that exactly two terms are non-degenerate. If eSe is non-degenerate it is the cricle group and then $eS\cap E$, $(Se\cap E)$, is an arc with multiplication (i) (or (ii)) again using Lemma 1. This, of course, is from Lemma A.

If, on the other hand both $eS\cap E$ and $Se\cap E$ are non-degenerate then one is a simple closed curve or an arc and the other is an arc. In this case since $eSe=e$, we see that multiplication must be

$$(e,e,f)\cdot(e,e',f')=(eef'e,e',f)=(e,e',f).$$

THEOREM 6. *Let S be a compact connected semigroup such that $S^2=S$. Suppose that K contains no non-degenerate compact connected group. If S contains one and only one idempotent not in K, it is arcwise connected if and only if K is arcwise connected.*

Proof. The Rees quotient $S/K=T$ is a compact connected semigroup with zero element. Furthermore, since $T^2=T$, we know that $T=TET$ (Corollary 1 of [5]). Letting e denote the non-zero idempotent of T we have $T=TeT$. Now the semigroup eTe contains an I-semigroup from e to k, the zero of T (Corollary of [13]). Let us denote this by $[k,e]$. Letting a denote the natural map from S to S/K we note that $a^{-1}[k,e]-K=A$ is homeomorphic to a half open interval. We also note that $K\cup A$ is a semigroup. If for some $a,b\in A$ the product ab is in K it follows that an arc exists from K to f the unit of $A\cdot(a^{-1}(e)=f)$. If on the other hand A is a semigroup then $\bar{A}$ is a compact connected semigroup with unit f, which is irreducible from $K\cap\bar{A}$ to f. Clearly $K\cap\bar{A}$ is the minimal ideal of $\bar{A}$. From Lemma 2.1 or Theorem 2.2 of [6] we know that $K\cap\bar{A}$ is a group and a continuum. Our hypothesis demands that $K\cap\bar{A}$ be degenerate. In any case there is an arc (indeed an I-semigroup), $[p,f]$ where $p\in K$. Now if x and y are points of S, there exist points c,d,r,s such that $x=cfd$ and $y=rfs$. This follows since $S^2=S$ implies $S=SES=SfS$ (Corollary 1 of [5]). The continuum $c[p,f]d\cup r[p,f]s\cup\cup K$ is arcwise connected yielding in particular an arc from x to y.

K is, of course, arcwise connected, whenever the same is true for S, since $K=SpS$, $p\in K$.

THEOREM 7. *Let S be compact and connected. Suppose $S^2=S$ and that S contains no arc. Then S is isomorphic to the cartesian product of two compact connected semigroups M and N such that multiplication in M satisfies* (i) *or* (ii) *and multiplication in N satisfies* (i) *or* (ii).

Proof. We assert that $S=K$. For if K is a proper subset of S, then since $S^2=S$, there is an idempotent e such that $e\notin K$ (Theorem 3 of [5]). eSe meets K and $S-K$, and, being a compact connected cemigroup with unit, contains an arc by Koch's theorem ([1]). We conclude that $S=K$. We also note that by Koch's result ([1]), the semigroup gSg must be degenerate, i.e. $gSg=g$, where g is any idempotent. (Otherwise gSg contains an arc.)

According to the Rees-Suschkewitch theorem for compact semigroups, the semigroup K is isomorphic to the semigroup $eSe\times(eS\cap E)\times\times(Se\cap E)$, $(e\in K\cap E)$, where multiplication is defined by $(t,q,f)(t',q',f')=(tqf't',q',f)$. Since $eSe=e$, we conclude that $K\cong eS\times Se$ with multiplication defined by $(q,f)(q',f')=(q',f)$. Taking $M=eS$ and $N=Se$ our proof is complete.

We note, of course, that M or N may well be degenerate.

THEOREM 8. *Let S be compact and connected. Suppose $S^2=S$ and that S contains no continuum N such that S/N is an arc. If S is irreducible between two points then multiplication in S must be either* (i) *or* (ii).

Proof. We assert that $K=S$. For if K were proper we know from Theorem 2.1 of [6] that the Rees quotient S/K is an arc. We conclude then that $S=K$. By the Rees-Suschkewitch Theorem, we know that K is homeomorphic to the cartesian product of the continua eSe, eS, and Se. Since $S=K$ is irreducible between two points, only one of these three continua is non-degenerate. If $K = eSe$ then K is a topological group using Lemma 1 and is an indecomposable continuum since it is homogeneous and irreducible between two points (Theorem 4 of [20]). If eSe is degenerate we have (i) or (ii).

THEOREM 9. *Let S be a compact, connected and one-dimensional semigroup such that $S^2 = S$. Suppose that S contains no continuum N such that S/N is arcwise connected. Then either S is a group being an indecomposable continuum or the multiplication in S is of type* (i) *or* (ii).

Proof. If K is a proper subset of S, form the Rees quotient S/K. Since S/K is one dimensional, with $(S/K)\cdot(S/K) = S/K$, and has a zero, it is arcwise connected by Theorem 3.3 of [6]. Since S is assumed to contain no such continuum, we conclude $S = K$. Since $S = K$ is one dimensional, it cannot be a non-degenerate cartesian product. Hence $S = K$ is isomorphic to one of eSe, Se, or eS. By Lemma 1, if $S = K = eSe$ then $S = K$ is a topological group and being such that it contains no subcontinuum N such that S/N is arcwise connected, it cannot be a simple closed curve. It follows that $S = K$ is an indecomposable continuum if $S = K = eSe$. If $S = K \cong eS$ then multiplication must be of type (i). If $S = K \cong Se$ we have type (ii).

We have seen then certain continua such that any continuous associative multiplication, subject to the condition $S^2 = S$, must be one of two trivial types, i.e. (i) or (ii). If $S^2 \neq S$ the situation is somewhat more complicated. For let N be any subcontinuum of S such that λ is a mapping from S to N. Then any multiplication on N may be extended to S. Explicitly let $\nu\colon N \times N \to N$ and $\lambda\colon S \to N$ be continuous with ν associative. Then $\theta\colon S \times S \to S$ defined by

$$\theta(s, t) = \nu\big(\lambda(s), \lambda(t)\big)$$

is associative and continuous, but not, of course, onto if N is proper.

Aside from describing all continuous functions of the form θ one can, however, in certain cases, describe products of three or more elements. Consider the following.

THEOREM 10. *Let S be a compact, connected semigroup containing no arcs. Suppose further that if N is any subcontinuum of S and M is any subcontinuum containing N then M is irreducible from N to some point. If K is non-degenerate, then $S^2 = K$.*

Proof. Suppose that K is a proper subset of S^2. Then S^2 is irreducible from K to, say, the point xy. Now Sy contains xy and meets K so that $Sy = S^2$. Also, $Sy^2 = (S^2)y = S(Sy) = SS^2 = S^3$, and in general we have $Sy^n = S^{n+1}$. However, from Corollary 1 of [3] we know that $\bigcap_n Sy^n = Se$ where e is some idempotent. We assert that $e \in K$. For if $e \notin K$ then eSe is irreducible from K to some point. But then eSe contains more than one point and consequently by [1], since eSe is a compact connected semigroup with unit, it contains an arc.

Now K is certainly not a non-degenerate cartesian product (being irreducible between two points). From the Rees-Suschkewitch theorem we conclude that K is composed entirely of right (left) zero elements. Taking e then to be a right zero we have $K \subseteq \bigcap_n S^{n+1} = \bigcap_n Sy^n = Se = e$ so that K is degenerate which is a contradiction. (If e is taken as a left zero one obtains the same contradiction by consideration of $\bigcap_n x^nS$.)

We note that if S satisfies the conditions of Theorem 10 then multiplication must satisfy one of the conditions (i) $abc = ab$ for all a, b, c in S or (ii) $abc = bc$ for all a, b, c in S. The severe restriction of indecomposability on multiplication is seen in the following.

THEOREM 11. *Let S be a compact connected semigroup with zero* 0. *If every subcontinuum of S which contains* 0 *is indecomposable then $S^3 = 0$.*

Proof. We show first that $S^5 = 0$. The continuum S^2 is indecomposable. Let xy_0 be in a composant of S^2 different from that of 0. Clearly then, we have $xS = S^2$. Now since the composant of S^2 which contains xy_0 is dense in S^2, we may take a sequence xy_i converging to 0 such that S^2 is irreducible from 0 to xy_i. We note that $xS = Sy_i = S^2$. Now let y_{m_i} be a convergent subsequence of y_i which converges to, say, the point y. Then xy_{m_i} converges to xy and $xy = 0$ by continuity. Now then we have, again by continuity, $xS = Sy_{m_i} = S^2 = Sy$ with $xy = 0$. Since $xS \cup Sy$ is indecomposable, we must have $Sx \supseteq yS$ or $Sx \subseteq yS$. If $Sx \supseteq yS$ then $0 = (Sx)(yS) \supseteq (yS)(yS)$. Multiplying by S we have $S0 = 0 = S(ySyS) = (Sy)(Sy)S = S^5$ so that $S^5 = 0$. If on the other hand $Sx \subseteq yS$ we have $0 = (Sx)(yS) \supseteq (Sx)(Sx)$ and multiplying $0S = 0 = (Sx)(Sx)S = S(xS)(xS) = S^5$, again $S^5 = 0$.

We now assert that $S^4 = 0$. Now one of yS and S^2y contains the other. If $yS \supseteq S^2y$ then $xyS \supseteq xS^2y = S^4$ but $xyS = 0$ so that $S^4 = 0$. If $yS \subseteq S^2y$ then $S^2yS \subseteq S^4y \subseteq S^5 = 0$. But $S^2yS = S^4$ so that $S^4 = 0$.

We show finally that $S^3 = 0$. Again, we have $yS \subseteq S^3$ or $S^3 \subseteq yS$. If $S^3 \supseteq yS$ then $S^4 \supseteq SyS = S^3 = 0$. Let us assume then that S^3 is a proper subset of yS. We note first that if $yS = S^2$ then $S^3 = xS^2 = xyS = 0$. Hence we may take $S^3 \subsetneq yS \subsetneq S^2$. Now let ys_i be a sequence converging to $s_3' \in S^3$ such that yS is irreducible from ys_i to s_3'. Such a sequence exists because yS is indecomposable. Clearly $Ss_i \supseteq yS$. Now let s_{m_i} be a convergent subsequence of s_i converging to s. We then have ys_{m_i} converging to ys so that $ys = s_3' \in S^3$. Now $Ss \supseteq yS$ so $S^2s \supseteq SyS = S^3$. But $S^2s = Sys = Ss_3' \subseteq S^4 = 0$. Hence $S^3 = 0$.

DEFINITION. By a *local one-parameter semigroup at* l we mean an arc $[p, l]$ such that for some open set V about l we have $([p, l] \cap V)^2 \subseteq [p, l]$.

EXAMPLE 1. (See Fig. 1.) We construct here a compact, connected, locally connected semigroup with unit and zero containing no local one-parameter semigroup at the unit.

Consider first the positive additive real numbers. Identify, modulo 1, the real numbers t greater than or equal to $\frac{1}{2}$. Explicitly for $t \geqslant \frac{1}{2}$, $t' \geqslant \frac{1}{2}$ define $t \equiv t'$ if $|t-t'|$ is an integer. The hyperspace of this de-

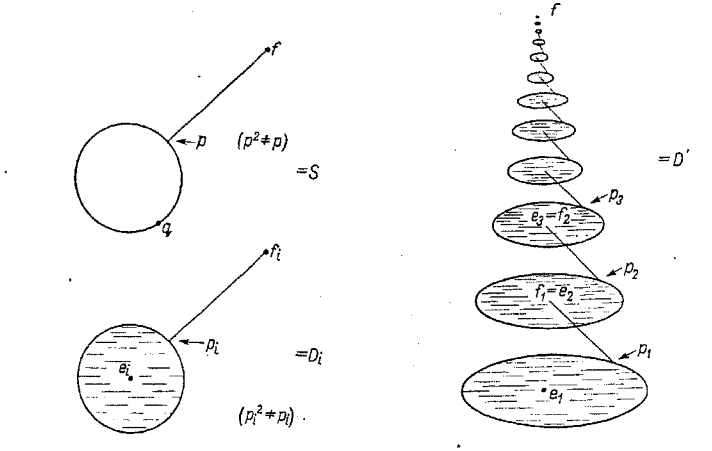

Fig. 1

composition is a simple closed curve C with a free arc which is a semigroup with unit element f. If we denote the identity of the cricle group, which is the minimal ideal, by q we note that q is not the single point of order 3. Denote this semigroup by S and the usual unit interval by $[0, 1]$. Using coordinatewise multiplication, form the semigroup $D = ([0, 1] \times C \cup \{1\} \times S)/\{0\} \times C$. We note that D is simply a disc with circle-group as boundary with a free arc eminating from a point which is not the unit of the group.

Let $D_1, D_2, D_3, \ldots$ be a sequence of semigroups each isomorphic to D. Let f_i be the unit and e_i be the zero element of D_i, for each i. Construct D' as the union of the semigroups D_i, such that $D_i \cap D_{i+1} = f_i = e_{i+1}$ and $D_i \cap D_d = \square$ if $d \neq i-1$, i, or $i+1$. Also, construct D' so that the sequence D_i converges to a single point f. To define multiplication in D', let x_i, x_j be two points of D' with $x_i \in D_i$, $x_j \in D_j$. If $i = j$ the product $x_i x_j$ is already defined (as in D). If $i < j$ define $x_i x_j = x_j x_i = x_i$. It follows that D' is a compact connected semigroup with identity f if we define finally $fd' = d'f = d'$ for each $d' \in D'$. We note, however, that D' contains no I-semigroup from e_i to $f_i = e_{i+1}$. With this in mind it is easy to see that although D' is locally connected it does not contain a local one-parameter semigroup at its unit f.

EXAMPLE 2. (See Fig. 2.) We construct here a compact connected semigroup with no arc at its unit element. First of all let $\mathfrak{S}$ be the usual complex unit disc considered as a semigroup. Let $[0, 1]$ be the usual unit interval and form $\mathfrak{S} \times [0, 1]$. Let $Z = \{(x, y)|\ x = e^{2\pi is},\ y = e^{-s},\ s \geqslant 0\}$. The semigroup $(\mathfrak{S} \times \{0\}) \cup Z = T$ may be described as the complex unit disc with an infinite half-ray spiraling upon the boundary. For each $i = 1, 2, 3, \ldots$ let D_i be a semigroup isomorphic with T. Denote the unit of D_i by f_i and its zero by e_i. Construct the semigroup D'' as

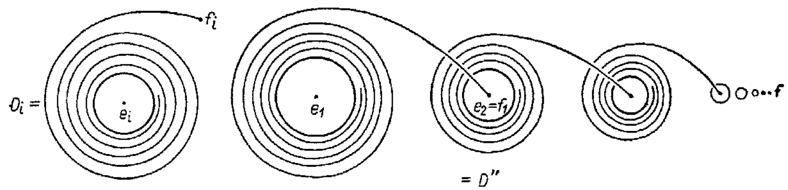

Fig. 2

the union of the D_i such that $D_i \cap D_{i+1} = f_i = e_{i+1}$ and $D_i \cap D_j = \square$ if $j \neq i-1$, i, or $i+1$. Also construct D'' so that the sequence D_i converges to a single point f. To define multiplication, let $x_i \in D_i$, $x_j \in D_j$. If $i = j$, the product $x_i x_j$ is already defined. If $i < j$, then define $x_i x_j = x_j x_i = x_i$. Also, define $fd = df = d$ for any $d \in D''$. It follows that D'' is a compact connected semigroup with zero e_1 and unit f. Clearly, D'' has no arc containing f.

EXAMPLE 3. (See Fig. 3.) We construct now a compact connected semigroup S with zero such that $S^2 = S$ having the property that it separates the plane. It is based upon a semigroup constructed by Faucett.

Let D denote the five element semigroup defined as follows: let the elements of D be θ, w, x, y, e. The multiplication is as follows: $xy = w$, $xe = x$, $ey = y$, $e^2 = e$, all other products being equal to θ. We note that $D^2 = D$. Let $I = [0, 1]$ be the usual unit interval. Form the semigroup $D \times I/J = D'$ where J is the ideal $D \times \{0\} \cup \{\theta\} \times I \cup \cup \{w\} \times \{1\}$. We note that D' is the union of a simple closed curve and a triod, joined at the branch point of the triod. It is clear that D' separates the plane. D' also shows that a translate of an arc need not be an arc. It is also not difficult to see that the simple closed curve in D' may be replaced by any locally connected continuum.

EXAMPLE 4. Let $\mathfrak{S}$ be an indecomposable continuum which is a topological group with unit e. Let I be the unit interval with the

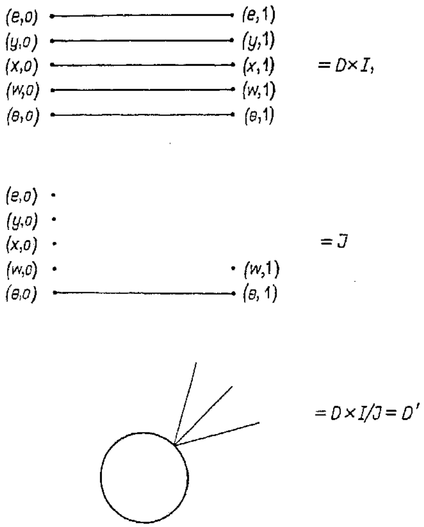

Fig. 3

multiplication $x\cdot y = \text{minimum}(x, y)$. Form $\mathfrak{S}\times I$ with coordinatewise multiplication. Let S denote

$$\mathfrak{S}\times\{0\}\cup\{e\}\times I\,.$$

Then S is a compact connected inverse semigroup (S also admits relative inverses) which is not arcwise connected (since its minimal ideal is not arcwise connected).

References

[1] R. J. Koch, *Arcs in partially ordered spaces*, Pacific Journ. Math. 9 No. 3 (1959), pp. 723-728.

[2] — *Ordered semigroups in partially ordered spaces* (to appear).

[3] — *Monothetic semigroups*, Proc. Amer. Math. Soc. 8 (1957), pp. 397-401.

[4] R. J. Koch and A. D. Wallace, *Semigroup structures on continua*, Trans. Amer. Math. Soc. 88 (1958), pp. 277-287.

[5] — *Maximal ideals in compact semigroups*, Duke Math. Journ. 21 (1954), pp. 681-686.

[6] R. P. Hunter, *On the semigroup structure of continua*, Trans. Amer. Math. Soc. 93 (1959), pp. 356-368.

[7] J. A. Green, *On the structure of semigroups*, Ann. of Math. 54 (1951), pp. 163-172.

[8] A. H. Clifford, *Semigroups admitting relative inverses*, Ann. of Math. 42 (1941), pp. 1037-1049.

[9] — *Semigroups containing minimal ideals*, Amer. Journ. Math. 70 (1948), pp. 521-526.

[10] W. M. Faucett, *Topological semigroups and continua with cut points*, Proc. Amer. Math. Soc. 6 (1955), pp. 747-753.

[11] J. B. Jones and G. S. Young, *Products in manifolds*, Notices, Amer. Math. Soc. 5 (1958), p. 602.

[12] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Coll. Pub. (1932).

[13] P. S. Mostert and A. L. Shields, *One parameter semigroups in a semigroup*, Trans. A. M. S. 96 (1960), pp. 510-517.

[14] — *On the structure of semigroups on a compact manifold with boundary*, Ann. of Math. 65 (1957), pp. 117-143.

[15] G. B. Preston, *Inverse semigroups*, Journ. London Math. Soc. 29 (1954), pp. 396-403.

[16] D. Rees, *On semigroups*, Proc. Cambridge Phil. Soc. 36 (1941), pp. 387-400.

[17] A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. 61 (1955), pp. 95-112.

[18] — *Cohomology, dimension, and semigroups*, Summa Brasil. Math. 3 Fasc. 5 August 1953.

[19] G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Coll. Pub., Vol. 28.

[20] C. E. Burgess, *Some theorems on n-homogeneous continua*, Proc. Amer. Math. Soc. 5 (1954), pp. 136-143.

[21] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton.

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