Cartesian products of Baire spaces

by

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1. Introduction. Following Bourbaki ([2], p. 75), a topological space is called a Baire space if every non-empty open set is of second category. Thus a Baire space is one which is of second category at every point, or equivalently, one in which the Baire category theorem is true. If Y is either an open or a dense subset of a space X, and if $E \subset Y$, then the category of E relative to Y is the same as the category of E relative to X. Hence if Z is dense in an open subset U of X then Z is a Baire subspace of X if and only if Z is of second category (relative to X, U, or Z) at each of its points.

A family $\boldsymbol{B}$ of non-empty open sets in a topological space will be called a *pseudo-base* if every non-empty open set contains at least one member of $\boldsymbol{B}$. A pseudo-base $\boldsymbol{B}$ is called *locally countable* if each member of $\boldsymbol{B}$ contains only countably many members of $\boldsymbol{B}$.

We shall denote Cartesian products by $X \times Y$ and by $\boldsymbol{P}\{X_\alpha\colon \alpha \in A\}$. If $E \subset X \times Y$ the section of E corresponding to any $x \in X$ will be denoted by $E(x) = \{y\colon (x, y) \in E\}$.

The following proposition was proved by Kuratowski and Ulam [9] for metric spaces. In the case of the product of two intervals the result had already been noted by L. E. J. Brouwer ([4], footnote p. 218).

(1.1) *If X and Y are topological spaces and Y has a countable pseudo-base, and if E is nowhere dense (resp. of first category) in $X \times Y$, then $E(x)$ is nowhere dense (resp. of first category) in Y for all x except a set of first category in X.*

Proof. Let $\{V_n\}$ be a countable pseudo-base in Y, let G be a dense open set in $X \times Y$, and let H_n be the X-projection of $(X \times V_n) \cap G$ $(n = 1, 2, \ldots)$. Then H_n is open and dense in X. For any $x \in \bigcap H_n$ each of the sets $G(x) \cap V_n$ is non-empty, hence $G(x)$ is dense open in Y. By taking G disjoint to E the principal statement follows. The parenthetical statement is a corollary.

The following generalization of another theorem of Kuratowski and Ulam is due essentially to Sikorski [14].

THEOREM 1. *If X and Y are topological spaces, if at least one of them has a locally countable pseudo-base, and if $A \subset X$ and $B \subset Y$, then*

(1.2) *$A \times B$ is of first category in $X \times Y$ if and only if A is of first category in X or B is of first category in Y, and*

(1.3) *$A \times B$ is of first category at (x, y) if and only if A is of first category at x or B is of first category at y.*

Proof. Suppose that Y has a locally countable pseudo-base $\boldsymbol{B}$, that $A \times B$ is of first category, and that A is of second category. For each $V \in \boldsymbol{B}$, $A \times (B \cap V)$ is of first category in $X \times V$. Since V has a countable pseudo-base, it follows from (1.1) that for some $x \in A$, $(A \times (B \cap V))(x) = B \cap V$ is of first category in V and therefore in Y. Hence B is of first category at each point of the set $\bigcup \{V: V \in \boldsymbol{B}\}$, which is dense in Y. By the Banach category theorem [1] it follows that B is of first category in Y. Similarly, if X has a locally countable pseudo-base, if $A \times B$ is of first category, and if B is of second category, then A is of first category. Conversely, if either A or B is of first category then $A \times B$ is of first category, without any restriction on X or Y. Thus (1.2) is proved. (1.3) is a corollary.

The following theorem is an immediate consequence of (1.3).

THEOREM 2. *If X and Y are Baire spaces, and if at least one of them has a locally countable pseudo-base, then $X \times Y$ is a Baire space.*

The essential content of Theorem 1, namely the "only if" parts of (1.2) and (1.3), can also be deduced from Theorem 2, with the help of the Banach category theorem and our earlier remarks about relativization. For, under the hypotheses of Theorem 1, suppose that both A and B are of second category. Then there exist non-empty open sets U and V such that A is of second category at each point of U, and B is of second category at each point of V. Hence $A_1 = A \cap U$ and $B_1 = B \cap V$ are Baire subspaces of X and Y respectively, and at least one of them has a locally countable pseudo-base. Assuming Theorem 2, it follows that $A_1 \times B_1$ is of second category relative to itself. Since $A_1 \times B_1$ is dense in the open subset $U \times V$ of $X \times Y$, it follows that $A_1 \times B_1$, and therefore $A \times B$, is of second category in $X \times Y$. Thus the "only if" part of (1.2) is re-obtained, and that of (1.3) follows.

In connection with Theorem 2 it should be noted that a product space can be a Baire space only if each of the factors is a Baire space.

Theorems 1 and 2 generalize easily to products of finitely many spaces each of which has a locally countable pseudo-base. But, as Kuratowski and Ulam remarked, Theorem 1 does not generalize to infinite products, even when each space has a countable base. (For example, let X be the interval $[0, 1]$ and let A be the interval $[0, \frac{1}{2}]$. Then A^{∞} is nowhere dense in X^{∞}, but A is of second category at each of its points.) Nevertheless, in this case Theorem 2 can be so generalized, despite the fact that it is essentially equivalent to Theorem 1.

THEOREM 3. *The Cartesian product of any family of Baire spaces, each of which has a countable pseudo-base, is a Baire space.*

This result may be compared with a theorem of Bourbaki ([3], p. 4, ex. 7), according to which the Cartesian product of any family of complete metric spaces is a Baire space. The two theorems overlap, but neither includes the other. In § 3 we make use of Theorem 3 to obtain a category analogue of the zero-one law. In § 5 we shall consider a generalization of Bourbaki's theorem.

In (1.1) the hypothesis that Y has a countable pseudo-base cannot be relaxed even to a locally countable base, as Kuratowski and Ulam showed by a simple example. It is much more difficult to show that the countability hypothesis in Theorems 1 and 2 cannot be omitted. Nevertheless, assuming the continuum hypothesis, this question is settled in § 4, where we construct an example of a Baire space whose square is not a Baire space. Whether Theorems 1 and 2 are true for arbitrary metric spaces still remains an open question (cf. [14]).

Before proceeding to the proof of Theorem 3, let us make a few remarks about the notion of a pseudo-base. It is obvious that any base is a pseudo-base, and that if a space has a countable pseudo-base then it has a countable dense subset. The Stone-Čech compactification of the space of positive integers furnishes an example of a space which has a countable dense set of isolated points, and therefore a countable pseudo-base, but no countable base [13]. An uncountable set X in which a subset is defined to be closed if and only if it is either finite or equal to X is an example of a space which has a countable dense subset but no countable pseudo-base. The property of possessing a countable pseudo-base is therefore logically intermediate between that of having a countable base and that of having a countable dense subset. For metrizable spaces all three properties are equivalent. It is easy to show that if a space X has a countable dense subset D, and if the first countability axiom is satisfied at each point of D, then X has a countable pseudo-base. The first example above, however, shows that even such a space need not have a countable base.

2. Proof of Theorem 3. From the proof of (1.1) we can also draw the following conclusion.

(2.1) *If X and Y are topological spaces, and Y has a countable pseudo-base, then for any sequence $\{G_n\}$ of dense open sets in $X \times Y$ there exists a set X_0 of first category in X such that for any $x \in X - X_0$ each of the sets $G_n(x)$ $(n = 1, 2, \ldots)$ is dense open in Y.*

(2.2) *If each of two spaces X and Y has a countable pseudo-base, then so does $X \times Y$.*

Proof. Obvious.

Now consider any sequence $\{X_i\}$ of spaces, each of which has a countable pseudo-base. For any $0 \leqslant m < n$ write

$$X^{(n)} = \mathop{P}_{i=1}^{n} X_i\,, \quad X^{(m,n)} = \mathop{P}_{i=m+1}^{n} X_i\,, \quad Y^{(n)} = \mathop{P}_{i=n+1}^{\infty} X_i\,.$$

As usual, we shall identify product spaces that arise from different groupings of factors. For example, $X^{(n)} = X^{(m)} \times X^{(m,n)}$ and $X^{(m)} \times Y^{(m)} = X^{(n)} \times Y^{(n)}$, for $0 < m < n$. As a result of this convention we can make the following assertions.

(2.3) *If* $x \in X, y \in Y$, *and* $G \subset X \times Y \times Z$, *then* $G(x)(y) = G(x, y) = \{z\colon (x, y, z) \in G\}$. *If* $\{y\} \times Z \subset G(x)$ *then* $\{(x, y)\} \times Z \subset G$.

For any fixed m, the following result follows from (2.2) and Theorem 2 by induction on n.

(2.4) *If* $0 \leqslant m < n$ *then* $X^{(m,n)}$ *has a countable pseudo-base. If each of the spaces* X_i *is a Baire space then so is* $X^{(m,n)}$.

(2.5) *The space* $X = \mathop{P}_{i=1}^{\infty} X_i$ *has a countable pseudo-base.*

Proof. By (2.4), $X^{(n)}$ has a countable pseudo-base, say $\{U(n, i)\}$, for each n. The sets $U(n, i) \times Y^{(n)}$, where n and i are arbitrary positive integers, constitute a countable pseudo-base in X.

We are now prepared to prove Theorem 3 for denumerable products [11].

(2.6) *If each of the spaces* X_i *is a Baire space then so is* $X = \mathop{P}_{i=1}^{\infty} X_i$.

Proof. Let $\{G_n\}$ be a decreasing sequence of dense open sets in X, and let G be any non-empty open set. We need to show that $G \cap \bigcap G_n$ is non-empty. Choose n_1 so that $G \cap G_1$ contains a basic open set $U_1 \times Y^{(n_1)}$, where U_1 is a non-empty open subset of $X^{(n_1)}$. By (2.4), U_1 is of second category, and by (2.1) applied to $X^{(n_1)} \times Y^{(n_1)}$ there exists a point $z_1 \in U_1$ such that $G_n(z_1)$ is dense open in $Y^{(n_1)}$ for every n. Moreover,

$$\{z_1\} \times Y^{(n_1)} \subset U_1 \times Y^{(n_1)} \subset G \cap G_1\,.$$

Proceeding by induction on k, let us suppose that we have defined integers $0 = n_0 < n_1 < \ldots < n_k$ and points $z_i \in X^{(n_{i-1}, n_i)}$ $(i = 1, \ldots, k)$ such that

(a) $\{(z_1, \ldots, z_k)\} \times Y^{(n_k)} \subset G \cap G_k$, and

(b) $G_n(z_1, \ldots, z_k)$ is dense open in $Y^{(n_k)}$ for $n = 1, 2, \ldots$.

Then for some integer $n_{k+1} > n_k$ the set $G_{k+1}(z_1, \ldots, z_k)$ contains a set of the form $U_{k+1} \times Y^{(n_{k+1})}$, where U_{k+1} is a non-empty open subset of $X^{(n_k, n_{k+1})}$. By (2.1) applied to $X^{(n_k, n_{k+1})} \times Y^{(n_{k+1})}$ there exists a point $z_{k+1} \in U_{k+1}$ such that each of the sets $G_n(z_1, \ldots, z_k)(z_{k+1})$ $(n = 1, 2, \ldots)$ is dense open in $Y^{(n_{k+1})}$. Hence, by (2.3), condition (b) is satisfied with k replaced by $k+1$. Moreover, $\{z_{k+1}\} \times Y^{(n_{k+1})} \subset U_{k+1} \times Y^{(n_{k+1})} \subset G_{k+1}(z_1, \ldots, z_k)$. Hence, by (2.3), $\{(z_1, \ldots, z_{k+1})\} \times Y^{(n_{k+1})} \subset G_{k+1}$. By hypothesis (a), the left member of this is also contained in G. Hence (a) is satisfied with k replaced by $k+1$.

Therefore sequence $\{n_k\}$ and $\{z_k\}$ can be so defined that (a) is satisfied for every positive integer k. Such a sequence $\{z_k\}$ defines a point $x \in X$ such that $x \in G \cap G_k$ for every k. Hence $G \cap \bigcap G_n$ is non-empty.

Now consider an uncountable family $\{X_\alpha\colon \alpha \in A\}$ of spaces each of which has a countable pseudo-base, and let X denote their Cartesian product.

(2.7) *Any disjoint family of basic open sets in* X *is countable* [1].

Proof. For each $\alpha \in A$ let D_α be a countable dense subset of X_α. Assign positive weights with sum 1 to the points of D_α. For any set $E \subset X_\alpha$ let $\mu_\alpha(E)$ be the sum of the weights of the points of $D_\alpha \cap E$. Then μ_α is a measure defined for all subsets of X_α. Moreover, $\mu_\alpha(X_\alpha) = 1$, and $\mu_\alpha(U) > 0$ for every non-empty open set U in X_α. Let (X, μ) denote the product of the measure spaces $\{(X_\alpha, \mu_\alpha)\colon \alpha \in A\}$ ([7], § 38 (2)). Then μ is defined and positive for every basic open set in X. Since $\mu(X) = 1$, it follows that any disjoint family of basic open sets in X is countable.

To complete the proof of Theorem 3, assume now that each of the spaces X_α is a Baire space and let $\{G_n\}$ be any sequence of dense open sets in X. For each n let $\{U_{n,m}\colon m = 1, 2, \ldots\}$ be a maximal disjoint family of basic open sets contained in G_n. (By (2.7) such a family must be countable.) Then the set $H_n = \bigcup_{m=1}^{\infty} U_{n,m}$ is an open set contained in G_n and dense in X. $U_{n,m}$ is a cylinder set based on a product of finitely many of the X_α, say on $P\{X_\alpha\colon \alpha \in A_{n,m}\}$. Let A_0 denote the countable set $\bigcup_{n,m} A_{n,m}$. Put $X_0 = P\{X_\alpha\colon \alpha \in A_0\}$, and $Y_0 = P\{X_\alpha\colon \alpha \in A - A_0\}$. Then each of the sets H_n is a cylinder set based on X_0, say $H_n = K_n \times Y_0$. Since H_n is dense open in X, K_n is dense open in X_0, for each n. From (2.6) it follows that $\bigcap K_n$ is dense in X_0. Hence $\bigcap H_n$, and therefore $\bigcap G_n$, is dense in X. Therefore X is a Baire space.

3. A category analogue of the zero-one law. Let X be the Cartesian product of a family $\{X_\alpha\colon \alpha \in A\}$ of sets. A set $E \subset X$ will be called a *tail set* if whenever $x = \{x_\alpha\}$ and $y = \{y_\alpha\}$ are points of X, and $x_\alpha = y_\alpha$ for all but a finite number of values of α, then E contains

[1] For an alternative proof, not depending on measure theory, see E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math. 34 (1947), pp. 127-143.

both x and y or neither. This definition can be cast in a more convenient form. For each finite subset J of A let $X(J) = \boldsymbol{P}\{X_a\colon a \in J\}$ and $Y(J) = \boldsymbol{P}\{X_a\colon a \in A-J\}$. Then E is a tail set if and only if for each finite set $J \subset A$ there is a set $B \subset Y(J)$ such that $E = X(J) \times B$.

The zero-one law of Kolmogoroff in the theory of probability states that in the product of any family of normalized measure spaces, any measurable tail set is of measure zero or one ([7], § 46 (3)). (The theorem is usually stated for the product of a sequence of spaces, but is easily seen to hold more generally.) A subset E of a topological space is said to have the property of Baire if E can be represented in the form $G+P$, where G is open, P is of first category, and "+" denotes symmetric difference. In a Baire space the complement of a set of first category is called *residual*. In view of the well-known analogy between measure and category, the following theorem (announced for sequential products in [11]) may be described as a category analogue of the zero-one law.

THEOREM 4. *If X is the Cartesian product of a family $\{X_a\colon a \in A\}$ of Baire spaces, each of which has a countable pseudo-base, then X is a Baire space, and any tail set having the property of Baire in X is either of first category or residual.*

It has already been shown that X is a Baire space, hence it suffices to prove the following lemma. Note that in this lemma the spaces X_a are no longer assumed to be Baire spaces.

(3.1) *If X is the Cartesian product of a family $\{X_a\colon a \in A\}$ of spaces, each of which has a countable pseudo-base, and if E is any tail set having the property of Baire in X, then either E or $X-E$ is of first category in X.*

Proof. Suppose that $X-E$ is of second category. Then there exists an open set G of second category and a set P of first category such that $X-E = G+P$. Let $\{G_i\}$ be a maximal disjoint family (countable by (2.7)) of basic open sets contained in G. Then $G - \bigcup G_i$ is nowhere dense. Since G is of second category, at least one of the sets G_i must be of second category, say $G_i = U \times Y(J)$, where U is an open subset of $X(J)$. U must be of second category in $X(J)$, by (1.2). By definition, the tail set E can be represented in the form $E = X(J) \times B$, where $B \subset Y(J)$. Hence $E \cap G_i = (X(J) \times B) \cap (U \times Y(J)) = U \times B$. But $E \cap G_i \subset E \cap G = E \cap ((X-E)+P) = E \cap P \subset P$. Therefore $U \times B$ is of first category in $X(J) \times Y(J)$. Since $X(J)$ has a countable pseudo-base, and U is of second category, it follows from (1.2) that B is of first category in $Y(J)$. Hence $E = X(J) \times B$ is of first category in X.

4. A Baire space whose square is not a Baire space. The following result shows that Theorems 1, 2, 3, and 4 cannot be proved without some restriction in place of the countability hypothesis.

THEOREM 5. *Assuming the continuum hypothesis, there exists a completely regular Baire space whose Cartesian product with itself is of first category.*

Let I denote the unit interval, $\boldsymbol{M}$ the field of Lebesgue measurable subsets of I, $\boldsymbol{N}$ the class of nullsets in $\boldsymbol{M}$, and $\boldsymbol{A}$ the Boolean algebra $\boldsymbol{M}/\boldsymbol{N}$. Let m denote Lebesgue measure on either $\boldsymbol{M}$ or $\boldsymbol{A}$. Let X be the Boolean space corresponding to $\boldsymbol{A}$. X is a compact Hausdorff space, therefore a Baire space. Let φ denote a Boolean isomorphism of $\boldsymbol{A}$ onto the field $\boldsymbol{F}$ of closed open subsets of X. The set function $m(\varphi^{-1}(A))$ is countably additive on $\boldsymbol{F}$, hence it can be extended to a measure μ on the σ-field generated by $\boldsymbol{F}$. It is easy to verify that $\mu(X) = 1$, and that $\mu(A) = 0$ implies that A is nowhere dense. Moreover, any set of first category in X is nowhere dense ([10], [8], [12]). For any set $E \subset X \times X$ and any $x \in X$ let us now denote the two sections of E determined by x by $E_x = \{y\colon (x, y) \in E\}$ and $E^x = \{y\colon (y, x) \in E\}$. Let $D = \{(x, y)\colon x = y\}$.

(4.1) *There exists an F_σ set E of first category in $X \times X$ such that* (i) *$D \subset E$, and* (ii) *for each $x \in X$, E_x and E^x differ from X by nowhere dense sets.*

Proof. Let G be an open subset of the square $I \times I$ with the property that if A and B are any two subsets of I with $m(A) > 0$ and $m(B) > 0$ then $(m \times m)(G \cap (A \times B)) > 0$, and such that $(m \times m)(G) < \varepsilon$. (It suffices to take a dense open subset W of $(-1, 1)$ with $m(W) < \varepsilon$ and define $G = \{(x, y)\colon 0 < x < 1, 0 < y < 1, x-y \in W\}$ ([6], Th. 1).) Represent G as a disjoint union of rectangles $I_n \times J_n$. Let ψ be the mapping of $\boldsymbol{M}$ onto $\boldsymbol{F}$ obtained by composing the canonical mapping of $\boldsymbol{M}$ onto $\boldsymbol{A}$ with φ. Put $U_n = \psi(I_n)$ and $V_n = \psi(J_n)$. Then the set $H = \bigcup_n (U_n \times V_n)$ is an open subset of $X \times X$ with $(\mu \times \mu)(H) < \varepsilon$. To show that H is dense in $X \times X$ consider any two non-empty closed open sets U and V in X. Choose A and B in $\boldsymbol{M}$ such that $\psi(A) = U$ and $\psi(B) = V$. Then $m(A) > 0$, $m(B) > 0$, and therefore $(m \times m)((I_n \times J_n) \cap (A \times B)) > 0$ for some n. Since $H \cap (U \times V) \supset (U_n \times V_n) \cap (U \times V) = (U_n \cap U) \times (V_n \cap V)$, it follows that $(\mu \times \mu)(H \cap (U \times V)) \geqslant \mu(U_n \cap U) \cdot \mu(V_n \cap V) = m(I_n \cap A) \cdot m(J_n \cap B) = (m \times m)((I_n \times J_n) \cap (A \times B)) > 0$. Hence H is dense in $X \times X$. Let $\{H_i\}$ be a sequence of dense open sets in $X \times X$ such that $(\mu \times \mu)(H_i) \to 0$. Then $F = (X \times X) - \bigcap H_i$ is an F_σ set of first category in $X \times X$, with $(\mu \times \mu)(F) = 1$. By Fubini's theorem there exists a set $N \subset X$ with $\mu(N) = 0$ such that $\mu(F_x) = \mu(F^x) = 1$ for every $x \in X - N$. Hence $X - F_x$ and $X - F^x$ are nowhere dense for every $x \in X - N$. Since N is nowhere dense in X, both of the sets $\bar{N} \times X$ and $X \times \bar{N}$ are closed and nowhere dense in $X \times X$. The set D is also closed and nowhere dense in $X \times X$, since X has no isolated points. Hence the set $E = F \cup (\bar{N} \times X) \cup (X \times \bar{N}) \cup D$ has all the properties required in (4.1).

To complete the proof of Theorem 5, note that A and F have power $\mathfrak{c}$. Hence, assuming the continuum hypothesis, there exists a mapping $\gamma \to W_\gamma$ of the ordinals $0 \leqslant \gamma < \Omega$ of first and second class onto the class of all non-empty closed open subsets of X. Assume also a well-ordering of X. Let x_0 be the first element of W_0. Then $(x_0, x_0) \in E$. If $0 < \gamma < \Omega$ and if x_α has been so defined for $0 \leqslant \alpha < \gamma$ that the condition

$$(\mathrm{P}_\gamma) \quad x_\alpha \in W_\alpha \quad \text{and} \quad (x_\alpha, x_\beta) \in E \quad \text{for all} \quad 0 \leqslant \alpha < \gamma, \ 0 \leqslant \beta < \gamma,$$

is satisfied, let x_γ be the first point that belongs to the set $W_\gamma \cap \bigcap_{0 \leqslant \alpha < \gamma} (E_{x_\alpha} \cup E^{x_\alpha})$. (Note that this set is of second category, therefore non-empty.) Then condition $(\mathrm{P}_{\gamma+1})$ is satisfied. Hence x_γ is defined by transfinite induction for all $0 \leqslant \gamma < \Omega$, and the set $Z = \{x_\gamma: 0 \leqslant \gamma < \Omega\}$ is a dense subset of X such that $Z \times Z \subset E$. Z is of second category at every point, since only nowhere dense sets are of first category in X. Therefore Z, considered as a subspace of X, is a completely regular Baire space. Because $Z \times Z$ is of first category and dense in $X \times X$ it is also of first category in itself.

5. Products of complete spaces. A topological space is called *quasi-regular* if every non-empty open set contains the closure of some non-empty open set [12]. It is convenient to introduce also the following definition: a topological space X is called *pseudo-complete* if X is quasi-regular and if there exists a sequence $\{\boldsymbol{B}(n)\}$ of pseudo-bases in X with the property that whenever $U_n \in \boldsymbol{B}(n)$ and $U_n \supset \overline{U}_{n+1}$ then $\bigcap U_n \neq 0$. Evidently pseudo-completeness is a topological property of X.

Any complete metric space is pseudo-complete. (Take for $\boldsymbol{B}(n)$ the class of all spherical neighborhoods of radius $\leqslant 1/n$.) Likewise any locally compact Hausdorff space is pseudo-complete. (Take for $\boldsymbol{B}(n)$ all non-empty open sets with compact (= bicompact) closure. Note that a locally compact Hausdorff space is regular and therefore quasi-regular.) Hence both parts of the usual statement of Baire's category theorem ([2], p. 76) are included in the following formulation:

(5.1) *Any pseudo-complete space is a Baire space.*

Proof. Let $\{G_n\}$ be any sequence of dense open sets, and let G be any non-empty open set. Since the space is quasi-regular, there exists a set $U_1 \in \boldsymbol{B}(1)$ such that $\overline{U}_1 \subset G \cap G_1$. Having chosen U_{n-1} let U_n be so chosen that $U_n \in \boldsymbol{B}(n)$ and $\overline{U}_n \subset U_{n-1} \cap G_n$. Then $\bigcap U_n \neq 0$. Since $U_n \subset G \cap G_n$, it follows that $G \cap \bigcap G_n \neq 0$.

According to Čech [5], a space is called *topologically complete* if it is homeomorphic to a G_δ subset of some compact Hausdorff space. Čech showed that any complete metric space is topologically complete, and that any topologically complete space is a Baire space. The latter result is also a consequence of (5.1) and the following lemma.

(5.2) *Let Y be a quasi-regular space with a pseudo-base $\boldsymbol{B}$ consisting of sets whose closure is countably compact. Then any dense G_δ subspace X of Y is pseudo-complete.*

Proof. As a dense subspace of a quasi-regular space, X is quasi-regular. Let $X = \bigcap G_n$, where G_n is open in Y. For each n let $\boldsymbol{B}(n)$ be the class of all sets of the form $H \cap X$, where $H \in \boldsymbol{B}$ and $\overline{H} \subset G_n$. Evidently $\boldsymbol{B}(n)$ is a class of non-empty open sets relative to X. Any non-empty open set relative to X has the form $G \cap X$, where G, and therefore $G \cap G_n$, is non-empty and open in Y. Since Y is quasi-regular, there exists a set $H \in \boldsymbol{B}$ such that $\overline{H} \subset G \cap G_n$. Hence $H \cap X \in \boldsymbol{B}(n)$ and $H \cap X \subset G \cap X$. Thus $\boldsymbol{B}(n)$ is a pseudo-base in X, for each n.

Suppose that $U_n \in \boldsymbol{B}(n)$ and that $U_n \supset \overline{U}_{n+1} \cap X$ (= the closure of U_{n+1} relative to X), for each n. Let $U_n = H_n \cap X$, where $H_n \in \boldsymbol{B}$ and $\overline{H}_n \subset G_n$. Since $U_n \supset U_{n+1}$, $\{\overline{U}_n\}$ is a decreasing sequence of non-empty closed subsets of the countably compact set $\overline{H}_1$. Hence $\bigcap \overline{U}_n \neq 0$. Since $\overline{H}_n \subset G_n$, we have $\overline{U}_n \subset G_n$, and therefore $\bigcap \overline{U}_n \subset \bigcap G_n = X$. Hence $\bigcap U_n \supset \bigcap (\overline{U}_{n+1} \cap X) = (\bigcap \overline{U}_{n+1}) \cap X = \bigcap \overline{U}_n \neq 0$, and so X is pseudo-complete.

On the other hand, a compact T_1-space is not necessarily a Baire space. For example, in a countably infinite set X define a subset to be closed if and only if it is either finite or equal to X.

THEOREM 6. *The Cartesian product of any family of pseudo-complete spaces is pseudo-complete.*

Proof. Let X be the product of a family $\{X_\alpha: \alpha \in A\}$ of pseudo-complete spaces. Any basic open set U in X is of the form $U = (\boldsymbol{P}_{i=1}^{n} U_i) \times Y$, where U_i is a non-empty open subset of X_{α_i} $(i = 1, \ldots, n)$ and $Y = \boldsymbol{P}\{X_\alpha: \alpha \in A - \{\alpha_1, \ldots, \alpha_n\}\}$. Since X_{α_i} is quasi-regular, there exist non-empty open sets V_i in X_{α_i} such that $\overline{V}_i \subset U_i$ $(i = 1, \ldots, n)$. Hence $(\boldsymbol{P}_{i=1}^{n} V_i) \times Y$ is a non-empty open set in X whose closure is contained in U. Thus X is quasi-regular.

Let $\boldsymbol{B}_\alpha(n)$ be a sequence of pseudo-bases in X_α with the property that $U_\alpha(n) \in \boldsymbol{B}_\alpha(n)$, $U_\alpha(n) \supset \overline{U}_\alpha(n+1)$, implies $\bigcap_n U_\alpha(n) \neq 0$. We may assume that $X_\alpha \in \boldsymbol{B}_\alpha(n)$ for all n and α. Let $\boldsymbol{B}(n)$ denote the family of all sets of the form $\boldsymbol{P}\{U_\alpha: \alpha \in A\}$, where $U_\alpha \in \boldsymbol{B}_\alpha(n)$ for every $\alpha \in A$, and $U_\alpha = X_\alpha$ for all but a finite number of values of α. Evidently $\boldsymbol{B}(n)$ is a pseudo-base in X, for each n. Suppose that $U_n \in \boldsymbol{B}(n)$ and that $U_n \supset \overline{U}_{n+1}$ for every n. If $U_n = \boldsymbol{P}\{U_\alpha(n): \alpha \in A\}$ then $\boldsymbol{P}\{\overline{U}_\alpha(n+1): \alpha \in A\} \subset \overline{U}_{n+1} \subset U_n = \boldsymbol{P}\{U_\alpha(n): \alpha \in A\}$. It follows that $\overline{U}_\alpha(n+1) \subset U_\alpha(n)$ for all n and α. Hence, for each $\alpha \in A$, there exists a point $x_\alpha \in \bigcap_n U_\alpha(n)$. The point $x = \{x_\alpha\}$ belongs to $\bigcap U_n$.

The theorem of Bourbaki mentioned in § 1 is an immediate consequence of this theorem and the preceding lemmas, as is also the following more general result.

COROLLARY. *The Cartesian product of any family of spaces topologically complete in the sense of Čech is a Baire space.*

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Sur un problème de la logique à n valeurs

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Dans la logique à n valeurs chaque proposition admet une des n valeurs $0, 1, 2, \ldots, n-1$ et chaque fonction logique d'un nombre fini k de propositions peut être déterminée par une fonction $f(x_1, x_2, \ldots, x_k)$ de k variables, définie pour $x_i = 0, 1, \ldots, n-1$ $(i = 1, 2, \ldots, k)$ et ne prenant que les valeurs $0, 1, \ldots, n-1$. Désignons par $\varphi(x, y)$ et $\psi(x, y)$ des fonctions de deux variables, définies pour $x = 0, 1, \ldots, n-1$, $y = 0, 1, \ldots, n-1$ comme il suit: $\varphi(x, y)$, resp. $\psi(x, y)$ est le reste de la division du nombre $x+y$, resp. xy par le nombre n. (Pour $n = 2$ ces fonctions déterminent la somme et le produit logique.)

Le but de cette Note est de trouver quel doit être le nombre naturel $n > 1$ pour que toute fonction logique d'un nombre fini de propositions dans la logique à n valeurs se réduise (par superpositions) aux trois fonctions

$$(1) \qquad 1\,(^1)\,,\ \varphi(x, y) \text{ et } \psi(x, y)\,.$$

Je démontrerai notamment que, pour qu'il en soit ainsi, il faut et il suffit que n soit un nombre premier.

En langage mathématique ce théorème peut être exprimé comme il suit:

THÉORÈME. *Pour que toute fonction d'un nombre fini de variables, définie pour les valeurs $0, 1, \ldots, n-1$ de ces variables, où n est un nombre naturel > 1, et ne prenant que les valeurs $0, 1, \ldots, n-1$, puisse être exprimée (par superpositions) à l'aide des fonctions* (1), *il faut et il suffit que n soit un nombre premier.*

Démonstration. Soit n un nombre premier. Désignons par $F_k^{(n)}$ la famille de toutes les fonctions $f(x_1; x_2, \ldots, x_k)$ de k variables définies pour $x_i = 0, 1, \ldots, n-1$ $(i = 1, 2, \ldots, k)$ et ne prenant que les valeurs $0, 1, \ldots, n-1$. La famille $F_k^{(n)}$ est évidemment formée de n^{n^k} fonctions distinctes. D'autre part soit $P_k^{(n)}$ la famille de tous les polynômes en

(1) C'est-à-dire la fonction dont la valeur est le nombre 1 pour $x = 0, 1, \ldots, n-1$.