Concerning the measurable boundaries of a real function

by

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1. While it is known that the equivalence classes of extended real-valued measurable functions on certain types of measure spaces form a complete lattice, this basic fact has found little application to problems concerning real functions.

It is our contention that this property of measurable functions greatly simplifies various discussions in the literature. A first application, [3], was to the definition and elementary properties of the upper and lower limits in measure of a sequence of measurable functions, a notion which was introduced by D. E. Menchov [5].

In this note we indicate how the complete lattice property of the measurable functions simplifies the theory of approximation of arbitrary real functions by means of measurable functions.

2. H. Blumberg [1] has defined the upper and lower measurable boundaries u and l of f, an extended real-valued function on the real line, as follows:

For every y, let $E_y = \{x\colon f(x) > y\}$.

For every ξ, let $u(\xi) = \inf\{y\colon$ exterior metric density of E_y at ξ is zero$\}$.

The lower measurable boundary l is defined in the obviously analogous way.

Blumberg showed that u and l are measurable and used these functions to obtain several other facts that pertain to arbitrary functions. The definitions and methods of Blumberg are restricted to functions on very special measure spaces, since they use such special devices as density theorems, the Vitali covering theorem, and approximate continuity.

We shall give a definition that makes sense in any totally σ-finite measure space. This definition agrees with the Blumberg definition for Lebesgue measure on the real line, and in the general situation, yields proofs simpler than those given by Blumberg for the special case.

3. We first say a few words about terminology and notation. Let $(X, \mathcal{S}, \mu)$ be a measure space. A null set is a set of measure zero.

Throughout the paper, we shall assume that $\mathcal{S}$ contains all subsets of null sets. Two measurable sets S and T are *equivalent* if their symmetric difference is a null set; the class of all sets equivalent to S is designated by $[S]$, and the collection of equivalence classes is called the *measure ring associated with the measure space*. A function is a *null function* if it vanishes everywhere except on a null set; two measurable functions f and g are called *equivalent* if $f-g$ is null; the class of measurable functions equivalent to f is designated by $[f]$; and the lattice of equivalence classes of measurable functions is designated by $\mathcal{M}$.

4. We now make a preliminary remark regarding the relation between the measure ring $\mathcal{R}$ associated with $(X, \mathcal{S}, \mu)$ and the lattice $\mathcal{M}$.

LEMMA 1. *The measure ring of a measure space $(X, \mathcal{S}, \mu)$ is a complete lattice if and only if $\mathcal{M}$, the family of equivalence classes of extended real-valued measurable functions on X, is a complete lattice.*

Proof of sufficiency. Let $[E_a]$, $a \in \mathfrak{A}$, be a family of members of $\mathcal{R}$, and let $f \in \sup_a [\chi_{E_a}]$, where χ_S is the characteristic function of the set S. Now, $0 \leqslant f(x) \leqslant 1$ almost everywhere, and $\{x\colon 0 < f(x) < 1\}$ is a null set. Thus, there is a subset E of X such that $f(x) = \chi_E(x)$ almost everywhere. Since f is a measurable function, E is a measurable set. Now, for every a in $\mathfrak{A}$, $E_a - E$ is a null set, since $[\chi_{E_a}] \leqslant [f]$. Moreover, if F is a measurable subset of E such that $E_a - F$ is null for every a in $\mathfrak{A}$, then

$$[\chi_{E_a}] \leqslant [\chi_F] \leqslant [\chi_E] = \sup_a [\chi_{E_a}],$$

so that $[\chi_F] = [\chi_E]$ and $[F] = [E]$. Hence, $[E] = \sup_a [E_a]$.

Proof of necessity. Let f_a, $a \in \mathfrak{A}$, be a family of measurable functions. We may assume, without loss of generality, that the f_a are non-negative. For each a, let the increasing sequence $\{f_a^n\}$, converging to f_a, be defined as follows:

$$f_a^n(x) = \begin{cases} \dfrac{k-1}{2^n}, & \text{if } \dfrac{k-1}{2^n} \leqslant f_a(x) < \dfrac{k}{2^n}, \quad k = 1, \ldots, n2^n; \\ n, & \text{if } f_a(x) \geqslant n. \end{cases}$$

Let $E_a^{nk} = \{x\colon (k-1)/2^n \leqslant f_a(x) < k/2^n\}$, $k = 1, 2, \ldots, \nu-1$, and let $E_a^{n\nu} = \{x\colon f_a(x) \geqslant n\}$, where $\nu = n2^n+1$. For fixed n, let E_{nk} be an element of $\sup_a [E_a^{nk}]$, for $k = 1, 2, \ldots, \nu$, let $F_{nk} = E_{nk} - \bigcup_{j=k+1}^{\nu} E_{nj}$ for every $k = 1, 2, \ldots, \nu-1$, and let $F_{n\nu} = E_{n\nu}$. Let the simple functions g_n be defined in the following manner:

$$g_n(x) = \sum_{k=1}^{\nu} \frac{k-1}{2^n} \chi_{F_{nk}}(x).$$

Then $[g_n] = \sup_a [f_a^n]$. Moreover, since $[f_a^n] \leqslant [f_a^{n+1}]$, for every a, n, it follows that

$$[g_1] \leqslant [g_2] \leqslant \ldots \leqslant [g_n] \leqslant \ldots$$

Hence $\lim g_n = g$ exists, almost everywhere, and is measurable. Finally,

$$[g] = \sup_{a,n} [f_a^n] = \sup_a [f_a].$$

A subset of X that intersects each measurable set of finite measure in a null set is called a *local null set*. We remark that the theorem obtained from Lemma 1 by replacing the null sets by the local null sets can be proved in exactly the same manner.

It is known (see [4], p. 169, or [3]) that the measure ring of a totally finite measure space is complete, and the totally σ-finite case follows immediately. We state this formally in the interest of completeness.

LEMMA 2. *The measure ring of a totally σ-finite measure space is a complete lattice.*

It is easily seen that there are measure spaces which do not have this property. For example, if $(X, \mathcal{S}, \mu)$ is a measure space for which the corresponding measure ring has no maximal element, then the latter is not complete.

A somewhat less trivial example is that for which X is uncountable, $\mathcal{S}$ is the set of all countable sets and complements of countable sets, and for every S in $\mathcal{S}$, $\mu(S)$ is the number of elements in S.

5. We now turn to our main consideration. Let $(X, \mathcal{S}, \mu)$ be a totally σ-finite measure space. Let f be an extended real-valued function on X. By a *measurable majorant* of f we understand any measurable g such that $g(x) \geqslant f(x)$ almost everywhere. (In all of the work that follows, the assumption that subsets of null sets be measurable can be eliminated if, in the above definition, the condition $g(x) \geqslant f(x)$ almost everywhere, be replaced by $g(x) < f(x)$ on a subset of a null set.)

LEMMA 3. *Let $(X, \mathcal{S}, \mu)$ be a totally σ-finite measure space, let f be an arbitrary extended real-valued function on X, and let $\mathcal{U}$ be the set of measurable majorants of f. If $u \in \inf_{g \in \mathcal{U}} [g]$, then u is a measurable majorant of f.*

Proof. Let μ^* be the outer measure generated by μ. Suppose that $u(x) < f(x)$ on a set of positive outer measure. Then there exist a positive number ε and a set E^* of positive outer measure such that $u(x)+\varepsilon < f(x)$ on E^*. Thus, for each measurable majorant g of f, there is a measurable set E_g such that $u(x)+\varepsilon < g(x)$ on E_g and such that E^*-E_g is a null set. Let H be a measurable cover of E^*. Then $H-E_g$ is of measure zero for each g in $\mathcal{U}$. Since $u(x)+\varepsilon < g(x)$ on E_g, it follows that the restriction of g to H is a measurable majorant of the restriction of $u+\varepsilon$ to H. Define the measurable function v as follows:

$$v(x) = \begin{cases} u(x)+\varepsilon, & \text{if } x \in H; \\ u(x), & \text{otherwise.} \end{cases}$$

Certainly $[v] > [u]$, but $[v] \leqslant [g]$ for every g in $\mathcal{U}$. Since we are thus led to a contradiction, our original premise is false, and the lemma is established.

6. We turn now to a consideration of the measurable boundaries of Blumberg.

THEOREM 1. *Let $X = [0,1]$ and let $(X, \mathcal{S}, \mu)$ be the Lebesgue measure space. If f is any extended real-valued function and $\mathcal{U}$ is the set of all measurable majorants of f, then the Blumberg upper measurable boundary of f belongs to* $\inf_{g\in\mathcal{U}}[g]$.

Proof. We first remark that Blumberg has shown, in [1], that his upper measurable boundary function, u_B, is a measurable majorant of f. Hence, if u belongs to $\inf_{g\in\mathcal{U}}[g]$, then $f(x) \leqslant u(x) \leqslant u_B(x)$ almost everywhere. Suppose that f is bounded. Then the upper integral of f is defined, and

$$\overline{\int} f\,d\mu \leqslant \int u\,d\mu \leqslant \int u_B\,d\mu\,.$$

But Blumberg has shown, in [3], that

$$\overline{\int} f\,d\mu = \int u_B\,d\mu\,.$$

Thus,

$$\int u\,d\mu = \int u_B\,d\mu\,,$$

whence

$$[u] = [u_B]\,.$$

The unbounded case follows by applying the order preserving one-one mapping $\tan^{-1}$ to the range of f.

Theorem 1 is the motivation for the following definition.

DEFINITION 1. Let $(X, \mathcal{S}, \mu)$ be a totally σ-finite measure space. If f is an extended real-valued function on X and if $u \in \inf_{\mathcal{U}}[g]$, where $\mathcal{U}$ is the set of all measurable majorants of f, then u is called an *upper measurable boundary* of f.

7. In the proof of Theorem 1 we used the fact that for bounded functions on the real line, $\overline{\int} f\,d\mu = \int u_B\,d\mu$. We now demonstrate the obvious generalization of this fact.

THEOREM 2. *Let $(X, \mathcal{S}, \mu)$ be a totally finite measure space and let f be a bounded real-valued function defined on X. If u is an upper measurable boundary of f, then $\overline{\int} f\,d\mu = \int u\,d\mu$.*

Proof. By definition of the upper integral,

$$\overline{\int} f\,d\mu = \inf_{\pi} \sum_{i=1}^{n_\pi} \sup_{x\in A_i} f(x)\cdot\mu(A_i)\,,$$

where the infimum is taken over all finite partitions of X into disjoint measurable sets. Thus, if π: $\{A_1, \ldots, A_{n_\pi}\}$ is an arbitrary partition of X and if $M_i = \sup_{x\in A_i} f(x)$, $i = 1, \ldots, n_\pi$, then it is clear that

$$s(x) = \sum_{i=1}^{n_\pi} M_i \chi_{A_i}(x)$$

is a measurable majorant of f. Hence $s(x) \geqslant u(x)$ almost everywhere, and

$$\int u\,d\mu \leqslant \int s\,d\mu = \sum_{i=1}^{n_\pi} M_i\mu(A_i)\,.$$

It now follows that

$$\int u\,d\mu \leqslant \inf_{\pi} \sum_{i=1}^{n_\pi} M_i\mu(A_i) = \overline{\int} f\,d\mu\,.$$

On the other hand, the relation $f(x) \leqslant u(x)$ almost everywhere implies that

$$\overline{\int} f\,d\mu \leqslant \overline{\int} u\,d\mu = \int u\,d\mu\,,$$

and the desired equality is proved.

8. As another application of his measurable boundaries, Blumberg gave a short proof (for the bounded case) of a theorem of Saks and Sierpiński [6]. We now do this for the general case.

THEOREM 3. *Let $(X, \mathcal{S}, \mu)$ be a totally σ-finite measure space and let μ_* be the inner measure engendered by μ. If u is an upper measurable boundary of a bounded function f defined on X, then for every $\varepsilon > 0$,*

$$\mu_*(\{x\colon\ |f(x)-u(x)| > \varepsilon\}) = 0\,.$$

Proof. Suppose, on the contrary, that there exists a positive number ε such that

$$\mu_*(\{x: |f(x)-u(x)|>\varepsilon\})>0 .$$

Then, there is a measurable set E of positive measure on which $|f(x)-u(x)|>\varepsilon$. Now, by Lemma 3, $f(x)\leqslant u(x)$ almost everywhere, so that $u(x)>f(x)+\varepsilon$ for every x in F, a non-null measurable subset of E. We define the function v as follows:

$$v(x)=\begin{cases} u(x)-\dfrac{\varepsilon}{2}, & \text{if} \quad x\in F;\\ u(x), & \text{otherwise.}\end{cases}$$

Obviously, $f(x)\leqslant v(x)$ almost everywhere and $[v]<[u]$. This is impossible, in view of the definition of $[u]$.

For unbounded functions, the measurable boundary of a real-valued function can be infinite on a set positive measure as is shown by the example below.

Example. Let $X=[0,1]$ and let $(X, \mathcal{S}, \mu)$ be the Lebesgue measure space. Let $\{A_n\}$ be a disjoint sequence of non-measurable sets, each of outer measure one, such that $X=\bigcup_{n=1}^{\infty} A_n$. If $f(x)=\sum_{n=1}^{\infty} n\chi_{A_n}(x)$, and u is an upper measurable boundary of f, then f is real-valued but u is almost everywhere infinite.

9. The original theorem of Saks and Sierpiński has a real-valued function as the measurable approximant of a real-valued function. Indeed, their proof goes over to the totally σ-finite case. Another proof which can be given general form is the one in [2]. However, by slightly generalizing the definition of this paper, a proof can be obtained using upper measurable boundaries. Since this has independent interest, we proceed with the discussion.

Let $(X, \mathcal{S}, \mu)$ be a totally finite measure space. If A is any subset of X, let $\widetilde{A}$ denote a measurable cover of A. Let f be an extended real-valued function defined on E, a subset of X. A measurable majorant of f is a measurable function g defined on $\widetilde{E}$ such that $g(x)\geqslant f(x)$ almost everywhere on E. An upper measurable boundary of f is any u belonging to $\inf[g]$, where $[g]$ ranges over the set of all equivalence classes of measurable majorants of f.

Theorem 4. *Let $(X, \mathcal{S}, \mu)$ be a totally finite measure space, let E be an arbitrary subset of X, and let f be a bounded real-valued function on E. Then, for every positive ε,*

$$\mu^*(\{x: |f(x)-u(x)|<\varepsilon\}\cap E)=\mu^*(E) .$$

Proof. For every $\varepsilon>0$, let $E_\varepsilon=\{x: |f(x)-u(x)|<\varepsilon\}\cap E$. Suppose that the theorem is false. Then, for some positive number ε, $\mu^*(E_\varepsilon)<\mu^*(E)$. Thus, $\mu(\widetilde{E}_\varepsilon)<\mu(\widetilde{E})$, and the function v given by

$$v(x)=\begin{cases} u(x)-\tfrac{1}{2}\varepsilon, & \text{if} \quad x\in \widetilde{E}-\widetilde{E}_\varepsilon;\\ u(x), & \text{otherwise,}\end{cases}$$

would then be a measurable majorant of f satisfying $[v]<[u]$.

Theorem 5. *Let $(X, \mathcal{S}, \mu)$ be a totally σ-finite measure space, and let f be a real-valued function on X. For every positive number ε, there is a real-valued measurable function u such that*

$$\mu_*(\{x: |f(x)-u(x)|>\varepsilon\})=0 .$$

Proof. Let Y be a measurable subset of X of finite measure. For every positive integer n, let $E_n=\{x: |f(x)|\leqslant n\}\cap Y$, and let $B_n=E_n-\widetilde{E}_{n-1}$. For every n, let u_n be an upper measurable boundary of the restriction of f to B_n. Now define u in the following manner:

$$u(x)=\begin{cases} u_1(x), & \text{if} \quad x\in \widetilde{B}\cap Y;\\ u_n(x), & \text{if} \quad x\in \widetilde{B}_n\cap Y-\bigcup_{k=1}^{n-1}\widetilde{B}_k\cap Y,\quad n=2,3,\ldots\end{cases}$$

It is easily seen that u satisfies the required condition for all x in Y. Since X can be decomposed into a disjoint sequence of measurable sets of finite measure, the proof may be completed by employing the preceding argument on each of the members of this sequence.

References

[1] Henry Blumberg, *The measurable boundaries of an arbitrary function*, Acta Math. 65 (1935), p. 263-282.

[2] Casper Goffman, *Proof of a theorem of Saks and Sierpiński*, Bull. Amer. Math. Soc. 54 (1948), p. 950-952.

[3] — and Daniel Waterman, *The upper and lower limits in measure*, to appear.

[4] P. R. Halmos, *Measure theory*, New York 1950.

[5] D. E. Menchov, *On convergence in measure of trigonometric series*, Amer. Math. Soc. Translation No. 105, 1954.

[6] S. Saks and W. Sierpiński, *Sur une propriété générale de fonctions*, Fund. Math. 11 (1928), p. 105-112.

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