On convex metric spaces I

by

A. Lelek and **W. Nitka** (Wrocław)

§ 1. Introduction. In February 1959 Professor Borsuk presented to us the following three problems (for an explanation of the notions see §§ 3 and 4 of this paper):

I. *Is it true that every n-dimensional continuum* (connected compact metric space) *which is strongly convex and without ramifications must be topologically n-cell?*

II. *Has every n-cell ($n = 2, 3, \ldots$) with an arbitrary strongly convex metric (topology preserving) at least $n+1$ terminal points?*

III. *Does there exist a compact metric space whose every point is a ramification point or a frontier point?*

The purpose of this paper is to give a partial (for $n = 2$) positive solution of problem I (see § 9), a general (for $n = 2, 3, \ldots$) negative solution of problem II (see § 14) and a positive solution of problem III (see § 12).

§ 2. Betweenness and linearity. We consider a space X with a metric ϱ and write shortly that (X, ϱ) is a *metric space*. Let $p, q, r \in X$. We say (compare [1], p. 317) that the point q is *between* the points p and r (writing pqr) provided that

$$\varrho(p, r) = \varrho(p, q) + \varrho(q, r).$$

Evidently pqr is equivalent to rqp, and we have psr provided that pqr and psq or qsr.

We say that the set $A \subset X$ is *linear* if there exists an isometrical transformation $i\colon A \to \mathcal{E}^1$ of A into the set $\mathcal{E}^1$ of all real numbers, i. e. $\varrho(p, q) = |i(p) - i(q)|$ for every $p, q \in A$.

Hence

2.1. *The set $\{p, q, r\}$ composed of three points is linear if and only if one of the points p, q, r is between two others.*

Let us note that there exists a metric space $(\{p, q, r, s\}, \varrho)$ composed of four points which is not linear, but every proper subset of which is linear. Namely put: $\varrho(p, r) = \varrho(r, q) = \varrho(q, s) = \varrho(s, p) = 1$ and $\varrho(p, q) = \varrho(r, s) = 2$.

However, it is easy to prove that

2.2. *If* $\varrho(p,s) = \varrho(p,q)+\varrho(q,r)+\varrho(r,s)$, *then the set* $\{p,q,r,s\}$ *is linear.*

Now let p, q, r and s be arbitrary points of space X and $p \neq q$. We consider the following three conditions each of which gives another kind of regularity of metric ϱ:

(α) *If* prq *and* psq, *then the set* $\{p,q,r,s\}$ *is linear.*

(β) *If* pqr *and* pqs, *then the set* $\{p,q,r,s\}$ *is linear.*

(γ) *If* pqr *and* spq, *then the set* $\{p,q,r,s\}$ *is linear.*

It is easy to see that condition (β) is equivalent to the following condition:

(β') *If* rpq *and* spq, *then the set* $\{p,q,r,s\}$ *is linear.*

Let us note that every other condition similar to those above is trivially false, or trivially true, or equivalent to (α) or (β) or (γ). Therefore among the conditions of this kind only (α), (β) and (γ) may be interesting objects of study.

§ 3. Convexity. We use this notion in the well-known sense of Menger: a metric space (X,ϱ) is said to be *convex* if for every two points $p, r \in X$ there exists a point $q \in X$ such that $p \neq q \neq r$ and pqr (see [1], p. 41).

An arc contained in X is said to be a *segment* if it is linear. A segment with end-points p and q is denoted by $\overline{pq}$. It is known that

3.1. *If* (X,ϱ) *is a complete convex metric space, then each two distinct points of it are joined by a segment* (see [1], p. 41).

A metric space (X,ϱ) is said to be *strongly convex* (see [2]), and then ϱ is called a *strongly convex metric* or SC-*metric*, if it is 1° complete, 2° convex, and 3° condition (α) holds. It is not difficult to prove that 3.1 and (α) imply that if (X,ϱ) is a strongly convex metric space, then each two distinct points p, q of X are joined by exactly one segment $\overline{pq}$. Therefore a strongly convex metric space is called by other authors *convex with unique segments* (see [1], p. 49-50). Nearly all spaces considered in this paper will be strongly convex, whence we shall use the notion of segment $\overline{pq}$ as uniquely determined by its end-points p and q.

A metric space (X,ϱ) is said to be *without ramifications*, and then ϱ is called a *metric without ramifications* or WR-*metric*, if it is 1° complete, 2° convex, and 3° condition (β) holds. It is easy to see that if (X,ϱ) is a strongly convex metric space without ramifications and $\overline{pq}$ is a segment in X, then $\overline{pq}$ has unique prolongations "to the right" (and, seeing (β'), also "to the left"), i. e. if $\overline{pq} \subset \overline{pr}$ and $\overline{pq} \subset \overline{ps}$, then $\overline{pr} \subset \overline{ps}$ or $\overline{ps} \subset \overline{pr}$. Therefore a strongly convex metric space without ramifications may also be called a space in which the *prolongation is unique* (see [3], p. 36).

A metric space (X,ϱ) is said to be *without edges*, and then ϱ is called a *metric without edges* or WE-*metric*, if it is 1° complete, 2° convex, and 3° condition (γ) holds.

A metric which is both SC-metric and WR-metric, or both SC-metric and WE-metric, or at the same time SC-metric and WR-metric and WE-metric, etc., will be called shortly SC-WR-metric or SC-WE-metric or SC-WR-WE-metric, etc., respectively.

We shall use the following obvious propositions:

3.2. *If* (X,ϱ) *is a strongly convex metric space, then* pqr *is equivalent to* $q \in \overline{pr}$.

3.3. *If* (X,ϱ) *is a strongly convex metric space without ramifications,* $\overline{pq}$ *and* $\overline{pr}$ *are segments in* X *and* $\overline{pq} \cap \overline{pr} - (p) \neq 0$, *then* $\overline{pq} \subset \overline{pr}$ *or* $\overline{pr} \subset \overline{pq}$, *i. e. the sum* $\overline{pq} \cup \overline{pr}$ *is a segment.*

3.4. *If* (X,ϱ) *is a compact strongly convex metric space,* $p_i, q_i \in X$, $p_i \neq q_i$ *for* $i = 0, 1, \ldots$, $\lim\limits_{i\to\infty} p_i = p_0$ *and* $\lim\limits_{i\to\infty} q_i = q_0$, *then* $\operatorname*{Lim}\limits_{i\to\infty} \overline{p_i q_i} = \overline{p_0 q_0}$ (¹).

§ 4. Special kinds of points. A point p of a metric space (X,ϱ) is said to be a *passing point* if there exist two points $a, b \in X$ such that $a \neq p \neq b$ and apb. The set of all passing points of (X,ϱ) is denoted by $\mathcal{P}(X,\varrho)$. We put $\mathcal{T}(X,\varrho) = X - \mathcal{P}(X,\varrho)$ and call each point belonging to $\mathcal{T}(X,\varrho)$ a *terminal point* (compare [1], p. 53).

Therefore

4.1. *If* q *is a terminal point and* $q \in \overline{pr}$, *then* $q = p$ *or* $q = r$.

A segment $\overline{pq} \subset X$ is called *maximal* if $\overline{pq} \subset \overline{rs} \subset X$ implies $\overline{pq} = \overline{rs}$ for every segment $\overline{rs}$. The end-point of a maximal segment is called a *frontier point* and the set of all frontier points of (X,ϱ) is denoted by $\mathcal{F}(X,\varrho)$. Obviously

4.2. *Every terminal point of a compact convex metric space* (X,ϱ) *is frontier, i. e.* $\mathcal{T}(X,\varrho) \subset \mathcal{F}(X,\varrho)$.

However, it is evident that the theorem inverse to 4.2 is not true.

A point q of a strongly convex metric space (X,ϱ) is said to be a *ramification point* if there exist three points $p, r_1, r_2 \in X$ such that 1° $p \neq q \neq r_i$ for $i = 1, 2$, and 2° $\overline{pr_1} \cap \overline{pr_2} = \overline{pq}$. The set of all ramification points of (X,ϱ) is denoted by $\mathcal{R}(X,\varrho)$.

4.3. $\mathcal{R}(X,\varrho) \subset \mathcal{P}(X,\varrho)$.

Proof. For $q \in \mathcal{R}(X,\varrho)$ condition 2° implies $q \in \overline{pr_1}$. Thus pqr_1 from 3.2. Hence $q \in \mathcal{P}(X,\varrho)$ because $p \neq q \neq r_1$ from 1°.

(¹) For the definitions of topological limits: $\operatorname*{Li}\limits_{i\to\infty} A_i$, $\operatorname*{Ls}\limits_{i\to\infty} A_i$ and $\operatorname*{Lim}\limits_{i\to\infty} A$, see [4], pp. 241-245.

§ 5. Cones and ϱ-cones. Let $\mathcal{E}^1_+$ denote the set of real non-negative numbers, i. e. $\mathcal{E}^1_+ = \{t\colon t \in \mathcal{E}^1, t \geqslant 0\}$. The space obtained from the cartesian product $X \times \mathcal{E}^1_+$ by identifying the set $X \times (0)$ to one point will be called a *cone over* X and denoted by Cone(X). The point corresponding to set $X \times (0)$ in the identification space will be called a cone *vertex*.

If (X, ϱ) is a strongly convex metric space, $A \subset X$ and $v \in X$, then the set

(1) $$C_\varrho(A, v) = \bigcup_{a \in A} \overline{va}$$

will be called a *ϱ-cone over* A with *vertex* v. We obviously have

5.1. $A \subset C_\varrho(A, v)$.

The set $B_\varrho(A, v)$ of points $x \in C_\varrho(A, v)$ such that if vxx' and $x' \in A$, then $x = x'$, will be called a *base* of ϱ-cone $C_\varrho(A, v)$. Evidently

5.2. *If* $A \neq (v)$, *then* $v \notin B_\varrho(A, v)$.

5.3. $B_\varrho(A, v) \subset A$.

Now we establish (to the end of this paragraph) that (X, ϱ) is a strongly convex compact metric space, A is a closed subset of X and $A \neq (v)$. It is easy to verify that

5.4. $C_\varrho(A, v) = C_\varrho[B_\varrho(A, v), v]$.

Formula 5.4 means that every ϱ-cone is a ϱ-cone over the base of itself. It is also easy to prove that the set $B_\varrho(A, v)$ is irreducible with respect to equality 5.4, i. e. if $Z \subset B_\varrho(A, v)$ and $Z \neq B_\varrho(A, v)$, then $C_\varrho(A, v) \neq C_\varrho(Z, v)$. For such sets Z we have only the inclusion $C_\varrho(Z, v) \subset C_\varrho(A, v)$ by virtue of (1) and 5.3.

5.5. $X = C_\varrho(X, v)$.

Proof depends on 5.1.

5.6. *If* (X, ϱ) *is a metric space,* ϱ *is* SC-WR-*metric,* $a, b, v \in X$, $a \neq b$ *and the set* $\{a, b, v\}$ *is not linear, then there exist points* $a_1, b_1 \in \overline{ab}$ *such that* $a_1 \neq b_1$, vaa_1, vbb_1, aa_1b_1 *and*

$$B_\varrho(\overline{ab}, v) = \overline{a_1b_1}\,.$$

Proof. Let a_1 be the farthest point from a, belonging to segment $\overline{ab}$ and such that vaa_1. It exists because $\overline{ab}$ is compact. Similarly there exists a point $b_1 \in \overline{ab}$, farthest from b and such that vbb_1. It is not difficult to show, applying 3.3, that the points a_1 and b_1 have all the desired properties.

The set $R_\varrho(A, v)$ of points $x \in X$ such that there exists $y \in B_\varrho(A, v)$ satisfying vyx will be called a *rest* of ϱ-cone $C_\varrho(A, v)$. It is not difficult to see that $v \notin R_\varrho(A, v)$ and

5.7. $B_\varrho(A, v) = C_\varrho(A, v) \cap R_\varrho(A, v)$.

Now let S be a sphere with centre $p \in X$ and radius $\varepsilon > 0$, i. e. $S = \{x\colon x \in X, \varrho(p, x) = \varepsilon\}$.

5.8. $B_\varrho(S, p) = S$.

Proof. From 5.3 we have $B_\varrho(S, p) \subset S$. Let $x \in S$, pxx' and $x' \in S$. It follows that $\varrho(p, x) = \varrho(p, x') = \varepsilon$ and $\varrho(p, x') = \varrho(p, x) + \varrho(x, x')$, whence $\varrho(x, x') = 0$, that is $x = x'$.

5.9. $\{x\colon x \in X, \varepsilon \leqslant \varrho(p, x)\} \subset R_\varrho(S, p)$.

Proof. Let $\varepsilon \leqslant \varrho(p, x)$. By virtue of the convexity of X a point $y \in X$ exists such that pyx and $\varrho(p, y) = \varepsilon$. Thus $y \in S$. We infer from 5.8 that $y \in B_\varrho(S, p)$. Hence $x \in R_\varrho(S, p)$.

Let Q be a massive sphere with centre p and radius ε, i. e. $Q = \{x\colon x \in X, \varrho(p, x) \leqslant \varepsilon\}$.

5.10. *If* $Q \cap \mathcal{F}(X, \varrho) = 0$, *then* $Q = C_\varrho(S, p)$.

Proof. It follows from (1) that $C_\varrho(S, p) \subset Q$. Let $x \in Q$, $x \neq p$ and let $\overline{aq}$ be a maximal segment containing points p, x such that pxq. Since $q \in \mathcal{F}(X, \varrho)$, we have $q \notin Q$ by hypothesis. That is $\varepsilon < \varrho(p, q)$. Therefore there exists a point $r \in \overline{pq}$ such that $\varrho(p, r) = \varepsilon$, i. e. $r \in S$. Since $\varrho(p, x) \leqslant \varepsilon$, we have $x \in \overline{pr}$ and $\overline{pr} \subset C_\varrho(S, p)$ by virtue of (1). This gives $x \in C_\varrho(S, p)$. Hence also the inclusion $Q \subset C_\varrho(S, p)$ holds.

§ 6. Projections and natural homeomorphisms. Suppose that (X, ϱ) is a strongly convex metric space without ramifications. If $x \in C_\varrho(A, v)$ and $x \neq v$, then according to 5.4 there exists a point $y \in B_\varrho(A, v)$ such that vxy. If there existed another such point $y' \in B_\varrho(A, v)$ that vxy' and $y \neq y'$, the set $\{v, x, y, y'\}$ would be linear by virtue of (β) and we should have vyy' or $vy'y$, contrary to the definition of base $B_\varrho(A, v)$, according to 5.3. Therefore the point y is determined by x and putting $p_c(x) = y$ we obtain a mapping $p_c\colon C_\varrho(A, v) - (v) \to B_\varrho(A, v)$.

Now if $x \in R_\varrho(A, v)$, we infer from the definition of a rest (see § 5) that there exists a point $y \in B_\varrho(A, v)$ such that vyx. If there existed another such point $y' \in B_\varrho(A, v)$ that $vy'x$ and $y \neq y'$, the set $\{v, y, y', x\}$ would be linear by virtue of (α) and we should have vyy' or $vy'y$, which is impossible for the same reason as previously. Therefore the point y is determined by x and putting $p_r(x) = y$ we obtain a mapping $p_r\colon R_\varrho(A, v) \to B_\varrho(A, v)$.

It is easy to see from 5.7 and from the definition of the base of ϱ-cone that the mappings p_c and p_r are identities on $B_\varrho(A, v)$. They will be called *projections* on the base of ϱ-cone.

6.1. *If* (X, ϱ) *is a compact space,* ϱ *is* SC-WR-*metric,* $A \subset X$ *is closed,* $A \neq (v)$ *and* $B_\varrho(A, v)$ *is closed, then the projections* p_c *and* p_r *are continuous mappings* (and therefore they are *retractions*).

Proof. Let $\lim\limits_{i \to \infty} x_i = x_0$ and $x_i \in C_\varrho(A, v) - (v)$, where $i = 0, 1, \ldots$ Since the set $B_\varrho(A, v)$ is compact and $p_c(x_i) \in B_\varrho(A, v)$ for $i = 0, 1, \ldots,$

there exists in each infinite sequence $\{i_j\}$ of natural numbers a subsequence $\{i_{j_k}\}$ such that $y = \lim_{k\to\infty} p_c(x_{i_{j_k}})$ for any $y \in B_\varrho(A, v)$. We infer from the definition of projection p_c that $vx_{i_{j_k}}p_c(x_{i_{j_k}})$, whence vx_0y by virtue of 3.2 and 3.4. But we also have $vx_0p_c(x_0)$ and $p_c(x_0) \in B_\varrho(A, v)$. Since $v \neq x_0$, (β) implies that the set $\{v, x_0, y, p_c(x_0)\}$ is linear. Thus $vyp_c(x_0)$ or $vp_c(x_0)y$. It follows from the definition of base of ϱ-cone (see § 5) and from 5.3 that $y = p_c(x_0)$. Therefore $\lim_{i\to\infty} p_c(x_i) = p_c(x_0)$ and p_c is continuous. The proof that p_r is continuous is quite similar.

Putting

$$h_c(x) = \begin{cases} \big(p_c(x),\ \varrho(v, x)\big) & \text{for} \quad x \in C_\varrho(A, v) - (v)\,, \\ B_\varrho(A, v)\times(0) & \text{for} \quad x = v\,, \end{cases} \tag{2}$$

and

$$h_r(x) = \big(p_r(x),\ \varrho(v, x)\big) \quad \text{for} \quad x \in R_\varrho(A, v)\,, \tag{3}$$

we obtain the mappings h_c: $C_\varrho(A, v) \to \mathrm{Cone}[B_\varrho(A, v)]$ and h_r: $R_\varrho(A, v) \to B_\varrho(A, v)\times \mathcal{E}^1_+$.

6.2. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric, $A \subset X$ is closed, $A \neq (v)$ and $B_\varrho(A, v)$ is closed, then the mappings h_c and h_r are homeomorphisms* (called *natural homeomorphisms*).

Proof. We shall show that h_c is a homeomorphism. The proof for h_r is quite similar.

h_c is 1-1 mapping. Indeed, let $x_1, x_2 \in C_\varrho(A, v)$ and $h_c(x_1) = h_c(x_2)$. From (2) we have either $h_c(x_1) = B_\varrho(A, v)\times(0)$, and then $x_1 = x_2 = v$, or $p_c(x_1) = p_c(x_2) = y$ and $\varrho(v, x_1) = \varrho(v, x_2)$. Consider the last case. We have vx_1y, vx_2y and $v \neq y$ by virtue of 5.2. It follows from (α) that the set $\{v, x_1, x_2, y\}$ is linear. This implies vx_1x_2 or vx_2x_1, whence $x_1 = x_2$, because $\varrho(v, x_1) = \varrho(v, x_2)$.

h_c is continuous. Indeed, suppose $\lim_{i\to\infty} x_i = x_0$ and $x_i \in C_\varrho(A, v)$ for $i = 0, 1, \ldots$ If $x_0 = v$, then $\lim_{i\to\infty}\varrho(v, x_i) = 0$, whence $\lim_{i\to\infty} h_c(x_i) = B_\varrho(A, v)\times(0)$ by virtue of (2) and the compactness of $B_\varrho(A, v)$. Therefore $\lim_{i\to\infty} h_c(x_i) = h_c(x_0)$. Now if $x_0 \neq v$, we may assume that $x_i \neq v$ for $i = 0, 1, \ldots$ It follows from 6.1 that $\lim_{i\to\infty} p_c(x_i) = p_c(x_0)$. Evidently $\lim_{i\to\infty} x_i = x_0$ implies $\lim_{i\to\infty}\varrho(v, x_i) = \varrho(v, x_0)$. Therefore $\lim_{i\to\infty} h_c(x_i) = h_c(x_0)$ by virtue of (2).

6.3. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric, $A \subset X$ is closed, $A \neq (v)$ and $B_\varrho(A, v)$ is closed, then the projections p_c and p_r are open mappings* (²).

(²) We say that the mapping is *open* if it maps open sets onto open sets (compare [5], p. 48).

Proof. Let p: $B_\varrho(A, v)\times\mathcal{E}^1 \to B_\varrho(A, v)$ be the projection $p(x, t) = x$ for $x \in B_\varrho(A, v)$ and $t \in \mathcal{E}^1$, and let $h_c' = h_c|[C_\varrho(A, v) - (v)]$. It is evident from (2) and (3) that $p_c = ph_c'$ and $p_r = ph_r$. However, h_c' and h_r are homeomorphisms by virtue of 6.2 and the projection p is obviously an open mapping. Hence p_c and p_r are open mappings.

§ 7. Properties of disk. By an *n-cell* we understand a homeomorphical image of the compact Euclidean n-dimensional massive sphere, $n = 1, 2, \ldots$ By a *disk* D we understand a 2-cell, i. e. a homeomorphical image of the set $\{(x, y)\colon x^2+y^2 \leqslant 1\}$, where x and y are real numbers. If h is that homeomorphism, the set $h\big(\{(x, y)\colon x^2+y^2 < 1\}\big)$ will be called the *interior* of disk D and denoted by $\mathrm{int}(D)$. The set $\mathrm{bd}(D) = D - \mathrm{int}(D)$ will be called a *boundary* of disk D.

7.1. *If X is a metric space of dimension $\leqslant 1$ and $D \subset X\times\mathcal{E}^1$ is a disk, then $\big[\big((x)\times\mathcal{E}^1\big)\cap D\big]^{[1]} \subset \mathrm{bd}(D)$ for every $x \in X$* (³).

Proof. Let $q_0 = (x_0, t_0) \in \mathrm{int}(D)$, $x_0 \in X$, $t_0 \in \mathcal{E}^1$ and let p: $X\times\mathcal{E}^1 \to X$ be a projection onto X, i. e. $p(x, t) = x$ for $x \in X$ and $t \in \mathcal{E}^1$. Put $p_d = p|D$. Hence $x_0 = p_d(q_0)$.

In an arbitrary neighbourhood of the point q_0 there exists a point q belonging to D and such that $p_d(q) \neq p_d(q_0)$. If this is not so, then some 2-dimensional neighbourhood U of q_0 in D is transformed by p_d into a single point x_0, thus we have $2 = \dim U = \dim p_d^{-1}(x_0) \leqslant \dim p^{-1}(x_0) = \dim[(x_0)\times\mathcal{E}^1]$, which is not possible. So let $q_1, q_2, \ldots$ be a sequence of points of D such that $\lim_{i\to\infty} q_i = q_0$ and $p_d(q_i) \neq p_d(q_0)$ for $i = 1, 2, \ldots$ Put $x_i = p_d(q_i)$ for $i = 1, 2, \ldots$ Therefore $x_i \neq x_0$ for $i = 1, 2, \ldots$, and since $\dim X \leqslant 1$, there exists for every $i = 1, 2, \ldots$ a neighbourhood V_i of x_0 in X such that

$$x_i \in X - \overline{V}_i\,, \quad \dim \mathrm{Fr}(V_i) \leqslant 0 \quad \text{for} \quad i = 1, 2, \ldots \tag{4}$$

and

$$(x_0) = \operatorname*{Lim}_{i\to\infty} \overline{V}_i\,. \tag{5}$$

It follows (see [4], p. 130) that $\mathrm{Fr}(V_i)$ separates X between the points x_0 and x_i, and therefore $p_d^{-1}[\mathrm{Fr}(V_i)]$ separates D between the points q_0 and q_i (connectedness being an invariant of continuous transformations). Thus there exists an irreducible separator $C_i \subset p_d^{-1}[\mathrm{Fr}(V_i)]$ of disk D between q_0 and q_i (see [5], p. 176). Whence C_i is a continuum (see [5], p. 333 and 335). But $\dim p_d(C_i) \leqslant \dim p_d p_d^{-1}[\mathrm{Fr}(V_i)] = \dim \mathrm{Fr}(V_i) \leqslant 0$

(³) For the definition of the *order* $\mathrm{ord}_a A$ of the set A at the point a see [5], pp. 200-201. We have $A^{[n]} = \{a\colon \mathrm{ord}_a A \leqslant n\}$. We shall often apply that if $A \subset B$, $\mathrm{ord}_p A \leqslant \mathrm{ord}_p B$.

by virtue of (4). However, $p_d(C_i)$ is a continuum, and thus it is a one-point set. We write $(\bar{x}_i) = p_d(C_i)$, where $i = 1, 2, \ldots$

Therefore we have $\bar{x}_i \in p_p p_d^{-1}[\mathrm{Fr}(V_i)] = \mathrm{Fr}(V_i) \subset \bar{V}_i$ for $i = 1, 2, \ldots$, whence, $\lim_{i\to\infty} \bar{x}_i = x_0$ by (5). But we have also $C_i \subset p_d^{-1}(\bar{x}_i) \subset p^{-1}(\bar{x}_i) = (\bar{x}_i) \times \mathcal{E}^1$, which implies that C_i is an arc of the form [4]:

$$C_i = (\bar{x}_i) \times [a_i, b_i],$$

where $a_i < b_i$ and $a_i, b_i \in \mathcal{E}^1$ for $i = 1, 2, \ldots$ The points $(\bar{x}_i, a_i)$ and $(\bar{x}_i, b_i)$ are end-points of arc C_i. Since C_i is an irreducible separator of disk D, $(\bar{x}_i, a_i) \in \mathrm{bd}(D)$ and $(\bar{x}_i, b_i) \in \mathrm{bd}(D)$ for $i = 1, 2, \ldots$

The condition $q_0 \in \mathrm{int}(D)$ implies that $0 < \varrho\big(q_0, \mathrm{bd}(D)\big)$ [5]. Thus a number $\varepsilon > 0$ exists such that

$$\varrho[(x_0, t_0), (\bar{x}_i, a_i)] > \varepsilon < \varrho[(x_0, t_0), (\bar{x}_i, b_i)]$$

for $i = 1, 2, \ldots$ Also, since $\lim_{i\to\infty} \bar{x}_i = x_0$, a number $\eta > 0$ and an index k exist such that $|a_i - t_0| > \eta < |b_i - t_0|$ for $i > k$.

But $\lim_{i\to\infty} q_i = q_0$ and C_i separates D between q_i and q_0. Whence, by virtue of the local connectedness of D, we obtain $\lim_{i\to\infty} \varrho(q_0, C_i) = 0$. Thus numbers c_i exist such that $a_i \leqslant c_i \leqslant b_i$ and $0 = \lim_{i\to\infty} \varrho[(x_0, t_0), (\bar{x}_i, c_i)] = \lim_{i\to\infty} |c_i - t_0|$. It follows that an index $l \geqslant k$ exists such that

$$a_i \leqslant t_0 - \eta < t_0 < t_0 + \eta \leqslant b_i \quad \text{for} \quad i > l.$$

Thus we have $[t_0 - \eta, t_0 + \eta] \subset \operatorname{Li}_{i\to\infty} [a_i, b_i]$ [1], and therefore (see [4], p. 242):

$$\begin{aligned}(x_0) \times [t_0 - \eta, t_0 + \eta] &\subset (\lim_{i\to\infty} \bar{x}_i) \times \operatorname{Li}_{i\to\infty} [a_i, b_i] \\ &= \big(\operatorname{Li}_{i\to\infty} (\bar{x}_i)\big) \times \big(\operatorname{Li}_{i\to\infty} [a_i, b_i]\big) = \operatorname{Li}_{i\to\infty} \big((\bar{x}_i) \times [a_i, b_i]\big) \\ &= \operatorname{Li}_{i\to\infty} C_i \subset D.\end{aligned}$$

Hence $q_0 = (x_0, t_0) \in (x_0) \times [t_0 - \eta, t_0 + \eta] \subset \big((x_0) \times \mathcal{E}^1\big) \cap D$, that is $1 < \mathrm{ord}_{q_0}\big((x_0) \times \mathcal{E}^1\big) \cap D$. Thus 7.1 is proved.

7.2. *If (X, ϱ) is a compact space, ϱ is SC-WR-metric, $\dim X \leqslant 2$, $D \subset X$ is a disk, $a \in X$, $a \neq d \in D$ and*

(6) $$\mathrm{ord}_d\, \overline{ad'} \cap D \leqslant 1 \quad \textit{for every} \quad d' \in D,$$

then $d \in \mathrm{bd}(D)$.

[4] $[a, b] = \{t\colon t \in \mathcal{E}^1,\ a \leqslant t \leqslant b\}$.

[5] $\varrho(A, B) = \inf_{a \in A,\, b \in B} \varrho(a, b)$, $\varrho(p, A) = \varrho(\{p\}, A)$.

Proof. Suppose on the contrary that $d \in \mathrm{int}(D)$. Let $D' \subset D$ be a disk such that

(7) $$d \in \mathrm{int}(D')$$

and $\delta(D') < \varrho(a, d)/2$ [6]. Hence, we have for every $d' \in D'$

$$\varrho(a, d') \geqslant \varrho(a, d) - \varrho(d, d') \geqslant \varrho(a, d) - \delta(D') > \varrho(a, d)/2,$$

which gives $\varrho(a, d)/2 \leqslant \varrho(a, D')$. Put $\varepsilon = \varrho(a, d)/3$ and let S be a sphere with centre a and radius ε.

Then $B_\varrho(S, a) = S$ by 5.8, and $D' \subset \{x\colon x \in X,\ \varepsilon \leqslant \varrho(a, x)\} \subset R_\varrho(S, a)$ by 5.9. We have also $0 < \varrho(S, D')$. Put $\eta = \varrho(S, D')$. Therefore if $h_r\colon R_\varrho(S, a) \to S \times \mathcal{E}^1_+ \subset S \times \mathcal{E}^1$ is a natural homeomorphism, then for every $x \in D'$ the segment $\overline{xp_r(x)}$ is transformed by h_r, as follows from (3), onto the segment

$$p_r(x) \times \big[\varepsilon, \varepsilon + \varrho\big(p_r(x), x\big)\big]$$

contained in $S \times \mathcal{E}^1$ and containing the segment $p_r(x) \times [\varepsilon, \varepsilon + \eta]$. Hence

$$p_r(D') \times [\varepsilon, \varepsilon + \eta] \subset h_r[R_\varrho(S, a)].$$

We have $\dim p_r(D') \leqslant 1$, because if it is not true,

$$\begin{aligned}\dim X \geqslant \dim R_\varrho(S, a) &= \dim h_r[R_\varrho(S, a)] \\ &\geqslant \dim \{p_r(D') \times [\varepsilon, \varepsilon + \eta]\} = \dim p_r(D') + 1 > 2,\end{aligned}$$

contrary to the hypothesis. Thus we can apply 7.1 for $X = p_r(D')$ and $D = h_r(D')$. We obtain in particular the following inclusion:

(8) $$\big[\big(p_r(d) \times \mathcal{E}^1\big) \cap h_r(D')\big]^{[1]} \subset \mathrm{bd}[h_r(D')].$$

We shall show that

(9) $$h_r(d) \in \big[\big(p_r(d) \times \mathcal{E}^1\big) \cap h_r(D')\big]^{[1]}.$$

Suppose the contrary. Therefore there exists a segment $p_r(d) \times [t, t'] \subset h_r(D')$ which contains $h_r(d)$ as an interior point of itself. Putting $t_0 = \varrho(a, d)$ we have $t < t_0 < t'$. It follows that for every t'' such that $t \leqslant t'' \leqslant t'$ a point $q(t'') \in D'$ exists such that $h_r[q(t'')] = \big(p_r(d), t''\big)$. Hence by (3) we have

(10) $$p_r[q(t'')] = p_r(d) \quad \text{and} \quad \varrho\big(a, q(t'')\big) = t'' \quad \text{for} \quad t \leqslant t'' \leqslant t'.$$

We infer from the definition of projection p_r (see § 6) that $p_r\big(q(t'')\big) \in \overline{aq(t'')}$ for $t \leqslant t'' \leqslant t'$. Whence, by (10), it follows that

[6] $\delta(A) = \sup_{p, q \in A} \varrho(p, q)$ is the *diameter* of the set A.

$0 \neq p_r(d) \in \overline{aq(t'')} \cap \overline{aq(t)} - (a)$. Thus $\overline{aq(t)} \subset \overline{aq(t'')}$ by virtue of 3.3 and (10), for $t \leqslant t'' \leqslant t'$. It follows that

(11) $$q(t_0) \in \overline{q(t)q(t')} \subset \overline{aq(t')} \cap D' \quad \text{and} \quad q(t) \neq q(t_0) \neq q(t') .$$

Since $p_r(d) \in \overline{aq(t_0)} \cap \overline{ad} - (a)$ by virtue of (10), we infer from 3.3 that $\overline{aq(t_0)} \subset \overline{ad}$ or $\overline{ad} \subset \overline{aq(t_0)}$. But $\varrho(a, q(t_0)) = t_0 = \varrho(a, d)$ by virtue of (10). Therefore $d = q(t_0)$. It follows from (11) that $\mathrm{ord}_d \overline{aq(t')} \cap D \geqslant \mathrm{ord}_d \overline{aq(t')} \cap D' = \mathrm{ord}_{q(t_0)} \overline{aq(t')} \cap D' > 1$, contrary to (6), because $q(t') \in D' \subset D$. Thus (9) is proved.

The formulae (8) and (9) give $h_r(d) \in \mathrm{bd}[h_r(D')]$, contrary to (7) and to the fact that h_r is a homeomorphism.

7.3. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric*, $\dim X \leqslant 2$, *$D \subset X$ is a disk, $a \in X$, $a \neq d \in D$ and $\overline{ad} \cap D = (d)$, then $d \in \mathrm{bd}(D)$.*

Proof. By 7.2 it is enough to prove (6). Let $d' \in D$. If $d \notin \overline{ad'}$, (6) is obvious. If $d \in \overline{ad'}$, then $\overline{ad'} = \overline{ad} \cup \overline{dd'}$, whence $\overline{ad'} \cap D = (\overline{ad} \cap D) \cup (\overline{dd'} \cap D) = (d) \cup (\overline{dd'} \cap D)$. Therefore $\mathrm{ord}_d \overline{ad'} \cap D \leqslant \mathrm{ord}_d \overline{dd'} = 1$.

7.4. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric*, $\dim X \leqslant 2$ *and $D \subset X$ is a disk, then*

$$D \cap \mathcal{F}(X, \varrho) \subset \mathrm{bd}(D) .$$

Proof. Let $d \in D \cap \mathcal{F}(X, \varrho)$. Therefore a point $a \in X$ exists such that $\overline{ad}$ is a maximal segment. Thus if $d' \in D$ and $d \in \overline{ad'}$, then $d = d'$. This gives $\mathrm{ord}_d \overline{ad'} \cap D = \mathrm{ord}_d \overline{ad} \cap D \leqslant \mathrm{ord}_d \overline{ad} = 1$, that is (6) holds and it follows from 7.2 that $d \in \mathrm{bd}(D)$.

7.5. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric*, $\dim X \leqslant 2$, *$D \subset X$ is a disk and $\overline{pq} \subset D$, then a point r exists such that*

$$r \in \mathrm{bd}(D) \quad \text{and} \quad q \in \overline{pr} \subset D .$$

Proof. By the compactness of (X, ϱ) there exists a maximal segment $\overline{ab}$ containing the points p and q such that pqb. Let r be a point of segment $\overline{pb}$ nearest to b and such that $\overline{pr} \subset D$. Evidently $\overline{pq} \subset \overline{pr}$, i. e. $q \in \overline{pr}$. It remains to prove that $r \in \mathrm{bd}(D)$. Suppose on the contrary

(12) $$r \in \mathrm{int}(D) .$$

Since $b \in \mathcal{F}(X, \varrho)$, 7.4 and (12) give $r \neq b$. We shall show that (6) holds for $a = b$ and $d = r$. Indeed, if $d' \in D$ and $r \notin \overline{bd'}$, we have $\mathrm{ord}_r \overline{bd'} = 0$. If $d' \in D$ and $r \in \overline{bd'}$, we have $r \in \overline{bp} \cap \overline{bd'} - (b)$, therefore from 3.3 we obtain $\overline{bp} \subset \overline{bd'}$ or $\overline{bd'} \subset \overline{bp}$. It follows from the definition of point r that $\mathrm{ord}_r \overline{bd'} \cap D \leqslant \mathrm{ord}_r \overline{pb} \cap D \leqslant 1$. Hence by 7.2 we have $r \in \mathrm{bd}(D)$, contrary to (12).

7.6. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric*, $\dim X \leqslant 2$, *$D \subset X$ is a disk and $p \in \mathrm{int}(D)$, then for every ε such that*

$$0 < \varepsilon < \varrho[p, \mathrm{bd}(D)]$$

the sphere $S = \{x\colon x \in X, \varrho(p, x) = \varepsilon\}$ is a simple closed curve contained in $\mathrm{int}(D)$, and also the massive sphere $Q = \{x\colon x \in X, \varrho(p, x) \leqslant \varepsilon\}$ is contained in $\mathrm{int}(D)$.

Proof. It is evident that $Q \cap \mathrm{bd}(D) = 0$. If there existed a point $x \in Q$ such that $x \notin D$, then, taking in the segment $\overline{px}$ the point d nearest to x and such that $d \in D$, we should have $d \in \mathrm{int}(D)$, $x \neq d$ and $\overline{xd} \cap D = (d)$, contrary to 7.3. Hence $Q \subset D$, and thus

(13) $$S \subset Q \subset \mathrm{int}(D) .$$

It remains to show that S is a simple closed curve. It is obvious that S separates the disk D between the points p and an arbitrary point $z \in \mathrm{bd}(D)$. But S is also an *irreducible* separator of D between these points. Indeed, if $q \in S$, then $\overline{pq} \subset D$ according to (13); hence, taking the segment $\overline{pr}$ from 7.5 such that $r \in \mathrm{bd}(D)$ and $q \in \overline{pr} \subset D$, we obtain the continuum $C = \overline{pr} \cup \mathrm{bd}(D)$ such that $C \cap S = (q)$, by virtue of (13) and 5.8, and $p, z \in C$. This means that no proper subset of S is a separator of D between the points p and z.

Now we shall prove that S is locally connected. Let $q \in S$ and (see § 6)

(14) $$p_c\colon C_\varrho(S, p) - (p) \to S$$

be a projection of ϱ-cone onto the base $B_\varrho(S, p) = S$, according to 5.8. Let

(15) $$q' \in C_\varrho(S, p) - S - (p) \quad \text{and} \quad p_c(q') = q .$$

It follows from (13) and 7.4 that $Q \cap \mathcal{F}(X, \varrho) \subset \mathrm{int}(D) \cap \mathcal{F}(X, \varrho) = 0$. Hence we have by 5.10

(16) $$Q = C_\varrho(S, p) .$$

Therefore (15) gives $q' \in Q - S - (p)$. It is a consequence of the convexity of (X, ϱ) that X is a connected and locally connected space (every massive sphere being connected); thus we can find a closed neighbourhood U of q' such that $U \subset Q - S - (p)$ and U is a locally connected continuum. But $B_\varrho(S, p)$ is closed, according to 5.8. It follows from 6.3 that p_c is an open mapping. Therefore we have $p_c(U) \subset S$, according to (14) and (16), and the locally connected continuum $p_c(U)$ is a closed neighbourhood of $p_c(q')$ in S; thus it is one of q by virtue of (15). This means that S is locally connected.

As a locally connected irreducible separator of disk D containing it in the interior by virtue of (13), S is a simple closed curve (see [5], p. 403).

§ 8. The base $B_\varrho(X, v)$. It is not difficult to verify that

8.1. *If (X, ϱ) is a compact space, then $B_\varrho(X, v) \subset \mathcal{F}(X, \varrho)$ for every $v \in X$.*

The aim of this paragraph is the following:

8.2. *If (X, ϱ) is a compact space, ϱ is* SC-WR-*metric,* $\dim X \leqslant 2$ *and $v \in X$, then the set $B_\varrho(X, v) \cup (v)$ is closed.*

Proof. Supposing the contrary, we have a sequence of points $p_0, p_1, \ldots$ such that

(17) $$p_i \in B_\varrho(X, v) \quad \text{for} \quad i = 1, 2, \ldots,$$

(18) $$\lim_{i\to\infty} p_i = p_0$$

and $p_0 \notin B_\varrho(X, v) \cup (v)$; hence a point s exists such that

(19) $$s \in B_\varrho(X, v)$$

and

(20) $$vp_0s \quad \text{and} \quad v \neq p_0 \neq s\,.$$

Therefore $v \notin \overline{sp_0} = \operatorname*{Lim}_{i\to\infty} \overline{sp_i}$, by 3.4, (18) and (20). It follows that for sufficiently great natural numbers i we have $v \notin \overline{sp_i}$. We may assume that it is true for all i. Similarly $s \notin \overline{vp_i}$ and $p_i \notin \overline{vs}$ by virtue of (17) and (19), because according to (18) and (20) the condition that $p_i \neq s$ for $i = 1, 2, \ldots$ may also be added. Therefore we have, by virtue of 2.1,

(21) $$\{s, p_i, v\} \text{ is not linear for } i = 1, 2, \ldots$$

Hence, putting $a = s$ and $b = p_i$ in 5.6, we infer that there exist points $a_1, b_1 \in \overline{sp_i}$ such that $B_\varrho(\overline{sp_i}, v) = \overline{a_1b_1}$, vsa_1 and vp_ib_1. It follows from the last two betweennesses and from (17) and (19) that $a_1 = s$ and $b_1 = p_i$. Therefore

(22) $$B_\varrho(\overline{sp_i}, v) = \overline{sp_i} \quad \text{for} \quad i = 1, 2, \ldots$$

Hence, applying 6.2, we infer that ϱ-cones $C_\varrho(\overline{sp_i}, v)$ are disks for $i = 1, 2, \ldots$ We write

(23) $$D_i = C_\varrho(\overline{sp_i}, v) \quad \text{for} \quad i = 1, 2, \ldots$$

If we had $\overline{sp_0} \cap \overline{vp_1} \neq 0$, then choosing $y \in \overline{sp_0} \cap \overline{vp_1}$ we should get $y \neq v$ by virtue of (20), and thus $y \in \overline{vs} \cap \overline{vp_1} - (v)$. Hence, applying 3.3, we should obtain vsp_1 or vp_1s, contrary to (21). Therefore $\overline{sp_0} \cap \overline{vp_1} = 0$.

Similarly $\overline{vp_0} \cap \overline{sp_1} = 0$. But since $\overline{sp_0} = \operatorname*{Lim}_{i\to\infty} \overline{sp_i}$ and $\overline{vp_0} = \operatorname*{Lim}_{i\to\infty} \overline{vp_i}$ by virtue of 3.4 and (18), a natural number $j > 1$ exists such that

(24) $$\overline{sp_j} \cap \overline{vp_1} = 0 = \overline{vp_j} \cap \overline{sp_1}\,.$$

We shall prove that

(25) $$\overline{sp_j} \cap \operatorname{bd}(D_1) = \overline{sp_1} \cap \operatorname{bd}(D_j) = (s)\,.$$

Indeed, we infer from (23) that $\operatorname{bd}(D_1) = \overline{vp_1} \cup \overline{sp_1} \cup \overline{vs}$. Thus $\overline{sp_j} \cap \operatorname{bd}(D_1) = (\overline{sp_j} \cap \overline{sp_1}) \cup (\overline{sp_j} \cap \overline{vs})$, by (24). Hence if we had $\overline{sp_j} \cap \operatorname{bd}(D_1) - (s) \neq 0$, then we should get 1° $\overline{sp_j} \cap \overline{sp_1} - (s) \neq 0$ or 2° $\overline{sp_j} \cap \overline{vs} - (s) \neq 0$. For 1°, 3.3 would give $p_j \in \overline{sp_1}$ or $p_1 \in \overline{sp_j}$, which contradicts (24). For 2°, 3.3 would give $p_j \in \overline{vs}$ or $v \in \overline{sp_j}$, which contradicts (21). Therefore $\overline{sp_j} \cap \operatorname{bd}(D_1) = (s)$. The proof that $\overline{sp_1} \cap \operatorname{bd}(D_j) = (s)$ is quite similar.

Now we shall prove that

(26) $$D_1 \cap D_j = \overline{vs}\,.$$

Indeed, (23) implies that $\overline{vs} \subset D_1 \cap D_j$. To show the inverse inclusion, let us suppose on the contrary that $D_1 \cap D_j - \overline{vs} \neq 0$. Then by (22) and (23) points c and d exist such that

(27) $$c \in \overline{sp_1}\,, \quad d \in \overline{sp_j}\,, \quad c \neq s \neq d\,, \quad \overline{vc} \cap \overline{vd} - (v) \neq 0\,.$$

Hence and from 3.3 we have (a) vcd or (b) vdc. It follows from (25) that $\overline{sp_1} \cap \overline{sp_j} = (s)$, and thus $c \neq d$ by (27).

Case (a). We have $c \in \overline{vd}$. Therefore if $d = p_j$, (27) would imply $c \in \overline{vp_j} \cap \overline{sp_1}$, contrary to (24). Thus $d \neq p_j$. But the conditions $s \neq d \neq p_j$, $c \neq d$ and $c \in \overline{vd}$ imply by virtue of (22) and (23) that $c \in \operatorname{int}(D_j)$. The segment $\overline{sp_1}$ contains the point c lying in $\operatorname{int}(D_j)$. Therefore if the point p_1 did not lie in D_j, we should obtain from (25) a segment $\overline{p_1c'} \subset \overline{p_1c}$ such that $c' \in \operatorname{int}(D_j)$ and $\overline{p_1c'} \cap D_j = (c')$, contrary to 7.3. Thus p_1 must belong to D_j. Hence $p_1 \in \operatorname{int}(D_j)$ by (25). But $p_1 \in D_j \cap B_\varrho(X, v) \subset D_j \cap \mathcal{F}(X, \varrho)$ by (17) and 8.1. This contradicts 7.4.

Case (b). We have $d \in \overline{vc}$. Therefore if $c = p_1$, (27) would imply $d \in \overline{vp_1} \cap \overline{sp_j}$, contrary to (24). Thus $c \neq p_1$. But the conditions $s \neq c \neq p_1$, $d \neq c$ and $d \in \overline{vc}$ imply by virtue of (22) and (23) that $d \in \operatorname{int}(D_1)$. It follows, as in the case (a), that $p_j \in \operatorname{int}(D_1)$ and $p_j \in D_1 \cap \mathcal{F}(X, \varrho)$, which contradicts 7.4.

Thus (26) is proved. Consequently $D = D_1 \cup D_j$ is a disk and $p_0 \in \operatorname{int}(D)$ by virtue of (20) and (26). However, $\operatorname{Lim} \overline{p_0p_i} = (p_0)$ according to (18) and 3.4. Hence there exists a natural number m such that $\overline{p_0p_m} \cap D \subset \operatorname{int}(D)$. If we had $p_m \notin D$, then choosing on the segment $\overline{p_0p_m}$ a point q nearest to p_m and such that $q \in D$ we should have $q \in \operatorname{int}(D)$, $q \neq p_m$ and $\overline{p_mq} \cap D = (q)$, contrary to 7.3. Hence $p_m \in D$, i. e. $p_m \in \operatorname{int}(D)$. But $p_m \in D \cap B_\varrho(X, v) \subset D \cap \mathcal{F}(X, \varrho)$ by (17) and 8.1. This contradicts 7.4.

§ 9. Characterization of the disk. The following theorem is a partial solution, for $n=2$, of problem I (see § 1):

THEOREM. *If (X, ϱ) is a compact strongly convex metric space without ramifications and* $\dim X = 2$, *then X is a disk.*

Proof. Since $\dim X = 2$, there exist points $a, b, v \in X$ such that $\{a, b, v\}$ is not linear. It follows from 5.6 and 6.2 that ϱ-cone $D = C_\varrho(\overline{ab}, v)$ is a disk. Let $p \in \operatorname{int}(D)$ and $0 < \varepsilon < \varrho[p, \operatorname{bd}(D)]$. It follows from 7.6 that the sphere S with centre p and radius ε is a simple closed curve and the same massive sphere Q is contained in $\operatorname{int}(D)$. From 7.4 and 8.1 we obtain $Q \cap B_\varrho(X, p) \subset Q \cap \mathcal{F}(X, \varrho) = 0$, whence

$$B_\varrho(X, p) \subset \{x\colon x \in X, \varepsilon < \varrho(p, x)\} \subset R_\varrho(S, p),$$

according to 5.9. Hence p is not a limit point of the set $B_\varrho(X, p)$. It follows from 8.2 that $B_\varrho(X, p) \cup (p)$ is a closed set, and thus $B_\varrho(X, p)$ is such, i. e. it is a compact set. Let

$$p_r\colon R_\varrho(S, p) \to B_\varrho(S, p)$$

be the projection on the base. From 5.8 we have $B_\varrho(S, p) = S$. Thus the base $B_\varrho(S, p)$ is a closed set and, consequently, p_r is a continuous mapping, according to 6.1. Therefore, putting $h = p_r | B_\varrho(X, p)$, we obtain a continuous mapping

$$h\colon B_\varrho(X, p) \to S.$$

We shall prove that h is a homeomorphism onto S. By virtue of the compactness of $B_\varrho(X, p)$ it is enough to show that h is a 1-1 mapping onto S.

Let $x_1, x_2 \in B_\varrho(X, p)$. If $h(x_1) = h(x_2) = y \in S$, then from the definition of h (compare § 6, the definition of p_r) we obtain pyx_1 and pyx_2. Thus $y \in \overline{px_1} \cap \overline{px_2} - (p)$, and $\overline{px_1} \subset \overline{px_2}$ or $\overline{px_2} \subset \overline{px_1}$ by virtue of 3.3. Therefore $x_1 = x_2$, because these points belong to the base $B_\varrho(X, p)$. Hence h is a 1-1 mapping.

Let $y \in S$. Since X is compact, there exists a point $x \in X$ such that pyx and that pxx' implies $x = x'$ for every $x' \in X$. Therefore $x \in B_\varrho(X, p)$ and $h(x) = p_r(x) = y$, according to the definition of p_r (see § 6). Hence h is a mapping onto S.

Thus, S being a simple closed curve, $B_\varrho(X, p)$ is the same. It follows from 5.5 that $X = C_\varrho(X, p)$ and from 6.2—that

$$h_c\colon C_\varrho(X, p) \to \operatorname{Cone}[B_\varrho(X, p)]$$

is a homeomorphism. But, since $B_\varrho(X, p)$ is a simple closed curve, $\operatorname{Cone}[B_\varrho(X, p)]$ is topologically a plane (compare the definition of cone in § 5). It is easy to see, according to (2), that $h_c[C_\varrho(X, p)] = h_c(X)$ is a subset of $\operatorname{Cone}[B_\varrho(X, p)]$ bounded by $h_c[B_\varrho(X, p)]$, i. e. by a simple closed curve. This means that $h_c(X)$ is a disk. Thus also X is a disk.

§ 10. Finite sums of metric spaces. Let $(X_1, \varrho_1), \ldots, (X_n, \varrho_n)$ be compact spaces such that if $p, q \in X_i \cap X_j$ then $\varrho_i(p, q) = \varrho_j(p, q)$ for $i, j = 1, \ldots, n$. Let $p, q \in X_1 \cup \ldots \cup X_n$ be arbitrary points. We say that the finite sequence $(a_0, a_1, \ldots, a_m)$ of points of $X_1 \cup \ldots \cup X_n$ is a *passage from p to q* if $a_0 = p$, $a_m = q$ and both a_{i-1} and a_i belong to the same space $X_{j(i)}$ for $i = 1, \ldots, m$.

We put

$$s(a_0, a_1, \ldots, a_m) = \sum_{i=1}^{m} \varrho_{j(i)}(a_{i-1}, a_i) \tag{28}$$

and define the function ϱ_S as follows:

1° $\varrho_S(p, q) =$ the minimum of the function s in the set of all passages from p to q, provided that there exists a passage from p to q,

2° $\varrho_S(p, q) = \delta_1(X_1) + \ldots + \delta_n(X_n)$, if there exists no a passage from p to q.

This definition is correct, because in the case 1° the minimum of the function s exists, the spaces $X_1, \ldots, X_n$ being compact. In 2° the symbol δ_i denotes the diameter of the space (X_i, ϱ_i) for $i = 1, \ldots, n$.

It is not difficult to verify that

10.1. *The function ϱ_S is a metric of the sum $X_1 \cup \ldots \cup X_n$, i. e. $(X_1 \cup \ldots \cup X_n, \varrho_S)$ is a metric space.*

10.2. *Let $\varrho_1, \ldots, \varrho_n$ be* SC-*metrics and for every two points p and q of the sum $X_1 \cup \ldots \cup X_n$ let there exist a passage $(a_0, \ldots, a_m)$ from p to q such that the function s considered in the set of all passages from p to q has its minimum* only *at $(a_0, \ldots, a_m)$. Then ϱ_S is* SC-*metric.*

Let us note that the metric ϱ_S can be incompatible with some of the metrics ϱ_i, i. e. ϱ_S is *not* an *extension* of ϱ_i.

§ 11. The SC-metric ϱ_ν. Let (X, ϱ) be a metric space such that $\delta(X) \leqslant 1$. Let p_1 and p_2 be arbitrary points of the cartesian product $X \times \mathcal{I}$, where $\mathcal{I} = \{t\colon 0 \leqslant t \leqslant 1\}$, i. e. $p_1 = (x_1, t_1)$, $p_2 = (x_2, t_2)$, $x_1, x_2 \in X$ and $t_1, t_2 \in \mathcal{I}$. We put

$$\varrho_\nu(p_1, p_2) = [1 + \min(t_1, t_2)]\,\varrho(x_1, x_2) + |t_1 - t_2|. \tag{29}$$

We shall prove that

11.1. *The function ϱ_ν is a metric of $X \times \mathcal{I}$, i. e. $(X \times \mathcal{I}, \varrho_\nu)$ is a metric space.*

Proof. By (29) it is sufficient to prove the triangle inequality. So, putting $p = (x, t)$, we must show that

$$\varrho_\nu(p_1, p_2) \leqslant \varrho_\nu(p_1, p) + \varrho_\nu(p, p_2).$$

We shall consider two cases:

Case 1: $t < \min(t_1, t_2)$. Then we have $t = \min(t_1, t) = \min(t, t_2)$. The hypothesis $\delta(X) \leqslant 1$ implies $\varrho(x_1, x_2) < 2$. Applying the formula

$$\min(t_1, t_2) = (t_1 + t_2 - |t_1 - t_2|)/2 ,$$

which is always true, we obtain from (29)

$$\begin{aligned}\varrho_\nu(p_1, p_2) &= (1+t)\varrho(x_1, x_2) + [\min(t_1, t_2) - t]\varrho(x_1, x_2) + |t_1 - t_2| \\ &< (1+t)\varrho(x_1, x_2) + 2\min(t_1, t_2) - 2t + |t_1 - t_2| \\ &= (1+t)\varrho(x_1, x_2) + t_1 + t_2 - 2t \\ &\leqslant (1+t)[\varrho(x_1, x) + \varrho(x, x_2)] + (t_1 - t) + (t_2 - t) \\ &= \varrho_\nu(p_1, p) + \varrho_\nu(p, p_2) .\end{aligned}$$

Remark 1. In Case 1 the point p is not between the points p_1 and p_2 (compare § 2).

Case 2: $\min(t_1, t_2) \leqslant t$. Then we have $\min(t_1, t_2) \leqslant \min(t_1, t)$ and $\min(t_1, t_2) \leqslant \min(t, t_2)$. Applying (29) we obtain

$$\begin{aligned}&\varrho_\nu(p_1, p_2) \\ &\leqslant [1 + \min(t_1, t_2)][\varrho(x_1, x) + \varrho(x, x_2)] + |t_1 - t| + |t - t_2| \\ &= [1 + \min(t_1, t_2)]\varrho(x_1, x) + |t_1 - t| + [1 + \min(t_1, t_2)]\varrho(x, x_2) + |t - t_2| \\ &\leqslant \varrho_\nu(p_1, p) + \varrho_\nu(p, p_2) .\end{aligned}$$

Remark 2. The first of the above inequalities changes to $<$ provided that $\varrho(x_1, x_2) < \varrho(x_1, x) + \varrho(x, x_2)$ or $|t_1 - t_2| < |t_1 - t| + |t - t_2|$. Thus in Case 2 the point p is not between the points p_1 and p_2 provided that x is not between x_1 and x_2 or t is not between t_1 and t_2 (compare § 2).

It is evident from (29) that

11.2. *The metric ϱ_ν preserves the natural topology of $X \times \mathcal{I}$ and the metric space $(X \times \mathcal{I}, \varrho_\nu)$ is bounded.*

Now we shall prove the following:

11.3. *Suppose that (X, ϱ) is a strongly convex space. Let $p_1 = (x_1, t_1)$, $p_2 = (x_2, t_2)$, $t_1 \leqslant t_2$ and*

$$A = \{p:\ p = (x, t_1),\ x \in \overline{x_1 x_2}\} , \qquad B = \{p:\ p = (x_2, t),\ t_1 \leqslant t \leqslant t_2\} . \tag{30}$$

Then the sum $A \cup B$ is a segment from p_1 to p_2 in the metric space $(X \times \mathcal{I}, \varrho_\nu)$.

Proof. The sets A and B are arcs with end-points $(x_1, t_1), (x_2, t_1)$ and $(x_2, t_1), (x_2, t_2)$ respectively and $A \cap B = \{(x_2, t_1)\}$. Hence $A \cup B$ is an arc from p_1 to p_2.

According to (29) and (30), the functions i_1: $A \to \mathcal{E}^1$ and i_2: $B \to \mathcal{E}^1$, defined as

$$\begin{aligned}i_1(p) &= \varrho_\nu(p_1, p) = (1+t_1)\varrho(x_1, x) \quad &\text{for}\quad p \in A ,\\ i_2(p) &= \varrho_\nu(p, p_2) = t_2 - t &\text{for}\quad p \in B ,\end{aligned}$$

are isometrical transformations (see § 2), because the segment $\overline{x_1 x_2}$ is a linear subset of the space (X, ϱ) (see § 3). Therefore the arcs A and B are segments in the space $(X \times \mathcal{I}, \varrho_\nu)$. But from (29) we obtain

$$\begin{aligned}\varrho_\nu(p_1, p_2) &= (1+t_1)\varrho(x_1, x_2) + (t_2 - t_1) \\ &= \varrho_\nu(p_1, (x_2, t_1)) + \varrho_\nu((x_2, t_1), p_2) ,\end{aligned}$$

i. e. the point (x_2, t_1) is between the points p_1 and p_2. This implies (see [1], p. 44) that $A \cup B$ is a segment from p_1 to p_2.

11.4. *If ϱ is SC-metric, then ϱ_ν is SC-metric.*

Proof. Let $p_1 = (x_1, t_1)$ and $p_2 = (x_2, t_2)$. We may establish that $t_1 \leqslant t_2$.

Since the segment is a linear set (see § 3), 11.3 implies that it is enough to show that $p = (x, t) \in X \times \mathcal{I}$ and $p_1 p p_2$ imply $p \in A$ or $p \in B$ (compare 2.1 and the definition of SC-metric in § 3). According to remarks 1 and 2 in the proof of 11.1, we may assume that $t_1 \leqslant t \leqslant t_2$ and $x_1 x x_2$ (i. e. $x \in \overline{x_1 x_2}$ by 3.2). Then we have

$$\begin{aligned}\varrho_\nu(p_1, p_2) &= \varrho_\nu(p_1, p) + \varrho_\nu(p, p_2) \\ &= (1+t_1)\varrho(x_1, x) + (t - t_1) + (1+t)\varrho(x, x_2) + (t_2 - t) \\ &= (1+t_1)[\varrho(x_1, x) + \varrho(x, x_2)] + (t - t_1)\varrho(x, x_2) + (t_2 - t_1) \\ &= (1+t_1)\varrho(x_1, x_2) + (t_2 - t_1) + (t - t_1)\varrho(x, x_2) \\ &= \varrho_\nu(p_1, p_2) + (t - t_1)\varrho(x, x_2) ,\end{aligned}$$

i. e. $(t - t_1)\varrho(x, x_2) = 0$. Hence $t = t_1$ or $x = x_2$. It follows from (30) that $p \in A$ or $p \in B$ respectively.

The following is an important property of SC-metric ϱ_ν:

11.5. *Let ϱ be SC-metric. In order that the point $p \in X \times \mathcal{I}$ be a terminal point of $(X \times \mathcal{I}, \varrho_\nu)$ it is necessary and sufficient that $p = (x, 1)$, where x is a terminal point of (X, ϱ), i. e.*

$$\mathcal{T}(X \times \mathcal{I}, \varrho_\nu) = \mathcal{T}(X, \varrho) \times (1) .$$

Proof that $\mathcal{T}(X \times \mathcal{I}, \varrho_\nu) \subset \mathcal{T}(X, \varrho) \times (1)$. Let $p \in \mathcal{T}(X \times \mathcal{I}, \varrho_\nu)$ and $p = (x, t)$. Putting $p_1 = (x_1, t)$ and $p_2 = (x, 1)$, where $x \neq x_1 \in X$, we obtain $p \neq p_1$ and $p_1 p p_2$ by (29). Whence, by the definition of terminal point (see § 4), we have $p = p_2$, i. e. $t = 1$. Now if $x', x'' \in X$ and $x' x x''$, then putting $p' = (x', 1)$ and $p'' = (x'', 1)$ we obtain $p' p p''$ by (29). Hence $p = p'$ or $p = p''$. This implies $x = x'$ or $x = x''$ respectively. Thus $x \in \mathcal{T}(X, \varrho)$, i. e. $p = (x, t) = (x, 1) \in \mathcal{T}(X, \varrho) \times (1)$.

Proof that $\mathcal{T}(X, \varrho)\times(1)\subset\mathcal{T}(X\times\mathcal{I}, \varrho_\nu)$. Let $p=(x,1)$, where $x\in\mathcal{T}(X,\varrho)$. Suppose $p_1=(x_1,t_1)$, $p_2=(x_2,t_2)$ be arbitrary points such that $t_1\leqslant t_2$ and p_1pp_2. It is enough to prove (see § 4) that $p=p_1$ or $p=p_2$.

We conclude from 11.3 and 11.4 that $\overline{p_1p_2}=A\cup B$. Thus 1° $p\in A$ or 2° $p\in B$. We shall consider these two cases:

1° There is $(x,1)\in A$. According to (30) we have $t_1=1$ and $x\in\overline{x_1x_2}$. Hence $t_2=1$ and x_1xx_2. This implies that $x=x_1$ or $x=x_2$, because x is a terminal point. Thus $p=(x,1)=(x_1,t_1)=p_1$ or $p=(x,1)=(x_2,t_2)=p_2$.

2° We have $(x,1)\in B$, i. e. $t_2=1$ and $x_2=x$, according to (30). Thus $p=(x,1)=(x_2,t_2)=p_2$.

§ 12. The SC-metric of a disk with 2 terminal points. Applying 11.4 and 11.5 for $X=\mathcal{I}$ and for ordinary metric $\varrho(x_1,x_2)=|x_1-x_2|$, we obtain a SC-metric $\bar\varrho=\varrho_\nu$ of the disk $D=\mathcal{I}\times\mathcal{I}$ such that

$$\mathcal{T}(D,\bar\varrho)=\mathcal{T}(\mathcal{I},\varrho)\times(1)=\{(0,1),(1,1)\}.$$

Hence only the points $(0,1)$ and $(1,1)$ are terminal.

Furthermore, it is easy to see, by 11.3, that

$$\mathcal{F}(D,\bar\varrho)=(0)\times\mathcal{I}\cup\mathcal{I}\times(1)\cup(1)\times\mathcal{I}$$

and

$$\mathcal{R}(D,\bar\varrho)=D-\mathcal{I}\times(1)-(0,0)-(1,0).$$

Therefore $D=\mathcal{R}(D,\varrho)\cup\mathcal{F}(D,\bar\varrho)$. This gives a solution of problem III (see § 1).

We have also $\operatorname{int}(D)\subset\mathcal{R}(D,\bar\varrho)$. But $\operatorname{int}(D)$ is topologically a plane $\mathcal{E}^2$. Thus in this way we may obtain a SC-metric of $\mathcal{E}^2$ such that each point is a ramification point.

Remark. The intersection $\mathcal{R}(D,\bar\varrho)\cap\mathcal{F}(D,\bar\varrho)$ is the sum of sides $(0)\times\mathcal{I}$ and $(1)\times\mathcal{I}$ without their end-points. The question of the existence of a metric ϱ of D such that $D=\mathcal{R}(D,\varrho)\cup\mathcal{F}(D,\varrho)$ and we have $\mathcal{R}(D,\varrho)\cap\mathcal{F}(D,\varrho)=0$, remains open.

§ 13. The SC-metrics of a disk with 0 and 1 terminal point. Let X_1 be the square with vertices $(0,1)$, $(1,1)$, $(1,2)$ and $(0,2)$, X_2—the rectangle with vertices $(0,2)$, $(1,2)$, $(1,5)$ and $(0,5)$, and X_3—the square with vertices $(0,5)$, $(1,5)$, $(1,6)$ and $(0,6)$. Then $D=X_1\cup X_2\cup X_3$ is a disk. We shall prove that

13.1. *There exists a* SC-*metric* ϱ_* *of* D *such that* $\mathcal{T}(D,\varrho_*)=0$.

Proof. Consider X_1 as the cartesian product $\{(x,1)\colon 0\leqslant x\leqslant 1\}\times\mathcal{I}$. Then we can put in X_1 the metric $\varrho_1=\varrho_\nu$ (see § 11), where ϱ is an ordinary Euclidean metric, and in X_3—the metric ϱ_3 obtained from ϱ_1 by the symmetry of the square X_1 with respect to the straight line $y=\frac{7}{2}$. More exactly, for $p=(a,b)$ and $q=(c,d)$ we put:

$$\varrho_1(p,q)=[1+\min(b-1,d-1)]|a-c|+|b-d|,\quad\text{if}\quad p,q\in X_1,$$
$$\varrho_3(p,q)=[1+\min(6-b,6-d)]|a-c|+|b-d|,\quad\text{if}\quad p,q\in X_3.$$

At last we put in X_2 the ordinary Euclidean metric ϱ_2 with the coefficient 2, i. e.

$$\varrho_2(p,q)=2[(a-c)^2+(b-d)^2]^{1/2},\quad\text{if}\quad p,q\in X_2.$$

Therefore, if $p,q\in X_1\cap X_2$, then $b=d=2$, whence $\varrho_1(p,q)=2|a-c|=2[(a-c)^2]^{1/2}=\varrho_2(p,q)$. Similarly if $p,q\in X_2\cap X_3$, then $b=d=5$, whence $\varrho_3(p,q)=\varrho_2(p,q)$. Thus we can apply § 10, for $n=3$. Let $\varrho_*=\varrho_s$. Thus, in order to prove that ϱ_* is SC-metric it is enough to show that the hypotheses of 10.2 hold.

Indeed, by virtue of 11.4, ϱ_1, ϱ_2 and ϱ_3 are SC-metrics. It is not difficult to verify that ϱ_* is an extension of ϱ_i for $i=1,2,3$. Thus we must prove only that the hypothesis about the passages in 10.2 holds for points p,q such that (I) $p\in X_1-X_2$ and $q\in X_2-X_1$, or (II) $p\in X_1$ and $q\in X_3$, or (III) $p\in X_2-X_3$ and $q\in X_3-X_2$. Case (III) is similar to Case (I), because the spaces (X_1,ϱ_1) and (X_3,ϱ_3) are isometric ones; thus we shall consider only Case (I) and Case (II).

Case (I). Evidently we may assume that $a\leqslant c$. Thus we have $1\leqslant b<2<d\leqslant 5$. Let $(t,2)\in X_1\cap X_2$, i. e. $0\leqslant t\leqslant 1$ and the sequence $(p,(t,2),q)$ is a passage from p to q. We put $f(t)=s(p,(t,2),q)$ (compare § 10), whence by (28) we have:

$$\begin{aligned}f(t)&=\varrho_1(p,(t,2))+\varrho_2((t,2),q)\\&=[1+(b-1)]|a-t|+|b-2|+2[(t-c)^2+(2-d)^2]^{1/2}\\&=b|a-t|+(2-b)+2[(c-t)^2+(d-2)^2]^{1/2}.\end{aligned}$$

Therefore, if $t<a$, then

$$\begin{aligned}f(a)&=(2-b)+2[(c-a)^2+(d-2)^2]^{1/2}\\&<b(a-t)+(2-b)+2[(c-t)^2+(d-2)^2]^{1/2}=f(t),\end{aligned}$$

and if $c<t$, then

$$\begin{aligned}f(c)&=b(c-a)+(2-b)+2(d-2)\\&<b(t-a)+(2-b)+2[(c-t)^2+(d-2)^2]^{1/2}=f(t).\end{aligned}$$

Hence the function f can have its minima only in the interval $a\leqslant t\leqslant c$. Here, however, we have

$$f(t)=b(t-a)+(2-b)+2[(c-t)^2+(d-2)^2]^{1/2}.$$

But the function f can have an extremum in the open interval $a < t < c$ only at the point

$$t_0 = c - b(d-2)/(4-b^2)^{1/2},$$

provided that $a < t_0 < c$, and then that extremum is a minimum of f, because

$$\left(\frac{d^2 f}{dt^2}\right)_{t_0} = (4-b^2)^{3/2}/4(d-2) > 0 .$$

It follows that the function f has in the interval $a \leqslant t \leqslant c$ exactly one minimum. This means that in Case (I) the hypotheses of 10.2 are satisfied.

Before we consider Case (II) we shall prove the following lemma:

13.2. *If* $0 \leqslant u \leqslant 1$ *and* $0 < v \leqslant 1$, *then*

$$2[(u+v)^2+9]^{1/2} < 2(u^2+9)^{1/2}+v .$$

Proof. We have $2u+3v/4 \leqslant 2+\frac{3}{4} < 3 \leqslant (u^2+9)^{1/2}$, whence $8uv+3v^2 < 4v(u^2+9)^{1/2}$. Adding $4(u^2+9)+v^2$ to this inequality, we obtain $4(u^2+2uv+v^2+9) = 8uv+3v^2+4(u^2+9)+v^2 < 4v(u^2+9)^{1/2}+4(u^2+9)+v^2$, whence $2[(u+v)^2+9]^{1/2} < 2(u^2+9)^{1/2}+v$.

Case (II). As in Case (I) we may assume that $a \leqslant c$. Thus we have $1 \leqslant b \leqslant 2 < 5 \leqslant d \leqslant 6$. Let $(t_1, 2) \in X_1 \cap X_2$ and $(t_2, 5) \in X_2 \cap X_3$, i. e. $0 \leqslant t_1 \leqslant 1$, $0 \leqslant t_2 \leqslant 1$ and the sequence $(p, (t_1, 2), (t_2, 5), q)$ is a passage from p to q. We put $f(t_1, t_2) = s(p, (t_1, 2), (t_2, 5), q)$, whence by (28) we have:

$$\begin{aligned} f(t_1, t_2) &= \varrho_1(p, (t_1, 2)) + \varrho_2((t_1, 2), (t_2, 5)) + \varrho_3((t_2, 5), q) \\ &= b|a-t_1| + |b-2| + 2[(t_1-t_2)^2+9]^{1/2} + (7-d)|t_2-c| + |5-d| \\ &\geqslant |a-t_1| + (2-b) + 2[(t_1-t_2)^2+9]^{1/2} + |t_2-c| + (d-5). \end{aligned}$$

We shall prove that the function f has its minimum only at the point (a, c). Indeed, since

$$|a-c| \leqslant |a-t_2| + |t_2-c| \quad \text{and} \quad |a-t_2| \leqslant |a-t_1| + |t_1-t_2| ,$$

we have

$$\begin{aligned} f(a, c) &= (2-b) + 2[(a-c)^2+9]^{1/2} + (d-5) \\ &\leqslant (2-b) + 2[(|a-t_2| + |t_2-c|)^2+9]^{1/2} + (d-5) \\ &< (2-b) + 2[(a-t_2)^2+9]^{1/2} + |t_2-c| + (d-5) \\ &\leqslant (2-b) + 2[(|a-t_1| + |t_1-t_2|)^2+9]^{1/2} + |t_2-c| + (d-5) \\ &< (2-b) + 2[(t_1-t_2)^2+9]^{1/2} + |a-t_1| + |t_2-c| + (d-5) \leqslant f(t_1, t_2) , \end{aligned}$$

where the inequalities $<$ are the consequences of 13.2 for $u = |a-t_2|$, $v = |t_2-c|$ and for $u = |t_1-t_2|$, $v = |a-t_1|$ respectively, provided that $t_2 \neq c$ and $t_1 \neq a$ respectively. These inequalities must be written as equalities if $t_2 = c$ or $t_1 = a$ respectively. Therefore $f(a, c) < f(t_1, t_2)$ if $t_1 \neq a$ or $t_2 \neq c$. It follows that the function f, and thus also s, has its minimum only at the point (a, c). This means that also in Case (II) the hypotheses of 10.2 are satisfied.

We have thus proved that ϱ_* is SC-metric.

It follows from 11.5 and from the definition of ϱ_1 and ϱ_3 that

$$\mathcal{T}(X_1, \varrho_1) = \{(0, 2), (1, 2)\} \quad \text{and} \quad \mathcal{T}(X_3, \varrho_3) = \{(0, 5), (1, 5)\} .$$

Since ϱ_2 is an ordinary metric (with the coefficient 2) of the rectangle X_2, the terminal points of (X_2, ϱ_2) are the vertices of X_2. Thus

$$\mathcal{T}(X_2, \varrho_2) = \mathcal{T}(X_1, \varrho_1) \cup \mathcal{T}(X_3, \varrho_3) .$$

Putting in Case (II): $p = (0, 1)$, $q = (0, 6)$ or $p = (1, 1)$, $q = (1, 6)$, we conclude that the function $f(t_1, t_2)$ has its minimum at $(0, 0)$ or $(1, 1)$ respectively. This means that the straight line segments with end-points: $(0, 1)$, $(0, 6)$ and $(1, 1)$, $(1, 6)$ respectively, are segments in (D, ϱ_*). But obviously $\mathcal{T}(X_2, \varrho_2)$ is contained in the sum of interiors of those segments. Therefore no terminal point of (X_i, ϱ_i), $i = 1, 2, 3$, is a terminal point of (D, ϱ_*). Thus, since ϱ_* is an extension of every ϱ_i, $i = 1, 2, 3$, we obtain $\mathcal{T}(D, \varrho_*) = 0$, and 13.1 is proved.

Remark. It is not difficult to verify that all the following 6 arcs:

$$\begin{matrix} A_{0,16}, & A_{0,12} \cup B_{01,1}, & A_{1,12} \cup B_{01,1}, \\ A_{1,16}, & A_{1,56} \cup B_{01,6}, & A_{0,56} \cup B_{01,6}, \end{matrix}$$

where

$$A_{i,jk} = \{(x, y)\colon\ x = i,\ j \leqslant y \leqslant k\}, \quad B_{ij,k} = \{(x, y)\colon\ i \leqslant x \leqslant j,\ y = k\},$$

are segments (even maximal segments) in (D, ϱ_*) and each point of $\mathrm{bd}(D)$ is an interior point of at least one of them. The number 6 is *here* the *smallest* of the set of natural numbers n for which there exist n arcs having this property. The question of the existence of a SC-metric ϱ of D such that corresponding number is smaller than 6, remains open.

13.3. *There exists a* SC-*metric* ϱ' *of a disk with one terminal point.*

Proof. Let D' be the sum of the square X_1 and the triangle (contained in X_2) with vertices $(0, 2)$, $(1, 2)$ and $(0, 5)$, in the notation of 13.1. Then $D' \subset D$. Defining the metric ϱ' of D' as ϱ_* restricted to D', we obtain a strongly convex space (D', ϱ') such that $\mathcal{T}(D', \varrho') = \{(0, 5)\}$. Obviously D' is a disk.

§ 14. Some SC-metrics of an n-cell. Now we may show the following theorem, which contains the solution of problem II (see § 1):

THEOREM. *There exists for every $n = 2, 3, \ldots$ and $m = 0, 1, \ldots$ a SC-metric ϱ_m^n of $\mathcal{I}^n$ such that the n-cell $\mathcal{I}^n$ metrized by ϱ_m^n has exactly m terminal points, i. e.*

$$\overline{\overline{\mathcal{T}(\mathcal{I}^n, \varrho_m^n)}} = m .$$

Proof. Of course, it is sufficient to define such metrics in the spaces homeomorphic to $\mathcal{I}^n$, i. e. in an arbitrary n-cell.

For $n = 2$ we put $\varrho_0^2 = \varrho_*$, $\varrho_1^2 = \varrho'$, $\varrho_2^2 = \bar{\varrho}$ (see 13.1, 13.3 and § 12) and $\varrho_i^2 =$ an ordinary Euclidean metric of convex plane i-gonal figure for $i = 3, 4, \ldots$

Suppose the theorem is true for $n = k$. We can diminish the diameter (e. g. to 1) of the space $(\mathcal{I}^k, \varrho_m^k)$ without changing the number of terminal points, for instance by dividing every distance by the diameter. Then we may apply the results of § 11. Namely, let

$$\varrho_m^{k+1} = (\varrho_m^k)_\nu \quad \text{for} \quad m = 0, 1, \ldots$$

We conclude from 11.5 that

$$\mathcal{T}(\mathcal{I}^{k+1}, \varrho_m^{k+1}) = \mathcal{T}(\mathcal{I}^k \times \mathcal{I}, \varrho_m^{k+1}) = \mathcal{T}(\mathcal{I}^k, \varrho_m^k) \times (1) \quad \text{for} \quad m = 0, 1, \ldots$$

It follows that

$$\overline{\overline{\mathcal{T}(\mathcal{I}^{k+1}, \varrho_m^{k+1})}} = \overline{\overline{\mathcal{T}(\mathcal{I}^k, \varrho_m^k)}} = m \quad \text{for} \quad m = 0, 1, \ldots$$

References

[1] L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford 1953.
[2] K. Borsuk, *On a metrization of polytopes*, Fund. Math. 47 (1959), pp. 325-341.
[3] H. Busemann, *The geometry of geodesics*, New York 1955.
[4] C. Kuratowski, *Topologie I*, Warszawa 1952.
[5] — *Topologie II*, Warszawa 1952.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 12. 4. 1960

A generalization of the incompleteness theorem

by

A. Mostowski (Warszawa)

The aim of this paper is to prove the following generalization of the Gödel incompleteness theorem (cf. [1] and [9]):

Let a formula Φ with one numerical free variable be called free for a system S if for every n formulas $\Phi(\Delta_0), \Phi(\Delta_1), \ldots, \Phi(\Delta_n)$ are completely independent (i. e., every conjunction formed of some of the these formulas and of the negations of the remaining ones is consistent; $\Phi(\Delta_j)$ denotes here the formula obtained from Φ by substituting the j-th numeral for the variable of Φ). We shall prove that *free formulas exist for certain systems S and some of their extensions.* In fact we shall prove for a class of formal systems S a slightly more general result: *given a family of extensions of S satisfying certain very general assumptions, there exists a formula which is free for every extension of this family.*

The following circumstance deserves perhaps mentioning and justifies to a certain extent the length of the paper. Our considerations prove the existence of free formulas not only for systems based on the usual rules of proof but also for systems based on the rule ω. Thus they furnish another illustration of the parallelism noted already in [2] between these two kinds of systems. Our discussion of systems based on the rule ω rests on the remark due to J. R. Shoenfield that the decomposition of the Π_1^1 sets into constituents (cf. Kleene [5], theorem I, p. 417) can on several occasions be exploited in the same way as the recursive enumerability of the Σ_1^0 sets. Thus our paper can be considered as a test of this useful heuristic principle. From a result noted at the end of the paper it follows that no similar phenomenon occurs for Π_2^1 sets.

In view of these remarks the author hopes that his paper in spite of its rather special subject may throw some light on a more important and broader topic, to wit the constructive analogue of the theory of projective sets.

1. We consider a consistent theory T with standard formalization and infinite sequence $\Delta_0, \Delta_1, \ldots$ of its terms without free variables. The Gödel number of a formula Φ will be denoted by $\ulcorner\Phi\urcorner$. A k-ary relation R (i. e., a subset of $N_0^k = N_0 \times \ldots \times N_0$ where N_0 is the set of integers $\geqslant 0$)