Principles of reflection in axiomatic set theory *

by

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It is well-known that the axiom systems for set theory of Zermelo Z and of Zermelo-Fraenkel ZF are not categorical, even if we rule out any model which can be considered to be non-standard. Let R denote the function given by $R(\alpha)=\sum_{\beta<\alpha}\mathfrak{P}(R(\beta))$ (1). The sets $R(\alpha)$ with limit-number α are by all means standard models of Z; the sets $R(\alpha)$ with inaccessible α are by all means standard models of ZF. Thus the axioms of Z and ZF, though describing the state of the universe, do not include statements which establish properties of the universe not shared by sets, or partial universes. If we start with the idea of the impossibility of distinguishing, by specified means, the universe from partial universes we shall be led to the following axiom schemata, listed according to increasing strength. These axiom schemata will be called *principles of reflection* (2) since they state the existence of standard models (by models we shall mean, for the time being, models whose universes are sets) which reflect in some sense the state of the universe.

Q will denote any set theory of the ZF-type.

R_1^Q —The principle of sentential reflection over Q

φ is any sentence of set theory. If φ holds then there exists a standard model of Q in which φ holds also.

R_2^Q —The principle of unbounded sentential reflection over Q

φ is any sentence of set theory. If φ holds then there exist standard models of ZF of arbitrarily great cardinality (or, including arbitrary sets) in which φ holds.

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* A part of the material in this paper is contained in the author's Ph. D. thesis submitted to the Hebrew University. The author wishes to express his gratitude to Professor A. A. Fraenkel and Professor A. Robinson for their guidance and kind encouragement. This paper was written while the author was a Sloan Fellow of the School for Advenced Study at the M. I. T.

(1) $\mathfrak{P}(x)$ denotes the power-set of x.

(2) The principles of reflection are closely related to the notions of arithmetical equivalence and arithmetical extension of Tarski and Vaught [3].

In order to proceed and formulate stronger principles of reflection we need the notion of the standard complete model, i. e., a set u which is a complete set ($x \in u \supset x \subseteq u$) and with the set $\{\langle xy \rangle; x \in y . x, y \in u\}$ as $\in$-relation forms a standard model of Q.

R_3^Q —The principle of partial reflection over Q

$\varphi(x_1, \ldots, x_n)$ is any formula of set theory. If for given $x_1, \ldots, x_n$ $\varphi(x_1, \ldots, x_n)$ holds then there exists a standard complete model u of Q such that $x_1, \ldots, x_n \in u$ and the relation φ of the model holds between them.

R_4^Q —The principle of complete reflection over Q

$\varphi(x_1, \ldots, x_n)$ is any formula of set theory. There exists a standard complete model u of Q such that for each n-tuple $x_1, \ldots, x_n \in u$ the relation φ of the model holds between them if and only if the relation φ of the universe holds between them (3).

We shall use in this paper the notations and results of [1], some of which will be reviewed here in short. S denotes general set theory with the axiom of foundation. The notion of standard model will be that of [1]. $\mathrm{Scm}^{\mathrm{ZF}}(u)$ is the formula stating that u is a standard complete model (in short: scm) of ZF. $\mathrm{Scm}^{\mathrm{ZF}}(u)$ holds if and only if $u = R(a)$, where a is an inaccessible number— $\mathrm{In}(a)$. $\mathrm{Rel}(u, \varphi)$ denotes the relativization of the formula φ to the set u. The functions $P_\eta(a)$ are defined by transfinite induction as follows: $P_0(0)$ is the first inaccessible number; $P_0(\beta+1)$ is the first inaccessible number greater than $P_0(\beta)$; for limit-number a, $P_0(a) = \lim_{\beta<a} P_0(\beta)$. $P_\eta(\beta+1)$ (respectively $P_\eta(0)$) is the first inaccessible number σ greater than $P_\eta(\beta)$ (respectively the first inaccessible number) such that for each $\eta' < \eta$ $\sigma = P_{\eta'}(\gamma)$ for some limit-number γ.

The role of the principle of complete reflection is discussed in [1]. Regarding the hierarchy $\mathrm{S}, \mathrm{ZF}, \mathrm{ZM}, \mathrm{ZM}_2, \ldots$ described in [1] it has been proved there that we pass from a theory Q of this sequence to the theory Q′ following it by adding to Q the principle of complete reflection over Q — R_4^Q. In the present paper we shall discuss the strength of the other principles of reflection. This will be done here only for the case where Q is ZF, but the situation is very much the same for any other Q of the above mentioned sequence.

The principle of sentential reflection over ZF

R_1^{ZF} $\quad \varphi \supset (\exists u)\big(\mathrm{Scm}^{\mathrm{ZF}}(u) . \mathrm{Rel}(u, \varphi)\big)$

or, equivalently

$\varphi \supset (\exists a)\big(\mathrm{In}(a) . \mathrm{Rel}(R(a), \varphi)\big)$

where φ is any sentence.

(3) Another principle of reflection which is apparently stronger than R_4^Q has been proved in [1] to be equivalent to R_4^Q.

Let ZF* denote the theory obtained from ZF by addition of R_1^{ZF}. In ZF*, assuming φ, we have $(\exists u)\big(\mathrm{Scm}^{\mathrm{ZF}}(u) . \mathrm{Rel}(u, \varphi)\big)$ and hence, as is well-known, we can prove the arithmetical statement asserting the consistency of $\mathrm{ZF} + \{\varphi\}$ — $\mathrm{Con}(\mathrm{ZF} + \{\varphi\})$. Thus we prove in ZF* $\varphi \supset \mathrm{Con}(\mathrm{ZF} + \{\varphi\})$ for any sentence φ, i. e., ZF* is essentially reflexive over ZF (see [1]) and hence ZF* is an essentially infinite extension of ZF.

We shall now give ZF* a characterization which will be very helpful when we deal with the problems of the consistency and the power of ZF*.

THEOREM 1. *The sentence φ is provable in ZF* using not more than n instances of R_1^{ZF} if and only if* (4)

$$(1) \quad (\exists a)\Big(\varphi . \bar{\bar{a}} \leqslant \boldsymbol{n} : \vee : \sim\varphi . \bar{\bar{a}} \leqslant \boldsymbol{n}-\boldsymbol{1} : . (a)\big(\mathrm{In}(a) . a \in a : \supset \mathrm{Rel}(R(a), \varphi)\big)\Big)$$

is provable in ZF.

In discussing this theorem the universe will also be called a *standard complete model* (scm). (1) asserts that there are at most n scm's in which φ does not hold.

Proof. We shall first prove in ZF that every instance of R_1^{ZF} holds in all the scm's of ZF except at most one. Assume that there are two different scm's of ZF, u_1 and u_2, in which the negation $\varphi . \sim(\exists u)\big(\mathrm{Scm}^{\mathrm{ZF}}(u) . \mathrm{Rel}(u, \varphi)\big)$ of an instance of R_1^{ZF} holds. One of those models can be the universe but the treatment of that case is completely analogous to the treatment of the case that both models are sets. Since $u_1 = R(a_1)$, $u_2 = R(a_2)$ and $a_1 \neq a_2$, we can assume, without loss of generality, that $a_1 > a_2$. By assumption we have, for $i = 1, 2$, $\mathrm{Rel}\big(u_i, \varphi . \sim(\exists u)\big(\mathrm{Scm}^{\mathrm{ZF}}(u) . \mathrm{Rel}(u, \varphi)\big)\big)$ which is

$$\mathrm{Rel}(u_i, \varphi) . \sim(\exists u)\big(u \in u_i . \mathrm{Rel}\big(u_i, \mathrm{Scm}^{\mathrm{ZF}}(u)\big) . \mathrm{Rel}\big(u_i, \mathrm{Rel}(u, \varphi)\big)\big) .$$

By $u \in u_i$ it is easy to prove, using the methods of Shepherdson [2], that $\mathrm{Rel}\big(u_i, \mathrm{Scm}^{\mathrm{ZF}}(u)\big) \equiv \mathrm{Scm}^{\mathrm{ZF}}(u)$. Obviously, $\mathrm{Rel}\big(u_i, \mathrm{Rel}(u, \varphi)\big) \equiv \mathrm{Rel}(u_i . u, \varphi) \equiv \mathrm{Rel}(u, \varphi)$ (since $u \subseteq u_i$). Thus we have $\mathrm{Rel}(u_i, \varphi) . \sim(\exists u)\big(u \in u_i . \mathrm{Scm}^{\mathrm{ZF}}(u) . \mathrm{Rel}(u, \varphi)\big)$. For $i = 1$ we get, since $u_2 \in u_1$ and $\mathrm{Scm}^{\mathrm{ZF}}(u_2)$, $\sim\mathrm{Rel}(u_2, \varphi)$. For $i = 2$ we get $\mathrm{Rel}(u_2, \varphi)$ and thus we have a contradiction.

Now let φ be a sentence which is provable in ZF* from the n instances $\psi_1, \ldots, \psi_n$ of R_1^{ZF}. ψ_i, $1 \leqslant i \leqslant n$, is proved in ZF to hold in every scm of ZF except at most one, hence $\bigwedge_{i=1}^{n} \psi_i$ holds in every scm of ZF except at most n. In every model of ZF in which $\bigwedge_{i=1}^{n} \psi_i$ holds φ holds also, being

(4) Boldface digits denote the numbers of the formal system. $\boldsymbol{n}$ or $\boldsymbol{i}$ denotes the number of the formal system corresponding to the informal number n or i, respectively.

provable in ZF from $\psi_1, \ldots, \psi_n$, and hence φ holds in every scm of ZF except at most n.

Let φ be any sentence of set theory. We shall prove by induction that, assuming φ and using n instances of R_1^{ZF} we can prove in ZF*

$$(2)\qquad (\exists u_1)\ldots(\exists u_n)\Big(\bigwedge_{i=2}^{n} u_{i-1}\in u_i\,.\bigwedge_{i=1}^{n}\mathrm{Scm}^{ZF}(u_i)\,.\,\mathrm{Rel}(u_i,\varphi)\Big)\,.$$

We assume it for n and prove it for $n+1$. We substitute the conjunction of φ and (2) for φ in R_1^{ZF}. (2) is proved, by hypothesis, using φ and n instances of R_1^{ZF}. Hence, the $n+1$-th instance of R_1^{ZF} just mentioned gives us

$$(\exists u)\Big(\mathrm{Scm}^{ZF}(u)\,.\,\mathrm{Rel}\Big(u,\varphi\,.(\exists u_1)\ldots(\exists u_n)\Big(\bigwedge_{i=2}^{n}u_{i-1}\in u_i\,.\bigwedge_{i=1}^{n}\mathrm{Scm}^{ZF}(u_i)\,.\,\mathrm{Rel}(u_i,\varphi)\Big)\Big)\Big)$$

which is equivalent to

$$(\exists u)(\exists u_1)\ldots(\exists u_n)\Big(\bigwedge_{i=1}^{n}u_i\in u\,.\bigwedge_{i=2}^{n}u_{i-1}\in u_i\,.\,\mathrm{Scm}^{ZF}(u)\,.\,\mathrm{Rel}(u,\varphi)\,.\bigwedge_{i=1}^{n}\mathrm{Rel}\big(u,\mathrm{Scm}^{ZF}(u_i)\big)\,.\,\mathrm{Rel}\big(u,\mathrm{Rel}(u_i,\varphi)\big)\Big)\,.$$

Since $\mathrm{Scm}^{ZF}(u)$, u is complete and hence $\bigwedge_{i=2}^{n}u_{i-1}\in u_i$ and $u_n\in u$ imply $\bigwedge_{i=1}^{n-1}u_i\in u$, which can be omitted. Also $\mathrm{Rel}\big(u,\mathrm{Scm}^{ZF}(u_i)\big)\equiv\mathrm{Scm}^{ZF}(u_i)$ and $\mathrm{Rel}\big(u,\mathrm{Rel}(u_i,\varphi)\big)\equiv\mathrm{Rel}(u\,.\,u_i,\varphi)\equiv\mathrm{Rel}(u_i,\varphi)$; and thus we have

$$(\exists u)(\exists u_1)\ldots(\exists u_n)\Big(u_n\in u\,.\bigwedge_{i=2}^{n}u_{i-1}\in u_i\,.\,\mathrm{Scm}^{ZF}(u)\,.\,\mathrm{Rel}(u,\varphi)\,.\bigwedge_{i=1}^{n}\mathrm{Scm}^{ZF}(u_i)\,.\,\mathrm{Rel}(u_i,\varphi)\Big)\,.$$

We replace the bound variable u by u_{n+1} and get

$$(\exists u_1)\ldots(\exists u_{n+1})\Big(\bigwedge_{i=2}^{n+1}u_{i-1}\in u_i\,.\bigwedge_{i=1}^{n+1}\mathrm{Scm}^{ZF}(u_i)\,.\,\mathrm{Rel}(u_i,\varphi)\Big)\,.$$

Now assume that it is provable in ZF that φ holds in all the scm's of ZF except at most n. Assume $\sim\varphi$. Using n instances of R_1^{ZF} we prove (2) with φ replaced by $\sim\varphi$. Thus $\sim\varphi$ holds in the universe and in n other scm's, i. e., $\sim\varphi$ holds in at least $n+1$ scm's and we have a contradiction. Thus we proved φ in ZF* from n instances of R_1^{ZF}.

Corollary 1. *We can derive a contradiction in* ZF* *from n instances of R_1^{ZF} if and only if we can prove $\sim(\exists\beta)\big(\beta=P_0(\boldsymbol{n}-1)\big)$ in* ZF. *Hence* ZF* *is consistent if and only if every formula $(\exists\beta)\big(\beta=P_0(\boldsymbol{n})\big)$, $n=0,1,\ldots$, is consistent with* ZF.

Proof. If we can prove in ZF $\sim(\exists\beta)\big(\beta=P_0(\boldsymbol{n}-1)\big)$ then there exist at most n scm's of ZF, namely, the universe, $P_0(0),\ldots,P_0(\boldsymbol{n}-2)$. Thus the formula $(x)(x\neq x)$ holds in all the scm's of ZF except at most n and by Theorem 1 it is provable from n instances of R_1^{ZF}.

On the other hand if $(x)(x\neq x)$ is provable in ZF* from n instances of R_1^{ZF} it can be proved in ZF that $(x)(x\neq x)$ holds in all the scm's of ZF except at most n. Assume in ZF $(\exists\beta)\big(\beta=P_0(\boldsymbol{n}-1)\big)$; hence there exist at least $n+1$ scm's of ZF, namely, the universe, $P_0(0),\ldots,P_0(\boldsymbol{n}-1)$, and at least in one of them $(x)(x\neq x)$ holds, which is a contradiction. Thus $\sim(\exists\beta)\big(\beta=P_0(\boldsymbol{n}-1)\big)$ is proved in ZF.

Corollary 2. *We can prove in* ZF* $(\exists\beta)\big(\beta=P_0(\boldsymbol{n})\big)$, $n=0,1,\ldots$, *but if* ZF* *is consistent we cannot prove in it* $(\exists\beta)\big(\beta=P_0(\omega)\big)$.

Proof. ω and the function P_0 can easily be seen to be absolute with respect to standard complete models of ZF (by the methods of Shepherdson [2]). Hence, the first part of this Corollary follows directly from Theorem 1.

Given any finite n we cannot prove in ZF that $(\exists\beta)\big(\beta=P_0(\omega)\big)$ holds in all the scm's of ZF except at most n, since if $(\exists\beta)\big(\beta=P_0(\omega)\big)$ holds in any of the $n+1$ scm's $P_0(\boldsymbol{0}),\ldots,P_0(\boldsymbol{n})$ we have a contradiction which proves (in ZF) $\sim(\exists\beta)\big(\beta=P_0(\boldsymbol{n})\big)$, and this is, by Corollary 1, contrary to the assumption that ZF* is consistent.

Let $\boldsymbol{\Lambda}$ and $\boldsymbol{M}$ be any definite ordinal numbers, which are absolute (in ZF) with respect to scm's of ZF. We note that the scm's of ZF in which

$$(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda})\big)\supset(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda}+\boldsymbol{M})\big)$$

does not hold are exactly the scm's $R\big(P_0(\boldsymbol{\Lambda}+\mu)\big)$, $0<\mu\leqslant\boldsymbol{M}$. Hence we have:

Corollary 3. *In* ZF* *we can prove, using n instances of R_1^{ZF} the formulae $(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda})\big)\supset(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda}+\boldsymbol{n})\big)$, $n=1,2,\ldots$ If $(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda})\big)$ is consistent with* ZF* *then we cannot prove $(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda})\big)\supset(\exists\beta)\big(\beta=P_0(\boldsymbol{\Lambda}+\omega)\big)$ in* ZF*.

The principle of unbounded sentential reflection over ZF

R_2^{ZF} $\quad\varphi\supset(\exists u)\big(x\in u\,.\,\mathrm{Scm}^{ZF}(u)\,.\,\mathrm{Rel}(u,\varphi)\big)$

or, equivalently

$\varphi\supset(\exists\alpha)\big(\alpha>\beta\,.\,\mathrm{In}(\alpha)\,.\,\mathrm{Rel}\big(R(\alpha),\varphi\big)\big)$

where φ is any sentence.

Let ZF** denote the theory obtained from ZF by the addition of R_2^{ZF}. We shall now give ZF** a characterization similar to that which we gave ZF.

Given a topological space A we call a subset B of A a *discrete* set if every $x \in B$ has a neighbourhood C such that $C \cdot B = \{x\}$. For $B \subseteq A$ we define the internal derivative of B, $Id(B)$, to be the set of all the members of B which are accomulation points of B, i. e. (5), $x \in Id(B) \equiv : x \in B \, . \, x \in \mathfrak{C}(B - \{x\})$. Obviously B is discrete if and only if $Id(B) = 0$. This can be generalized as follows:

LEMMA. *In the topological space A which satisfies the separability condition T_1 (i. e., for every point x, $\{x\}$ is a closed set) a subset B of A is the union of n discrete sets if and only if $Id^n(B) = 0$ (where Id^n is the n-th iteration of the operation Id).*

Proof. We prove the Lemma by induction on n. Assume it for $n-1$. Let $B_1, \ldots, B_n$ be discrete subsets of A, $B = \sum_{i=1}^{n} B_i$. Let $x \in Id^n(B)$. Since $Id^n(B) \subseteq B$, we can assume, without loss of generality, that $x \in B_n$. Since B_n is discrete, there exists an open set C such that $C \cdot B_n = \{x\}$.

We shall now prove that if D is an open set and E is any set then $D \cdot Id(E) = Id(D \cdot E)$. Obviously $Id(D \cdot E) \subseteq D \cdot Id(E)$. Now let $z \in D \cdot Id(E)$, hence $z \in \mathfrak{C}(E - \{z\})$. We want to show that $z \in \mathfrak{C}(D \cdot E - \{z\})$. Assume $z \notin \mathfrak{C}(D \cdot E - \{z\})$. Then, by $z \in D$, $z \notin \mathfrak{C}(D \cdot E - \{z\}) + (A - D)$. But $\mathfrak{C}(D \cdot E - \{z\}) + (A - D)$ is a closed set containing $E - \{z\}$, contradicting $z \in \mathfrak{C}(E - \{z\})$. Thus we proved $D \cdot Id(E) = Id(D \cdot E)$. By iterating we get $D \cdot Id^n(E) = Id^n(D \cdot E)$.

Since $x \in Id^n(B)$, $x \in C$ and C is open, there is a point y, $y \neq x$, $y \in C$ and $y \in Id^{n-1}(B)$. By the condition T_1, $\{x\}$ is closed and hence $C - \{x\}$ is open. Thus we have, by substituting $C - \{x\}$ for D, B for E and $n-1$ for n in $D \cdot Id^n(E) = Id^n(D \cdot E)$ $(C - \{x\}) \cdot Id^{n-1}(B) = Id^{n-1}\big((C - \{x\}) \cdot B\big)$. Since $C \cdot B_n = \{x\}$, we have $(C - \{x\}) \cdot B = \sum_{i=1}^{n-1} (C - \{x\}) \cdot B_i$, and since $(C - \{x\}) \cdot B_i$, $i = 1, \ldots, n-1$, are obviously discrete, we have, by the assumption of the induction,

$$Id^{n-1}\big((C - \{x\}) \cdot B\big) = 0 \,,$$

contradicting

$$y \in (C - \{x\}) \cdot Id^{n-1}(B) = Id^{n-1}\big((C - \{x\}) \cdot B\big).$$

On the other hand if $Id^n(B) = 0$ then $B - Id(B)$ is obviously discrete. By the assumption of the induction, $Id^{n-1}\big(Id(B)\big) = Id^n(B) = 0$ implies that $Id(B)$ is a union of $n-1$ discrete sets, which completes the proof.

(5) $\mathfrak{C}(X)$ is the topological closure of X.

We shall now regard the ordinal numbers, with the class On of all the ordinal numbers added as the largest ordinal number, in the order topology. We shall see how to write in the language of set theory that the property $\mathrm{Rel}\big(R(\alpha), \varphi\big)$ holds for all the inaccessible ordinals α (On is taken as an inaccessible number, $R(On)$ as the universe and hence $\mathrm{Rel}\big(R(On), \varphi\big)$ is φ) except for a family which is the union of n discrete families. First we shall define by induction formulae $\Phi_n(\alpha)$, $n = 0, 1, \ldots$, asserting that α is in the nth internal derivative of the family of the inaccessible numbers which do not satisfy $\mathrm{Rel}\big(R(\alpha), \varphi\big)$ $\big(Id^0(X) = X\big)$. For an ordinal number α

$$\Phi_0(\alpha) \equiv : \mathrm{In}(\alpha) \, . \sim \mathrm{Rel}\big(R(\alpha), \varphi\big) \,,$$

$$\Phi_n(\alpha) \equiv : \mathrm{In}(\alpha) \, . \sim \mathrm{Rel}\big(R(\alpha), \varphi\big) \, . \, (\beta)\Big(\beta < \alpha \supset (\exists \gamma)\big(\beta < \gamma < \alpha \, . \, \Phi_{n-1}(\gamma)\big)\Big) \,,$$

$$\Phi_0(On) \equiv \sim \varphi \,,$$

$$\Phi_n(On) \equiv : \sim \varphi \, . \, (\beta)(\exists \gamma)\big(\gamma > \beta \, . \, \Phi_{n-1}(\gamma)\big) \,.$$

By the Lemma to say formally that the property $\mathrm{Rel}\big(R(\alpha), \varphi\big)$ holds for all inaccessible numbers except for the members of a family which is the union of n discrete families is to say $\sim (\exists \alpha) \Phi_n(\alpha) \, . \sim \Phi_n(On)$.

THEOREM 2. *The sentence φ is provable in ZF** from not more than n instances of R_2^{ZF} if and only if*

*"The family of the ordinals α for which φ does not hold in the scm $R(\alpha)$ of ZF is a union of **n** discrete families"*

is provable in ZF.

Proof. We shall prove first that every instance of R_2^{ZF} holds in every scm $R(\alpha)$ of ZF except for a discrete family of inaccessible ordinals α. We have to prove that if

$$\mathrm{Rel}\Big(R(\gamma), \sim\Big(\varphi \subset (\beta)(\exists \alpha)\big(\alpha > \beta \, . \, \mathrm{In}(\alpha) \, . \, \mathrm{Rel}\big(R(\alpha), \varphi\big)\big)\Big)\Big)$$

holds for a set d of ordinals γ it does not hold for its limit δ.

$$\mathrm{Rel}\Big(R(\gamma), \sim\Big(\varphi \supset (\beta)(\exists \alpha)\big(\alpha > \beta \, . \, \mathrm{In}(\alpha) \, . \, \mathrm{Rel}\big(R(\alpha), \varphi\big)\big)\Big)\Big)$$

$$\equiv : \mathrm{Rel}\big(R(\gamma), \varphi\big) \, . \, \mathrm{Rel}\Big(R(\gamma), \sim (\beta)(\exists \alpha)\big(\alpha > \beta \, . \, \mathrm{In}(\alpha) \, . \, \mathrm{Rel}\big(R(\alpha), \varphi\big)\big)\Big) \,.$$

It can be seen, as we have already mentioned earlier and noting that the ordinal numbers of the scm $R(\delta)$ are the ordinal numbers smaller than δ (see Shepherdson [2]), that

$$\mathrm{Rel}\Big(R(\delta), \sim\Big(\varphi \supset (\beta)(\exists \alpha)\big(\alpha > \beta \, . \, \mathrm{In}(\alpha) \, . \, \mathrm{Rel}\big(R(\alpha), \varphi\big)\big)\Big)\Big)$$

$$\equiv : \mathrm{Rel}\big(R(\delta), \varphi\big) \, . \, (\exists \beta)\Big(\beta < \delta \, . \, (\alpha)\big(\beta < \alpha < \delta \, . \, \mathrm{In}(\alpha) : \supset \sim \mathrm{Rel}\big(R(\alpha), \varphi\big)\big)\Big) \,.$$

But this is in contradiction to the fact that δ is the limit of the set d of the γ's and the γ's satisfy $\mathrm{Rel}(R(\gamma), \varphi)$. The proof is the same if δ is On.

In analogy to Theorem 1 we have that if φ is provable in ZF** from n instances of $\mathrm{R}_2^{\mathrm{ZF}}$ we can prove in ZF that φ holds in all the scm's $R(\alpha)$ except for a family of inaccessible numbers α which is the union of n discrete families.

Given a formula $\psi(\alpha)$ $(\mu\alpha)\psi(\alpha)$ will denote the smallest ordinal α such that $\psi(\alpha)$, if there exists such an ordinal at all. Given any formula φ we define the functions P_n^φ as follows:

$$P_0^\varphi(0) = (\mu\alpha)\big(\mathrm{In}(\alpha).\mathrm{Rel}(R(\alpha), \varphi)\big),$$

$$P_0^\varphi(\beta+1) = (\mu\alpha)\big(\alpha > P_0^\varphi(\beta).\mathrm{In}(\alpha).\mathrm{Rel}(R(\alpha), \varphi)\big)$$

and, for limit-number β,

$$P_0^\varphi(\beta) = \sup_{\gamma<\beta} P_0^\varphi(\gamma);$$

$$P_{n+1}^\varphi(0) = (\mu\alpha)\big(\mathrm{In}(\alpha).(\exists\gamma)(\gamma \text{ is a limit number}.\ \alpha = P_n^\varphi(\gamma)).\mathrm{Rel}(R(\alpha), \varphi)\big),$$

$$P_{n+1}^\varphi(\beta+1) = (\mu\alpha)\big(\alpha > P_{n+1}^\varphi(\beta).\mathrm{In}(\alpha).(\exists\gamma)(\gamma \text{ is a limit number}\ .\ \alpha = P_n^\varphi(\gamma)).\mathrm{Rel}(R(\alpha), \varphi)\big)$$

and, for limit-number β,

$$P_{n+1}^\varphi(\beta) = \sup_{\gamma<\beta} P_{n+1}^\varphi(\gamma).$$

We shall now see that in ZF** we can prove $\varphi \supset (\gamma)(\exists\delta)\big(\delta = P_{n-1}^\varphi(\gamma)\big)$. using n instances of $\mathrm{R}_2^{\mathrm{ZF}}$. For $n = 1$, we have, by $\mathrm{R}_2^{\mathrm{ZF}}$, $\varphi \supset (\beta)(\exists\alpha)\big(\alpha > \beta .\mathrm{In}(\alpha).\mathrm{Rel}(R(\alpha), \varphi)\big)$, hence $(\gamma)(\exists\delta)\big(\delta = P_0^\varphi(\gamma)\big)$. Assume that $\varphi \supset (\gamma)(\exists\delta)\big(\delta = P_{n-2}^\varphi(\gamma)\big)$ is provable in ZF** by using $n-1$ instances of $\mathrm{R}_2^{\mathrm{ZF}}$.

Substituting $(\gamma)(\exists\delta)\big(\delta = P_{n-2}^\varphi(\gamma)\big)$ for φ in $\mathrm{R}_2^{\mathrm{ZF}}$ we obtain

$$(\gamma)(\exists\delta)\big(\delta = P_{n-2}^\varphi(\gamma)\big) \supset (\beta)(\exists\alpha)\Big(\alpha > \beta.\mathrm{In}(\alpha).(\gamma)\big(\gamma < \alpha \supset (\exists\delta)(\delta < \alpha\ .\ \delta = P_{n-2}^\varphi(\gamma))\big)\Big)$$

(here we use the facts that the relativizations of (γ) and $(\exists\delta)$ to $R(\alpha)$ are $(\gamma)(\gamma < \alpha \supset$ and $(\exists\delta)(\delta < \alpha.$, respectively, and also that the function P_{n-2}^φ is absolute with respect to scm's of ZF; these facts can be easily proved by the methods of Shepherdson [2]). Since the antecedent is proved from φ by using $n-1$ instances of $\mathrm{R}_2^{\mathrm{ZF}}$, we prove the consequent from φ by using n instances of $\mathrm{R}_2^{\mathrm{ZF}}$. From the consequent it follows that $(\beta)(\exists\alpha)\big(\alpha > \beta.\mathrm{In}(\alpha).P_{n-2}^\varphi(\alpha) = \alpha\big)$ and hence $(\gamma)(\exists\delta)\big(\delta = P_{n-1}^\varphi(\gamma)\big)$.

Now we assume that we can prove in ZF that φ holds in all the scm's $R(\alpha)$ of ZF except for a family of inaccessible numbers α which is the union of n discrete families. We shall prove φ from n instances of $\mathrm{R}_2^{\mathrm{ZF}}$ by contradiction. Assume $\sim\varphi$. By what we have just proved we have, by n instances of $\mathrm{R}_2^{\mathrm{ZF}}$, $(\gamma)(\exists\delta)\big(\delta = P_{n-1}^\varphi(\gamma)\big)$. Let B be the family of the inaccessible ordinals α such that $\mathrm{Rel}(\mathrm{R}(\alpha), \sim\varphi)$ holds. It is easy to see that $Id^i(B)$, $0 \leqslant i < n$, is the family which consists of On and the members of B which are in the range of P_i^φ, and that $Id^n(B)$ is the family consisting of the single member On. But, by assumption, we can prove in ZF that $Id^n(B) = 0$ and thus we have a contradiction.

In analogy to Corollaries 1-3 we have

Corollary 4. *We can derive a contradiction in* ZF** *from n instances of* $\mathrm{R}_2^{\mathrm{ZF}}$ *if and only if we can prove* $\sim(\alpha)(\exists\beta)\big(\beta = P_{n-1}(\alpha)\big)$ *in* ZF. *Hence* ZF** *is consistent if and only if every formula* $(\alpha)(\exists\beta)\big(\beta = P_n(\alpha)\big)$, $n = 0, 1, \ldots$, *is consistent with* ZF.

Corollary 5. *We can prove in* ZF** $(\alpha)(\exists\beta)\big(\beta = P_n(\alpha)\big)$, $n = 0, 1, \ldots$, *but if* ZF** *is consistent we cannot prove in it* $(\exists\beta)\big(\beta = P_\omega(0)\big)$.

Corollary 6. *We can prove in* ZF**, *using n instances of* $\mathrm{R}_2^{\mathrm{ZF}}$, *the formula* $(\alpha)(\exists\beta)\big(\beta = P_\Lambda(\alpha)\big) \supset (\alpha)(\exists\beta)\big(\beta = P_{\Lambda+n}(\alpha)\big)$. *If* $(\alpha)(\exists\beta)\big(\beta = P_\Lambda(\alpha)\big)$ *is consistent with* ZF** *we cannot prove in* ZF** $(\alpha)(\exists\beta)\big(\beta = P_\Lambda(\alpha)\big) \supset (\exists\beta)\big(\beta = P_{\Lambda+\omega}(0)\big)$.

The principle of partial reflection over ZF

$\mathrm{R}_3^{\mathrm{ZF}}$ $\quad \varphi \supset (\exists u)\big(x_1, \ldots, x_n \in u.\mathrm{Scm}^{\mathrm{ZF}}(u).\mathrm{Rel}(u, \varphi)\big)$

or, equivalently

$\varphi \supset (\exists\alpha)\big(x_1, \ldots, x_n \in R(\alpha).\mathrm{In}(\alpha).\mathrm{Rel}(R(\alpha), \varphi)\big)$

where φ is a formula with no free variables except $x_1, \ldots, x_n$.

Let ZF*** denote the theory obtained from ZF by the addition of $\mathrm{R}_3^{\mathrm{ZF}}$.

Theorem 3. *In* ZF*** *we can prove*:

"Let F be a function which is defined on all the ordinal numbers, the values of which are ordinal numbers and which is strictly increasing and continuous; let F be absolute with respect to scm's *of* ZF (*i. e., if* $\mathrm{Scm}^{\mathrm{ZF}}(u)$ *and* $\alpha, \beta \in u$ *then* $\beta = F(\alpha)$ *if and only if* $\mathrm{Rel}\big(u, \beta = F(\alpha)\big)$); *then F has a fixed point at an inaccessible number".*

Proof. By using $\mathrm{R}_3^{\mathrm{ZF}}$ with respect to the formula $(\gamma)(\exists\delta)\big(\delta = F(\gamma)\big)$ we get $(\exists\alpha)\Big(\mathrm{In}(\alpha).(\gamma)\Big(\gamma < \alpha \supset (\exists\delta)\big(\delta < \alpha.\mathrm{Rel}(R(\alpha), \delta = F(\gamma))\big)\Big)\Big)$. Since we have, for $\gamma, \delta < \alpha$, $\mathrm{Rel}\big(R(\alpha), \delta = F(\gamma)\big) \equiv \delta = F(\gamma)$, we have $(\gamma)\big(\gamma < \alpha \supset (\exists\delta)(\delta < \alpha.\delta = F(\gamma))\big)$, i. e., for $\gamma < \alpha$, $F(\gamma) < \alpha$, hence by continuity $F(\alpha) \leqslant \alpha$. On the other hand, since F is strictly increasing, we have $(\eta)\big(F(\eta) \geqslant \eta\big)$, hence $F(\alpha) = \alpha$.

The axiom M (see [1]) which is equivalent to R_4^{ZF} is like the formula in Theorem 3 only that the condition that F is absolute with respect to scm's of ZF is omitted in M. This accounts for the great strength of R_3^{ZF}, since in many proofs in which M is used the functions used turn out to be absolute with respect to scm's of ZF, as we shall see in the following corollary (⁶).

COROLLARY 7. *In* ZF*** *we can prove* $(\eta)(\alpha)(\exists\beta)\big(\beta = P_\eta(\alpha)\big)$, $(\alpha)(\exists\beta)\big(\beta = Q(\alpha)\big)$ (and also $(\eta)(\alpha)(\exists\beta)\big(\beta = Q_\eta(\alpha)\big)$ and "*There exist arbitrarily great Q*-numbers*", etc.).

Proof. These formulae can be proved in ZM (see [1]). In their proof the axiom M is used with respect to certain functions F. Those functions can be easily shown (in ZF) to be absolute with respect to scm's of ZF and hence we can use Theorem 3 and prove these formulae in ZF***.

We note that in contrast to R_1^{ZF} and R_2^{ZF} any finite number of instances of R_3^{ZF} can be replaced by a single instance of R_3^{ZF}. Let ψ be the conjunction of the instances $\psi_1, \ldots, \psi_m$ of R_3^{ZF} corresponding to the formulae $\varphi_1, \ldots, \varphi_m$ with no free variables except $x_1, \ldots, x_n$. Let φ be the formula $\bigvee_{i=1}^{m} x_{n+1} = \boldsymbol{i}.\varphi_i$. The instance of R_3^{ZF} corresponding to φ implies ψ, since ψ_i follows from it by substituting $\boldsymbol{i}$ for x_{n+1}.

References

[1] A. Lévy, *Axiom schemata of strong infinity in axiomatic set theory*, Pacific Journ. of Mathematics 10 (1960), p. 223-238..

[2] J. C. Shepherdson, *Inner models for set theory, Part I*, Journ. Symbolic Logic 16 (1951), p. 161-190.

[3] A. Tarski and R. L. Vaught, *Arithmetical extensions of relational systems*, Compositio Mathematica 13 (1957), p. 81-102.

(⁶) In the meanwhile Vaught has proved that ZF ($= S + \{R_4^S\}$) is properly stronger than $S + \{R_3^S\}$. In a similar way one can prove that ZM ($= ZF + \{R_4^{ZF}\}$) is properly stronger than ZF***.

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Reçu par la Rédaction le 18. 4. 1959

Maximal n-disjointed sets and the axiom of choice

by

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This note contains a generalization of a result of R. L. Vaught [1] concerning the equivalence of the existence of maximal disjointed sets with the axiom of choice. Our generalization arises naturally when the notion of a disjointed set is considered as a special case (namely, when $n = 2$) of the notion of an n-disjointed set.

Let n be an integer greater or equal to 2. A set x is said to be *n-disjointed* if any n distinct elements of x has an empty intersection. An n-disjointed subset y of x is said to be a maximal n-disjointed subset of x if y is not properly contained in any n-disjointed subset of x. Notice that if y is an n-disjointed set then y is an m-disjointed set for each m greater or equal to n; also, if y is an n-disjointed set then every subset of y is an n-disjointed set.

Consider the following two sentences:

ξ_n: *Every n-disjointed subset of a set x can be extended to a maximal n-disjointed subset of x.*

ν_n: *Every set x contains a maximal n-disjointed subset.*

It is quite clear that for each n the sentence ξ_n implies the sentence ν_n. We shall now show that the sentence ξ_2 is equivalent with the sentence ν_2. Let y be a 2-disjointed subset of x, and let z be the set of those elements t of x such that t does not intersect any member of y, i. e.,

$$z = \{t;\, t \in x \text{ and, for each } s \in y,\, t \cap s = 0\}.$$

By ν_2, there exists a maximal 2-disjointed subset w of z. We assert that $y \cup w$ is a maximal 2-disjointed subset of x containing y. Clearly, $y \cup w$ is 2-disjointed and $y \subseteq y \cup w \subseteq x$. Suppose that $t \in x$ and $y \cup w \cup \{t\}$ is also 2-disjointed, then $t \in z$, $w \cup \{t\} \subseteq z$ and $w \cup \{t\}$ is 2-disjointed. Since w is maximal in z, $t \in w$ and $t \in y \cup w$. This proves the maximality of $y \cup w$ in x. While the above argument for the case when $n = 2$ is quite simple, we do not know at present whether ν_n implies ξ_n for any $n \geqslant 3$.

Our generalization is contained in the following

THEOREM. *For each $n \geqslant 2$, the sentence ξ_n is equivalent with the axiom of choice.*