Undecidable and creative theories

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1. The basic method of proving that a formal system is undecidable (i.e., has an unsolvable decision problem) is the original method of Church [1], which requires that recursive functions or sets be representable in some sense in the system. Other methods are given in [5]; but in each case, it is shown that the decidability of the given system would imply the decidability of a system already seen to be undecidable by the basic method.

To formulate the precise results, we recall some definitions. By a theory, we shall mean a formal system, formalized within the first order predicate calculus with equality. We suppose Gödel numbers assigned to the terms and sentences (*) of each theory by one of the usual methods. We say that a theory is decidable if the set of (Gödel numbers of) theorems of the theory is recursive. A theory is axiomatizable if the set of theorems of the theory is recursively enumerable (*).

We shall suppose that in each theory $T$ a sequence of terms
\[ \overline{0}, \overline{1}, \overline{2}, \ldots \]
is fixed so that the Gödel number of $\overline{n}$ is a recursive function of $n$, and so that if $m \neq n$, then $\vdash \overline{m} \neq \overline{n}$.

Let $A(x)$ be a sentence of the theory $T$ containing no free variable other than $x$. We say that $A(x)$ strongly represents a set $K$ if
\[ n \in K \iff \vdash x \overline{A}(\overline{n}) \]
and
\[ n \notin K \iff \vdash x \overline{\neg A}(\overline{n}) \]
for all $n$. We say that $A(x)$ weakly represents $K$ if
\[ n \in K \iff \vdash x \overline{A}(\overline{n}) \]
\[ n \notin K \iff \vdash x \overline{\neg A}(\overline{n}) \]

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(*) We do not require (as in [5]) that a sentence contain no free variables.

(*) This is equivalent to more usual definitions of axiomatizability by Craig’s theorem.
for all $n$. If $A(x, y)$ contains no free variables other than $x$ and $y$, we say that $A(x, y)$ defines the function $f$ if
\[ \vdash A(x, y) \iff y = f(n) \]
for all $n$.

We can now state the basic result ([9], p. 49): A consistent theory $T$ is undecidable provided that:

1. Every recursive function is definable in $T$.
2. Every recursive set is strongly representable in $T$.

H. Putnam ([7], p. 53) has shown that (A) may be replaced by:

3. Every recursive set is weakly representable in $T$.

(1) It is easily seen that (A) implies (B). Actually, we may replace (B) by the weaker condition (C):

4. Every recursive set is weakly representable in $T$.

For let $J(m, n)$ be the predicate '$m$ is the number of a sentence $A(x)$ such that $A(n)$ is provable'. If $T$ were decidable, $J$ would be recursive. But by (C), every recursive set is of the form $\exists n J(m, n)$ for some $m$. By the usual diagonal argument, it follows that $J$ is not recursive.

We can now ask: in any of these cases, can we conclude that $T$ is, in some sense, effectively undecidable? In the case of a recursively enumerable set there is a precise formulation of the idea of being effectively non-recursive, viz., Post's notion of creativity (see e. g., [6]). We say that a theory is creative if the set of theorems of the theory is creative.

The question then becomes: if a consistent axiomatizable theory $T$ satisfies (A), (B) or (C), is it necessarily creative?

This question was first considered by S. Feferman [3], who showed that if $T$ satisfies some further conditions, which state that $T$ contains a formula '$x \leq y$' about which certain of the usual properties of $\leq$ can be proved, then $T$ is creative. The author [8] showed that Feferman's conditions could be replaced by weaker ones. Ehrenfeucht and Feferman [2] showed that if $T$ satisfies (A) and either the conditions of (3) or those of (5), then every recursively enumerable set is weakly representable in $T$. This conclusion implies that $T$ is creative; indeed, if any creative set is weakly representable in $T$ then $T$ is creative, as is easily seen.

In this article, we show that the additional conditions cannot be eliminated. We give an example of a consistent axiomatizable theory $T$ satisfying (A) (and hence (B) and (C)), which is not creative, and in which no non-recursive set is weakly representable.

We shall actually prove slightly more about $T$. We first recall that if $A$ is recursively enumerable, then $A$ has degree $\leq \mathbf{0}$; and $A$ has degree $\mathbf{0}'$ if and only if every recursively enumerable set is recursive in $A$. To a theory $T$ we assign the degree of the set of theorems of $T$. We shall prove that our theory $T$ has degree $\leq \mathbf{0}$ by [6]. Theorem 20, it follows that $T$ is not creative.

In § 2, we construct a recursively enumerable set $B$; no formal systems are involved in the construction. In § 3, we use $B$ to construct our theory $T$.

2. Concerning recursive functions, we use the notation of [4], as supplemented by footnote 3 of [5]. In particular, $\langle e \rangle$ designates the partial function which is recursive in $e$ with Gödel number $e$. It is defined at $x$ if and only if $T\vdash \langle e \rangle(x, y) = y$ for some $y$, where $\langle e \rangle$ is the number of the sequence $a(0), a(1), ..., a(y - 1)$; in this case, $\langle e \rangle(x) = U(y)$ for any such $y$. Also $\langle e \rangle_\alpha$ represents the exponent of the $(\alpha - 1)$-st prime in the prime-power expansion of $x$.

We shall sometimes identify a set with its characteristic function; this explains such notation as $A(e)$. We write $\langle e, v \rangle$ for $\langle e \rangle^v$; this may be thought of as the ordered pair of $e$ and $v$. We set
\[ A^{\langle e \rangle_0} = \langle \langle e, e \rangle \rangle \langle e \rangle \].

The set of all natural numbers is designated by $\mathbb{N}$; the empty set is designated by $\emptyset$.

**Lemma 1.** Let $Z$ be a recursively enumerable set such that for all $e$, $Z^e$ is finite or equal to $\mathbb{N}$. Then is a recursively enumerable subset $A$ of $Z$ having degree $\leq \mathbf{0}$ such that $Z^A = A^e$ is finite for all $e$.

**Proof.** We suppose that a recursive function enumerating a fixed pair of values for arguments $\leq \emptyset$. In addition to $A$, we shall enumerate a set $B$ which is not recursive in $A$; this will ensure that $A$ has degree $\mathbf{0}'$. As a number $x$ is placed in $B$, we shall define finite sets $Q(x)$ and $K(x)$ to be used later in the enumeration (1).

We now describe the 4th stage in the enumeration of $A$ and $B$. We say that a number is active (at this stage) if $x$ has previously been placed in $B$ and no member of $Q(x)$ has been placed in $A$.

**Step 1.** Place $A$ all $\langle e, v \rangle$ in $Z_4$ such that $\langle e, v \rangle \in B(x)$ for every active $x$.

**Step 2.** Let $A_4$ be the set of elements already placed in $A$, either in Step 1 of this stage or at an earlier stage. Let $(k, x, A) = f$.

(1) The purpose of $Q(x)$ and $K(x)$ is roughly the following. If $x$ is placed in $B$, it is to ensure that $\langle f \rangle(x) = e$ for $f(x)$. The members of $Q(x)$ and $K(x)$ are those $\langle e, v \rangle$ which, if later placed in $A$, would make this inequality false. If such an $\langle e, v \rangle$ appears in $Z$, we must establish an order of precedence between two things we wish to do: place $\langle e, v \rangle$ in $A$, and insure that the above inequality remains true. If $\langle e, v \rangle \in Q(x)$, placing $\langle e, v \rangle$ in $A$ takes precedence over this inequality; if $\langle e, v \rangle \in K(x)$, the inequality takes precedence.
place $x$ in $B$ if it has not already been placed in $B$, and if (a)-(e) below hold.

(a) \((\exists y)\exists z \in T[\overline{A}(y), f, \overline{z}]\).
If (a) holds, we set $y = \mu y\exists z \in T[\overline{A}(y), f, \overline{z}]$.
(b) $U(y) = 0$.
(c) There is no active $x'$ with $x' < x$ and $x'_k = f$.
Assuming (a)-(c) hold, we let $J_0$ be the set of $\langle x, v \rangle$ such that $\langle x, v \rangle \in y < y$ and $\langle x, v \rangle \in A_k$. Let $J$ be the smallest set such that $J_0 \subseteq J_0$ and such that for all active $x'$,

$$\overline{R}(x') \cap Z_k \cap J = \emptyset \implies Q(\overline{z}) \subseteq J.$$ 

Since $J$ is included in the union of $J_0$ and of the $Q(\overline{z})$ for active $x'$ at stage $k$, $J$ is finite and can be effectively found.

(d) $J \cap A_k = \emptyset$.
(e) $\langle x, v \rangle \in J$ if $\langle x, v \rangle \in J$ and $\langle x, v \rangle \in J$ for all $\langle x, v \rangle \in \overline{R}(x)$ such that $\langle x, v \rangle \in \overline{R}(x')$ and $\overline{R}(x_{k+1}) = \emptyset$. Then $x$ is active at every stage after it is placed in $B$.

Let $M(x, v)$ be the set of all $x$ in $B$ that $\langle x, v \rangle \in \overline{R}(x)$. If $\langle x, v \rangle \in \overline{R}(x)$, then $\langle x, v \rangle$ cannot be placed in any $\overline{R}(x)$ at the $k$-th (or any later) stage. Hence

$$\langle x, v \rangle \in Z \rightarrow M(x, v) \text{ is finite.} \tag{1}$$

We now show $Z^{n_0} - A^{n_0}$ is finite. If $v \in Z^{n_0} - A^{n_0}$, then $M(x, v)$ contains an effective $x$. Otherwise, in view of (1), we could choose $\langle x, v \rangle \in A_k$ so large that $\langle x, v \rangle \in Z_k$ and no member of $M(x, v)$ is active at the $k$-th stage; so $\langle x, v \rangle$ would be placed in $A$ at the $k$-th stage. Now for any $x_0 \in M(x, v)$, $x_0 < x$. Hence we need only show for any $x < x$, there are only finite many effective $x$ with $x_0 < x$. This follows from (e) of Step 2 above.

It remains to show that $B$ is not recursive in $A$. For this, we suppose that $B = \langle \overline{f} \rangle$ and derive a contradiction.

Case 1. There is an effective $x$ such that $x_0 = f$.

Suppose that $x$ is placed in $B$ at the $k$-th stage, and use the notation of Step 2 above. We have $f(x) = B(x) = 1$, $T[\overline{A}(y), f, \overline{z}]$, and $U(y) = 0$. It follows that $\overline{A}(y) = \overline{A}(y)$. Since $A_k \subseteq A$, it follows that $J_k \subseteq A_0 \neq \emptyset$, and hence that $J \cap A_0 \neq \emptyset$.

Let $\langle x, v \rangle$ be the first member of $A_k$ placed in $A$; say it is placed in $A$ at the $p$-th stage. Since $J \cap A_0 = \emptyset$ by (d), $p > k$. Now $\langle x, v \rangle \in Q(\overline{z})$, since $x$ is effective. Also $\langle x, v \rangle \in R(x)$, since $x$ is active at the $p$-th stage.
We now prove (d) and (e). Suppose that \( \langle e, v \rangle \in J \cap A_1 \). Then
\( \langle e, v \rangle \in S \cap A_1 \). Hence \( \langle e, v \rangle \in Q(x') \) where \( x' \) is effective. But if \( x' \) is effective, \( Q(x') \cap A_1 = \emptyset \); so we have a contradiction. This proves (d).
Now let \( \langle e, v \rangle \in J \cap S \) where \( e < j \), and \( \langle e, v \rangle \in A_1 \). Then \( \langle e, v \rangle \in S \). By (3), \( e \in F \); so by choice of \( F \), \( \langle e, v \rangle \in Z \). Hence by (iii), \( \langle e, v \rangle \in A_2 \). This proves (e), and completes the proof of Lemma 1.
Now let \( Z \) be the set of \( \langle e, v \rangle \) such that \( \langle e, v \rangle \) is defined for all \( x \leq v \). Clearly \( Z \) satisfies the hypothesis of Lemma 1. Hence there is a recursively enumerable subset \( A \) of \( Z \) of degree \( < \omega^0 \) such that \( \omega^0 - A^0 \) is finite for all \( z \). Let \( B \) consist of all pairs \( \langle e, z \rangle \) such that either \( \langle e, z \rangle \in A \) or \( z \leq \langle e \rangle \). For \( \langle e, z \rangle \in B \), define \( F(e, z) \) by
\[
F(e, z) = \langle (e)_2 \rangle \quad \text{if} \quad z \leq \langle e \rangle_1,
F(e, z) = \langle (e)_1 \rangle \quad \text{if} \quad z > \langle e \rangle_1.
\]
Then we readily verify:

- \( B \) is a recursively enumerable set of degree \( < \omega^0 \), and \( F \) is a partial recursive function with domain \( B \). For all \( e, F(e, z) \) is recursive.
- If \( f \) is recursive, \( \langle e, z \rangle \in B \) if and only if \( F(e, z) = z \).

3. We now construct a consistent axiomatizable theory \( T \) of degree \( < \omega^0 \) in which every recursive function is definable, but in which no non-recursive set is weakly representable.

The non-logical symbols of \( T \) are the constants \( 0, 1, 2, \ldots \), and the one-place function symbols \( \Phi_0, \Phi_1, \Phi_2, \ldots \). The non-logical axioms of \( T \) are all sentences of the following forms (where \( B \) and \( F \) are as in § 2) (ii).

(i) \( \langle e, m \rangle \in B \) and \( F(e, m) = n \).
(ii) \( \Phi(f(m)) = n \) where \( \langle e, m \rangle \in B \) and \( F(e, m) = n \).
(iii) \( \Phi(x) \) where \( \Phi(x) \) is any sentence.

where \( \not\equiv \) is the same \( e \) and \( t_i \) are distinct, and no \( x \) occurs in any of the terms \( U_i \) or \( V_i \).

Clearly \( T \) is axiomatizable. If \( f \) is recursive, there is an \( e \) such that \( F(e, m) = f(m) \) for all \( m \); then \( \Phi(x) \equiv y \) defines \( f \).

**Lemma 2.** Every sentence in \( T \) is equivalent to a quantifier-free sentence.

**Proof.** It is sufficient to prove this for sentences of the form
\[
(3a_1) \ldots (3a_n)(S_1 \land \ldots \land S_k)
\]

where each \( S_i \) is an equality or an inequality. We first eliminate all terms containing two or more \( \Phi(x) \)'s by, e.g., replacing
\[
\Phi_e(\Phi_f(x)) = U
\]
by
\[
(3y)(y = \Phi_e(x) \land \Phi_f(y) = U),
\]
and then bringing the new quantifier to the front. We then eliminate all inequalities not of the form \( x_i \neq y_i \) by replacing \( U \neq V \) by
\[
(3y)(3z)(y = U \land z = V \land y \neq z).
\]

Then we eliminate equality of the form
\[
\Phi_e(x) = \Phi_f(y)
\]
by replacing each such equality by
\[
(3y)(y = \Phi_e(x) \land y = \Phi_f(y)).
\]

We can thus suppose each \( B \) in (1) has one of the forms
\[
(2) \quad x_i = x_j, \\
(3) \quad x_i = U, \\
(4) \quad \Phi_e(x) = a_i, \\
(5) \quad \Phi_e(x) = U, \\
(6) \quad U = V, \\
(7) \quad x_i \neq x_j, \\
(8) \quad x_i \neq U, \\
(9) \quad U \neq V,
\]
where \( U \) and \( V \) contain no \( x_i \).

We eliminate (2) and (3) by omitting \( x_i = x_j \) or \( x_i = U \); omitting \( (3a) \); and replacing \( x_i \) everywhere by \( a_i \) or \( U \); this converts sentences under (4)-(9) into sentences under (4)-(9). If there are two sentences (4) or (5) with the same \( e \) and \( f \), we replace one by the equality of the right-hand sides. If this introduces new formulae under (2) or (3), we eliminate them as above. This process terminates, since an elimination of a sentence under (2) or (3) eliminates a quantifier.

Next bring all sentences (6) or (9) outside the quantifier. If an inequality \( x_i \neq x_j \) appears under (7), replace the entire sentence by \( 0 = 0 \). Otherwise, the quantified part is an axiom under (3), so can be replaced by \( 0 = 0 \). This concludes the proof of the lemma.

By a configuration, we mean a finite set \( D = \{ a_1, \ldots, a_n \} \) and a finite set of functions \( \psi_1, \ldots, \psi_k \), each having subsets of \( D \) as domain and range.
This may be considered as a partial semi-model (4) for $T$, in which $a_0, \ldots, a_m$ correspond to $\theta_0, \ldots, \theta_n$ and $y_0, \ldots, y_k$ correspond to $\phi_0, \ldots, \phi_n$. It will then assign a truth value to some of the sentences which contain no variables and no non-logical symbols other than $\theta_0, \ldots, \theta_n, \phi_0, \ldots, \phi_n$. A configuration is allowable if it does not assign falsehood to any axiom under (II).

**Lemma 3.** A sentence $S$ of $T$ containing no variables is unprovable if and only if $S$ is assigned falsehood by some allowable configuration.

**Proof.** If $S$ is not provable, it is false in some model $M$ of $T$. Since $S$ has no variables, some configuration 'includes' in $M$ assigns to $S$ the same truth value as $M$, namely falsehood. Since $M$ is a model, the configuration is allowable.

To prove the converse, it is clearly necessary to prove that any allowable configuration can be extended to a model of $T$. It is readily seen that such a configuration can be extended to a model for (I) and (II). Since every extension of a model for (I) and (II) is again a model for (I) and (II), we need only prove: any model $M$ for (I) and (II) can be extended to a model for (III).

By the completeness theorem, we need only show that $M$ can be a model for a finite number of axioms under (III). Now if we write these axioms with different bound variables, form their conjunction, and bring quantifiers to the front, we obtain a new axiom under (III). Thus we need only show that $M$ can be extended to a model for one axiom under (III).

Let $y_1, \ldots, y_k$ be the free variables in (III), and let $a_0, \ldots, a_n$ be individuals of $M$. Introduce new individuals $b_1, \ldots, b_n$. It is easy to see that the functions of $M$ can be extended to the enlarged domain so that the scope of (III) is true when $y_1, \ldots, y_k, x_1, \ldots, x_n$ are interpreted as $a_0, \ldots, a_n, b_1, \ldots, b_n$. If we do this for each $k$-tuple $a_0, \ldots, a_n$ and repeat the whole process infinitely often, we obtain the desired extension of $M$.

It follows from Lemma 3 that $T$ is consistent; for $\theta \neq \theta$ is assigned falsehood by any non-empty configuration.

We now show the decision problem for $T$ is reducible to that of the set $B$, so that $T$ has degree $< \theta$. Since the proof of Lemma 2 enables us to effectively obtain a quantifier free sentence equivalent to a given sentence, and since the quantifier-free sentence will be closed whenever the given sentence is closed, we may confine ourselves to closed quantifier-free sentences, i.e., sentences without variables. The set of provable sentences without variables is recursively enumerable since $T$ is axiomatizable. Hence we need only show that the set of unprovable sentences without variables is recursively enumerable in $B$.

Since a configuration is a finite object, we can enumerate all configurations. The allowable of a configuration $a_0, \ldots, a_n, y_0, \ldots, y_k$ may be expressed as follows: if $y_i(a_j) = a_j$ and $t = s$, then $F(a, s) = f$. Hence given a configuration and a decision method for $B$, we can decide if the configuration is allowable. Finally, given a configuration and a sentence $S$ without variables, we can decide whether or not the configuration assigns falsehood to $S$. Using Lemma 3, it follows that the set of unprovable sentences without variables is recursively enumerable in $B$.

Now consider a sentence $A(x)$ containing no free variable other than $x$. Let $\phi_0, \phi_1, \ldots, \phi_n$ include all the function symbols in $A(x)$. Let $B'$ be the set of $(\epsilon, e)$ in $B$ such that $\epsilon < e$. Just as above, we can show that the decision problem for the set of sentences

$A(\overline{0})$, $A(\overline{1})$, ...

is reducible to the decision problem for $B'$. But $B'$ is recursive, since $B'_{\epsilon}$ is recursive for all $\epsilon$. Hence the decision problem for the set (10) is solvable. This means that the set which $A(x)$ weakly represents is recursive.

**References**


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