Collections of convex sets which cover a Banach space

by

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1. Introduction. It is a well-known theorem of A. H. Stone ([3], p. 160) that each open cover of a metric space has a locally finite refinement (1). However, an example has recently been found ([4]) of a metric space $M$ with a base $B$ such that there is an open cover of $M$ which does not have a locally finite refinement consisting of elements in $B$. This example suggested the theorem which is the subject of this paper.

Theorem. For any cover $U$ of a reflexive, infinite-dimensional Banach space $B$, where $U$ consists of bounded convex sets, there is a point $x$ in $B$ such that each neighborhood of $x$ meets infinitely many elements of $U$. That is, $U$ is not locally finite.

In the proof of the Theorem which is given in section 3, the fact that each closed, convex set which does not contain 0 lies on one side of a hyperplane is used to reduce the problem to a finite-dimensional one. A rather technical consequence of Brouwer’s fixed point theorem, which is proved in section 2, completes the proof.

For convenience, only real Banach spaces will be considered.

2. Lemma 1 and notation. The subject of this section is a lemma which is similar to ([22], Proposition IV LD), but as will be noted, more information is required about a situation which is slightly different from that treated in this result.

First, some notation. Let $(a_i: i = 1, 2, \ldots)$ be a countable collection of linearly independent points of a Banach space. For each integer $s > 0$, let $I^s$ be the set of those elements of the form $\sum a_i x_i$ with $0 \leq a_i \leq 1$ for $i = 1, 2, \ldots, s$. Of course, $I^s$ is isomorphic to an $s$-dimensional cube.

Let $I' = \emptyset$ and $I^{-1} = \emptyset$. Let $C_i = (x \in I'; a_i = 0)$ and $C_i = (x \in I'; a_i = 1)$. If $U$ is a cover of $I'$, let $U = (U \in \mathcal{U}; U' \cap I' \neq \emptyset)$, but $U' \cap I^{-1} = \emptyset$ where the closure of a subset $A$ of $I'$ is written $A'$.

(1) A collection $\mathcal{V}$ of subsets of a topological space $X$ is a locally finite refinement of a cover $\mathcal{U}$ of $X$, if $\mathcal{V}$ is a cover for $X$, if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$, and if for each $x \in X$ there is a neighborhood $N$ of $x$ such that $N$ intersects only finitely many elements of $\mathcal{V}$. 
Lemma 1. Let $\mathcal{U}$ be a finite open cover of $\mathbb{I}$ such that (a) if $U \in \mathcal{U}$ and $U \cap C_i \neq \emptyset$, then $U \cap C_i \cap C_{i+1} = \emptyset$, and (b) if $U \in \mathcal{U}$, then $U \cap C_i \cap C_{i+1} = \emptyset$. Then there is a $\mathcal{U}$-cover for $t = 0, 1, \ldots, s$ such that $\bigcap \{U_t^i : i = 0, 1, \ldots, s \neq 0\} = \emptyset$.

Proof. Since each $U \in \mathcal{U}$ has the property that $U \cap C_{i+1} \neq \emptyset$, then $U_{s+1} = \emptyset$. Define $f(x)$ to be the point in $\mathbb{I}$ whose coordinates are the same as those of $x$, except for the $t$th coordinate. The coordinate of $f(x)$ is $x_{s+1} = -\min \{d(x, U_t^i), d(x, U_t^i) \}; (i, j) \neq 0$. It is easily seen that $f$ is a continuous function from $\mathbb{I}$ to $\mathbb{I}$. Let $f = f_{i=1}^s\{f(x) = d_{s+1}(x, U_t^i)\}$. Suppose that $s = (s_1, s_2, \ldots, s_n)$ is a fixed point of $f$. (Of course, this notation means $s = \sum_{i=1}^n s_i s_i$.) Then $s$ is a fixed point of each $f_i$. However, $s$ is a fixed point of $f_i$ if and only if $s \in \mathcal{U}$ or $s = 0$. Suppose $s_i = 0$ for some $1 \leq i < s$. Since $s \in \mathcal{U}$ implies that $s \in \mathcal{U}$ by virtue of (b), it follows that $s_{i+1} = 0$ because $s$ is a fixed point of $f_{i+1}$. Hence suppose that $s$ has the form $(s_1, s_2, \ldots, s_i, 0, \ldots, 0)$ where $s_i \neq 0$ for $1 \leq i < s$. Since there is an $U \in \mathcal{U}$ such that $s \in \mathcal{U}$, the Lemma follows.

3. Proof of the Theorem. The proof is by contradiction. It will be assumed that $\mathcal{U}$ is a locally finite collection of bounded, convex sets which cover $B$. One may even assume that each $U \in \mathcal{U}$ is open since (a) it suffices to prove the theorem for separable $B$ (b) for separable $B$, $\mathcal{U}$ must be countable, and (c) the $i$th member of $\mathcal{U}$ may be expanded by $i^n$, the resulting collection is a locally finite cover with open, bounded, convex sets. Using this, a countable number of linearly independent points $x_1, x_2, \ldots$ will be chosen such that the intersections of members of $\mathcal{U}$ with $\mathbb{I}$ form a collection which satisfies the conditions of Lemma 1. This will complete the proof, as the following argument shows. Suppose that the $x_i$ have been chosen in the manner indicated. Let $\mathcal{U}^i$, $i = 0, 1, \ldots, s$ be defined as in section 2, except we are interested only in that part of each $U \in \mathcal{U}$ which lies in $\bigcup \{U_t^i : s = 1, 2, \ldots \}$. Each $\mathcal{U}^i$ is a finite collection since $\mathcal{U}$ is locally finite. A collection $U_0, U_1, \ldots, U_s$ will be said to be a chain if $U_0 \in \mathcal{U}$ and $\bigcap \{U_t^i : i = 0, 1, \ldots, s \neq 0\} = \emptyset$. Lemma 1 states that there are arbitrarily long chains. Hence, since $\mathcal{U}$ is finite, a standard argument shows that there is an infinite chain $U_0, U_1, \ldots$, that is, for this collection $\bigcap \{U_t^i : i = 0, 1, \ldots, s \neq 0\} = \emptyset$ for $s = 0, 1, \ldots$. Because each $U_t$ is convex, $\mathcal{U}_t$ is weakly closed (11) p. 2), and hence weakly compact, since $U_t$ is bounded and $B$ is reflexive (11), p. 58). Therefore, there is an $x \in \bigcap \{U_t^i : i = 0, 1, \ldots\}$, and this contradicts the assumption that $\mathcal{U}$ is locally finite.

All that remains is to choose the $s_i$. Let $\mathcal{U}_i = \{U \in \mathcal{U} : 0 < U < U_i\}$. Pick an $x \in B$ such that $\|x\| = \max \{\text{diameter} (U) : U \in \mathcal{U}_i\}.$ Obviously $\mathcal{U}_i \cap \mathcal{U}_i$ has the required property with respect to $P$. Suppose that $s_1, s_2, \ldots, s_i$ is chosen such that $(U \cap C_i) : U \in \mathcal{U}_i$ and $U \cap I \neq \emptyset$ is a collection of Lemma 1. Also assume that for each $1 < i \leq t$ there is an infinite dimensional subspace $B_i$ such that $B_0 < B_1 < \ldots < B_t$ and $(\lim_{i \to t} \mathcal{U}_i) \cap (B_0 + \ldots + B_t) = 0$. Here, $B_0 + \ldots + B_t$ means the set of all $x \in B$ such that $x = u + v$, $u \in B_i$, and $v \in I \cap I$. Suppose that $B_i$ is chosen such that $s_i \neq s_i$ for $j < i$, but suppose that $s_i \neq s_i$, $i = 2, 3, \ldots, s$. Let us show how $B_{s+1}$ is chosen. By the definition of $\mathcal{U}_i$, $(\lim_{i \to t} \mathcal{U}_i) \cap (I^t \cap I) = 0$. Let $\pi$ be the natural projection of $\mathcal{U} + I^t$ onto $B_i$. Then $0 \neq \pi(\lim_{i \to t} \mathcal{U}_i \cap (B_0 + \ldots + B_t))$, hence for $U \in \mathcal{U}$ there is a hyper-space $h_U$ in $B_i$ such that $h_U \cap \pi(\mathcal{U} \cap (B_0 + \ldots + B_t)) = 0$. Define $B_{s+1}$ to be $\{h_0 \cup \mathcal{U} : \mathcal{U} \in \mathcal{U}\}$. Since $x \in \mathcal{U}$ is some $U \in \mathcal{U}_i$, $s_i \neq B_i$. Choose $s_{i+1}$ to be an element of $B_{s+1}$ such that $\|s_{i+1}\| = \max \{\text{diameter} (U) : U \in \mathcal{U}_i\}$. $i = 1, 2, \ldots, s_i$. It can now be shown easily that the conditions of Lemma 1 are satisfied, and hence the theorem follows as we have seen.

Remarks. A slightly stronger result has been proved than was claimed. It has been shown that there is a $t$-dimensional cube $I^t \subset B$ such that infinitely many members of $\mathcal{U}$ meet $I^t$. I do not know if it may be always chosen to be 0. That is, is there a point in infinitely many members of $\mathcal{U}$?

Moreover, it is easy to see that the same approach establishes the analogous result for a covering of an arbitrary infinite-dimensional normed linear space by open convex sets, if the family $\mathcal{V}$ of their closures has the following property. Whenever a subfamily of $\mathcal{V}$ has the finite intersection property, then it has a nonempty intersection.

References

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