Undecidable and creative theories

by

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1. The basic method of proving that a formal system is undecidable (i. e., has an unsolvable decision problem) is the original method of Church [1], which requires that recursive functions or sets be representable in some sense in the system. Other methods are given in [9]; but in each case, it is shown that the decidability of the given system would imply the decidability of a system already seen to be undecidable by the basic method.

To formulate the precise results, we recall some definitions. By a *theory*, we shall mean a formal system, formalized within the first order predicate calculus with equality. We suppose Gödel numbers assigned to the terms and sentences (¹) of each theory by one of the usual methods. We say that a theory is *decidable* if the set of (Gödel numbers of) theorems of the theory is recursive. A theory is *axiomatizable* if the set of theorems of the theory is recursively enumerable (²).

We shall suppose that in each theory T a sequence of terms

$$\bar{0}, \bar{1}, \bar{2}, \ldots$$

is fixed so that the Gödel number of $\bar{n}$ is a recursive function of n, and so that if $m \neq n$, then $\vdash_T \bar{m} \neq \bar{n}$.

Let $A(x)$ be a sentence of the theory T containing no free variable other than x. We say that $A(x)$ *strongly represents* a set K if

$$n \in K \to \vdash_T A(\bar{n})$$

and

$$n \notin K \to \vdash_T \neg A(\bar{n})$$

for all n. We say that $A(x)$ *weakly represents* K if

$$n \in K \leftrightarrow \vdash_T A(\bar{n}) .$$

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(¹) We do not require (as in [9]) that a sentence contain no free variables.

(²) This is equivalent to more usual definitions of axiomatizability by Craig's theorem.

for all n. If $A(x,y)$ contains no free variables other than x and y, we say that $A(x,y)$ *defines* the function f if

$$\vdash_T A(\bar{n}, y) \leftrightarrow y = \overline{f(n)}$$

for all n.

We can now state the basic result ([9], p. 49): *A consistent theory T is undecidable provided that:*

(A) *Every recursive function is definable in T.*

H. Putnam ([7], p. 53) has shown that (A) may be replaced by:

(B) *Every recursive set is strongly representable in T.*

(It is easily seen that (A) implies (B).) Actually, we may replace (B) by the weaker condition [3]:

(C) *Every recursive set is weakly representable in T.*

For let $J(m,n)$ be the predicate 'm is the number of a sentence $A(x)$ such that $A(\bar{n})$ is provable'. If T were decidable, J would be recursive. But by (C), every recursive set is of the form $\hat{n}J(m,n)$ for some m. By the usual diagonal argument, it follows that J is not recuesive.

We can now ask: in any of these cases, can we conclude that T is, in some sense, effectively undecidable? In the case of a recursively enumerable set there is a precise formulation of the idea of being effectively non-recursive, viz., Post's notion of creativity (see, e. g., [6]). We say that a theory is *creative* if the set of theorems of the theory is creative. The question then becomes: if a consistent axiomatizable theory T satisfies (A), (B), or (C), is it necessarily creative?

This question was first considered by S. Feferman [3], who showed that if T satisfies some further conditions, which state that T contains a formula '$x \leqslant y$' about which certain of the usual properties of $\leqslant$ can be proved, then T is creative. The author [8] showed that Feferman's conditions could be replaced by weaker ones. Ehrenfeucht and Feferman [2] showed that if T satisfies (A) and either the conditions of [3] or those of [8], then every recursively enumerable set is weakly representable in T. This conclusion implies that T is creative; indeed, if any creative set is weakly representable in T then T is creative, as is easily seen.

In this article, we show that the additional conditions cannot be eliminated. We give an example of a consistent axiomatizable theory T satisfying (A) (and hence (B) and (C)), which is not creative, and in which no non-recursive set is weakly representable.

We shall actually prove slightly more about T. We first recall that if A is recursively enumerable, then A has degree $\leqslant \mathbf{0}'$; and A has degree $\mathbf{0}'$

(3) This fact was pointed out to the author by R. L. Vaught. The author would like to thank Vaught for several helpful conversations on the material of this paper.

if and only if every recursively enumerable set is recursive in A. To a theory T we assign the degree of the set of theorems of T. We shall prove that our theory T has degree $< \mathbf{0}'$. By [6], Theorem 20, it follows that T is not creative.

In § 2, we construct a recursively enumerable set B; no formal systems are involved in the construction. In § 3, we use B to construct our theory T.

2. Concerning recursive functions, we use the notation of [4], as supplemented by footnote 2 of [5]. In particular, $\{e\}^a$ designates the partial function which is recursive in a with Gödel number e. It is defined at x if and only if $T_1^1(\bar{a}(y), e, x)$ for some y, where $\bar{a}(y)$ is the number of the sequence $a(0), a(1), \ldots, a(y-1)$; in this case, $\{e\}^a(x) = U(y)$ for any such y. Also $(x)_n$ represents the exponent of the $(n-1)$-st prime in the prime power expansion of x.

We shall sometimes identify a set with its characteristic function; this explains such notation as $A(x)$. We write $\langle e, v\rangle$ for $2^e 3^v$; this may be thought of as the ordered pair of e and v. We set

$$A^{(e)} = \hat{v}(\langle e, v\rangle \in A)\,.$$

The set of all natural numbers is designated by N; the empty sets is designated by $\emptyset$.

Lemma 1. *Let Z be a recursively enumerable set such that for all e, $Z^{(e)}$ is finite or equal to N. Then is a recursively enumerable subset A of Z having degree $< \mathbf{0}'$ such that $Z^{(e)} - A^{(e)}$ is finite for all e.*

Proof. We suppose that a recursive function enumerating Z is fixed, and let Z_k be the set of values of this function for arguments $\leqslant k$. In addition to A, we shall enumerate a set B which is not recursive in A; this will insure that A has degree $< \mathbf{0}'$. As a number x is placed in B, we shall define finite sets $Q(x)$ and $R(x)$ to be used later in the enumeration [4].

We now describe the kth stage in the enumeration of A and B. We say that x is *active* (at this stage) if x has previously been placed in B and no member of $Q(x)$ has been placed in A.

Step 1. Place in A all $\langle e, v\rangle$ in Z_k such that $\langle e, v\rangle \notin R(x)$ for every active x.

Step 2. Let A_k be the set of elements already placed in A, either in Step 1 of this stage or at an earlier stage. Let $(k)_0 = x$, $(x)_0 = f$. We

(4) The purpose of $Q(x)$ and $R(x)$ is roughly the following. If x is placed in B, it is to ensure that $\{f\}^A(x) \neq B(x)$ for $f = (x)_0$. The members of $Q(x)$ and $R(x)$ are those $\langle e, v\rangle$ which, if later placed in A, would make this inequality false. If such an $\langle e, v\rangle$ appears in Z, we must establish an order of precedence between two things we wish to do: place $\langle e, v\rangle$ in A, and insure that the above inequality remains true. If $\langle e, v\rangle \in Q(x)$, placing $\langle e, v\rangle$ in A takes precedence over this inequality; if $\langle e, v\rangle \in R(x)$, the inequality takes precedence.

place x in B if it has not already been placed in B, and if (a)-(e) below hold.

(a) $(\exists y)_{y \leqslant k} T_1^1(\bar{A}_k(y), f, x)$.

If (a) holds, we set $y = \mu y T_1^1(\bar{A}_k(y), f, x)$.

(b) $U(y) = 0$.

(c) There is no active x' with $x' < x$ and $(x')_0 = f$.

Assuming (a)-(c) hold, we let J_0 be the set of $\langle e, v\rangle$ such that $\langle e, v\rangle < y$ and $\langle e, v\rangle \notin A_k$. Let J be the smallest set such that $J_0 \subset J$, and such that for all active x',

$$R(x') \cap Z_k \cap J \neq \emptyset \rightarrow Q(x') \subset J .$$

Since J is included in the union of J_0 and of the $Q(x')$ for active x' at stage k, J is finite and can be effectively found.

(d) $J \cap A_k = \emptyset$.

(e) $\langle e, v\rangle \in J \;\&\; e \leqslant f \;\&\; (x)_{e+1} \neq 0 \rightarrow \langle e, v\rangle \in Z_k$.

If (a)-(e) hold, we let $Q(x)$ be the set of $\langle e, v\rangle \in J - Z_k$ such that $e \leqslant f$, and let $R(x)$ be the set of $\langle e, v\rangle \in J - Z_k$ such that $e > f$. This completes the description of A and B.

We say that x is *effective* if $x \in B$ and $Q(x) \cap A = \emptyset$. Then x is active at every stage after it is placed in B.

Let $M(e, v)$ be the set of such x in B that $\langle e, v\rangle \in R(x)$. If $\langle e, v\rangle \in Z_k$, then $\langle e, v\rangle$ cannot be placed in any $R(x)$ at the k-th (or any later) stage. Hence

(1) $$\langle e, v\rangle \in Z \rightarrow M(e, v) \text{ is finite.}$$

We now show $Z^{(e)} - A^{(e)}$ is finite. If $v \in Z^{(e)} - A^{(e)}$, then $M(e, v)$ contains an effective x. Otherwise, in view of (1), we could choose k so large that $\langle e, v\rangle \in Z_k$ and no member of $M(e, v)$ is active at the k-th stage; so $\langle e, v\rangle$ would be placed in A at the k-th stage. Now for any $x \in M(e, v)$, $(x)_0 < e$. Hence we need only show for any $f < e$, there are only finite many effective x with $(x)_0 = f$. This follows from (c) of Step 2 above.

It remains to show that B is not recursive in A. For this, we suppose that $B = \{f\}^A$ and derive a contradiction.

Case 1. There is an effective x such that $(x)_0 = f$.

Suppose that x is placed in B at the k-th stage, and use the notation of Step 2 above. We have $\{f\}^A(x) = B(x) = 1$, $T_1^1(\bar{A}_k(y), f, x)$, and $U(y) = 0$. It follows that $\bar{A}_k(y) \neq \bar{A}(y)$. Since $A_k \subset A$, it follows that $J_0 \cap A \neq \emptyset$, and hence that $J \cap A \neq \emptyset$.

Let $\langle e, v\rangle$ be the first member of J placed in A; say it is placed in A at the p-th stage. Since $J \cap A_k = \emptyset$ by (d), $p > k$. Now $\langle e, v\rangle \notin Q(x)$, since x is effective. Also $\langle e, v\rangle \notin R(x)$, since x is active at the p-th stage. Since $J \subset Q(x) \cup R(x) \cup Z_k$, it follows that $\langle e, v\rangle \in Z_k$. But $\langle e, v\rangle \notin A_k$; so $\langle e, v\rangle \in R(x')$ for some x' active at the k-th stage. Then $\langle e, v\rangle \in R(x') \cap Z_k \cap J$, so $Q(x') \subset J$. Hence no member of $Q(x')$ is placed in A before the p-th stage; so x' is active at the p-th stage. This is a contradiction, since $\langle e, v\rangle \in R(x')$ and $\langle e, v\rangle$ is placed in A at the p-th stage.

Case 2. There is no effective x with $(x)_0 = f$.

Let F be the set of e such that $Z^{(e)} = N$. Choose x so that

(2) $$(x)_0 = f ,$$

(3) $$e \leqslant f \rightarrow \big((x)_{e+1} \neq 0 \leftrightarrow e \in F\big) ,$$

(4) $$e \leqslant f \;\&\; e \notin F \;\&\; \langle e, v\rangle \in Z \rightarrow \langle e, v\rangle \in Z_x .$$

This is possible, since $Z^{(e)}$ is finite for $e \notin F$.

Suppose $x \in B$; say that x is put in B at the k-th stage. Since x is not effective, there is an $\langle e, v\rangle$ in $Q(x) \cap A$. Then $\langle e, v\rangle \notin Z_k$ (since $Q(x) \cap Z_k = \emptyset$), and hence $\langle e, v\rangle \notin Z_x$ (since $x = (k)_0 \leqslant k$). Also $e \leqslant (x)_0 = f$ and $\langle e, v\rangle \in A \subset Z$. Hence by (4), $e \in F$; so by (3), $(x)_{e+1} \neq 0$. Then by (e) of Step 2, $\langle e, v\rangle \in Z_k$, a contradiction.

It follows that $x \notin B$. Hence $0 = B(x) = \{f\}^A(x)$. Letting $y = \mu y T_1^1(\bar{A}(y), f, x)$, we then have $U(y) = 0$. Let S consist of all $\langle e, v\rangle$ such that either $\langle e, v\rangle \leqslant y$ and $\langle e, v\rangle \notin A$, or $\langle e, v\rangle \in Q(x')$, where x' is effective and $(x')_0 \leqslant y$. By (c) of Step 2, S is finite; and $\langle e, v\rangle \in S$ implies $e \leqslant y$. Now choose k with $(k)_0 = x$ so large that:

(i) $\bar{A}_k(y) = \bar{A}(y)$.

(ii) $y \leqslant k$.

(iii) $S \cap Z \subset Z_k$.

(iv) $\langle e, v\rangle < y \;\&\; \langle e, v\rangle \in A \rightarrow \langle e, v\rangle \in A_k$.

(v) No non-effective x' with $x' < x$ is active at the k-th stage.

(vi) If x' is non-effective, and $x' \in M(e, v)$ for some $\langle e, v\rangle \in Z \cap S$, then x' is not active at the k-th stage. (This is possible by (1).)

We now show that at the k-th stage, (a)-(e) of Step 2 are satisfied. This will give the desired contradiction, since $x \notin B$.

By (i) and (ii), (a) holds with y as above; so (b) holds. By (v), (c) holds (since no x' with $(x')_0 = f$ is effective).

Now we show $J \subset S$. By (iv), $J_0 \subset S$. Hence we need only show that, for x' active at the k-th stage,

$$R(x') \cap Z_k \cap S \neq \emptyset \rightarrow Q(x') \subset S .$$

Let $\langle e, v\rangle \in R(x') \cap Z_k \cap S$. Then $x' \in M(e, v)$ and $\langle e, v\rangle \in Z \cap S$. Hence by (vi), x' is effective. Also $(x')_0 < e \leqslant y$; whence $Q(x') \subset S$.

We now prove (d) and (e). Suppose that $\langle e, v\rangle \in J \cap A_k$. Then $\langle e, v\rangle \in S \cap A$. Hence $\langle e, v\rangle \in Q(x')$ where x' is effective. But if x' is effective, $Q(x') \cap A = \emptyset$; so we have a contradiction. This proves (d). Now let $\langle e, v\rangle \in J$, $e \leqslant f$, and $(x)_{e+1} \neq 0$. Then $\langle e, v\rangle \in S$. By (3), $e \in F$; so by choice of F, $\langle e, v\rangle \in Z$. Hence by (iii), $\langle e, v\rangle \in Z_k$. This proves (e), and completes the proof of Lemma 1.

Now let Z be the set of $\langle e, v\rangle$ such that $\{e\}(x)$ is defined for all $x \leqslant v$. Clearly Z satisfies the hypotheses of Lemma 1. Hence there is a recursively enumerable subset A of Z of degree $< \mathbf{0}'$ such that $Z^{(e)} - A^{(e)}$ is finite for all e. Let B consist of all pairs $\langle e, z\rangle$ such that either $\langle (e)_0, z\rangle \in A$ or $z \leqslant (e)_1$. For $\langle e, z\rangle \in B$, define $F(e, z)$ by

$$F(e, z) = \big((e)_2\big)_z \quad \text{if} \quad z \leqslant (e)_1\,,$$

$$F(e, z) = \{(e)_0\}(z) \quad \text{if} \quad z > (e)_1\,.$$

Then we readily verify:

B is a recursively enumerable set of degree $< \mathbf{0}'$, and F is a partial recursive function with domain B. For all e, $B^{(e)}$ is recursive. If $\{f\}$ is recursive, there is an e such that $F(e, z) = \{f\}(z)$ for all z.

3. We now construct a consistent axiomatizable theory T of degree $< \mathbf{0}'$ in which every recursive function is definable, but in which no non-recursive set is weakly representable.

The non-logical symbols of T are the constants $\overline{0}, \overline{1}, \overline{2}, \ldots$ and the one-place function symbols $\Phi_0, \Phi_1, \Phi_2, \ldots$ The non-logical axioms of T are all sentences of the following forms (where B and F are as in § 2) ([5]).

(I) $\overline{m} \neq \overline{n}$ where $m \neq n$.

(II) $\Phi_e(\overline{m}) = \overline{n}$ where $\langle e, m\rangle \in B$ and $F(e, m) = n$.

(III) $$(\exists x_1) \ldots (\exists x_n)\Big(\bigwedge_{i=1}^{n} x_{r_i} \neq x_{s_i} \;\&\; \bigwedge_{i=1}^{b} \Phi_{m_i}(x_{t_i}) = x_{v_i} \;\&\; \bigwedge_{i=b+1}^{c} \Phi_{m_i}(x_{t_i}) = U_i \;\&\; \bigwedge_{i=1}^{d} x_{p_i} \neq V_i\Big)$$

where $r_i \neq s_i$, the pairs (m_i, t_i) are distinct, and no x_j occurs in any of the terms U_i or V_i.

Clearly T is axiomatizable. If $\{f\}$ is recursive, there is an e such that $F(e, m) = \{f\}(m)$ for all m; then $\Phi_e(x) = y$ defines $\{f\}$.

LEMMA 2. *Every sentence in T is equivalent to a quantifier free sentence.*

Proof. It is sufficient to prove this for sentences of the form

(1) $$(\exists x_1) \ldots (\exists x_n)(S_1 \;\&\; \ldots \;\&\; S_k)$$

([5]) We use $\wedge$ with indices to indicate conjunctions of several sentences.

where each S_i is an equality or an inequality. We first eliminate all terms containing two or more Φ_e's by, e. g., replacing

$$\Phi_e\big(\Phi_f(x)\big) = U$$

by

$$(\exists y)\big(y = \Phi_f(x_i) \;\&\; \Phi_e(y) = U\big)\,,$$

and then bringing the new quantifier to the front. We then eliminate all inequalities not of the form $x_i \neq x_j$ by replacing $U \neq V$ by

$$(\exists y)(\exists z)(y = U \;\&\; z = V \;\&\; y \neq z)\,.$$

Then we eliminate equalities of the form

$$\Phi_e(x_i) = \Phi_e(x_j)$$

by replacing each such equality by

$$(\exists y)\big(y = \Phi_e(x_i) \;\&\; y = \Phi_e(x_j)\big)\,.$$

We can thus suppose each S_r in (1) has one of the forms

(2) $$x_i = x_j\,,$$

(3) $$x_i = U\,,$$

(4) $$\Phi_e(x_i) = x_j\,,$$

(5) $$\Phi_e(x_i) = U\,,$$

(6) $$U = V\,,$$

(7) $$x_i \neq x_j\,,$$

(8) $$x_i \neq U\,,$$

(9) $$U \neq V\,,$$

where U and V contain no x_i.

We eliminate (2) and (3) by omitting $x_i = x_j$ or $x_i = U$; omitting $(\exists x_i)$; and replacing x_i everywhere by x_j or U; this converts sentences under (4)-(9) into sentences under (4)-(9).

If there are two sentences (4) or (5) with the same e and i, we replace one by the equality of the right-hand sides. If this introduces new formulae under (2) or (3), we eliminate them as above. This process terminates, since an elimination of a sentence under (2) or (3) eliminates a quantifier.

Next bring all sentences (6) or (9) outside the quantifier. If an inequality $x_i \neq x_i$ appears under (7), replace the entire sentence by $\overline{0} \neq \overline{0}$. Otherwise, the quantified part is an axiom under (III), so can be replaced by $\overline{0} = \overline{0}$. This concludes the proof of the lemma.

By a *configuration*, we mean a finite set $D = \{a_0, \ldots, a_m\}$ and a finite set of functions $\psi_0, \ldots, \psi_k$, each having subsets of D as domain and range.

This may be considered as a partial semi-model [6] for T, in which $a_0, \dots, a_m$ correspond to $\overline{0}, \dots, \overline{m}$ and $\psi_0, \dots, \psi_k$ correspond to $\Phi_0, \dots, \Phi_k$. It will then assign a truth value to some of the sentences which contain no variables and no non-logical symbols other than $\overline{0}, \dots, \overline{m}, \Phi_0, \dots, \Phi_k$. A configuration is *allowable* if it does not assign falsehood to any axiom under (II).

LEMMA 3. *A sentence S of T containing no variables is unprovable if and only if S is assigned falsehood by some allowable configuration.*

Proof. If S is not provable, it is false in some model M of T. Since S has no variables, some configuration 'included' in M assigns to S the same truth value as M, namely falsehood. Since M is a model, the configuration is allowable.

To prove the converes, it is clearly sufficient to prove that any allowable configuration can be extended to a model of T. It is readily seen that such a configuration can be extended to a model for (I) and (II). Since every extension of a model for (I) and (II) is again a model for (I) and (II), we need only prove: any model M for (I) and (II) can be extended to a model for (III).

By the completeness theorem, we need only show that M can be to a model for a finite number of axioms under (III). Now if we write these axioms with different bound variables, form their conjunction, and bring quantifiers to the front, we obtain a new axiom under (III). Thus we need only show that M can be extended to a model for one axiom under (III).

Let $y_1, \dots, y_t$ be the free variables in (III), and let $c_1, \dots, c_t$ be individuals of M. Introduce new individuals $b_1, \dots, b_n$. It is easy to see that the functions of M can be extended to the enlarged domain so that the scope of (III) is true when $y_1, \dots, y_t, x_1, \dots, x_n$ are interpreted as $c_1, \dots, c_t, b_1, \dots, b_n$. If we do this for each t-tuple $c_1, \dots, c_t$, and repeat the whole process infinitely often, we obtain the desired extension of M.

It follows from Lemma 3 that T is consistent; for $\overline{0} \neq \overline{0}$ is assigned falsehood by any non-empty configuration.

We now show the decision problem for T is reducible to that of the set B, so that T has degree $< \mathbf{0}'$. Since the proof of Lemma 2 enables us to effectively obtain a quantifier free sentence equivalent to a given sentence, and since the quantifier-free sentence will be closed whenever the given sentence is closed, we may confine ourselves to closed quntifier-free sentences, i. e., sentences without variables. The set of provable sentences without variables is recursively enumerable since T is axiomatizable. Hence we need only show that the set of unprovable sentences without variables is recursively enumerable in B.

[6] As usual, a semi-model differs from a model in that the non-logical axioms need not be true in a semi-model. We confine ourselves to semi-models in which equality is represented by equality.

Since a configuration is a finite object, we can enumerate all configurations. The allowability of a configuration $a_0, \dots, a_m, \psi_0, \dots, \psi_k$ may be expressed as follows: if $\psi_e(a_i) = a_j$ and $\langle e, i \rangle \in B$, then $F(e, i) = j$. Hence given a configuration and a decision method for B, we can decide if the configuration is allowable. Finally, given a configuration and a sentence S without variables, we can decide whether or not the configuration assigns falsehood to S. Using Lemma 3, it follows that the set of unprovable sentences without variables is recursively enumerable in B.

Now consider a sentence $A(x)$ containing no free variable other than x. Let $\Phi_0, \Phi_1, \dots, \Phi_p$ include all the function symbols in $A(x)$. Let B' be the set of $\langle e, v \rangle$ in B such that $e \leqslant p$. Just as above, we can show that the decision problem for the set of sentences

$$A(\overline{0}),\ A(\overline{1}),\ \dots \tag{10}$$

is reducible to the decision problem for B'. But B' is recursive, since $B^{(e)}$ is recursive for all e. Hence the decision problem for the set (10) is solvable. This means that the set which $A(x)$ weakly represents is recursive.

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