A problem on Baire classes

by

G. Lederer (Reading)

Introduction. In what follows all sets are subsets of $[0,1]$, all functions are defined thereover and have real values.

E. Marczewski introduced the following definition:

A function $f(x)$ is *almost continuous with respect to a closed set F at a point x_0 in F*, if a function $g(x)$ exists, such that $g(x) = f(x)$ a. e. and if $\lim_{\substack{x\to x_0\\ x\in F}} g(x) = f(x_0)$.

He raised the following problem:

P. Let $f(x)$ be such that for every non-empty closed set F there is x_0 in F such that $f(x)$ is almost continuous w. r. t. F at x_0. Is then $f(x)$ a. e. equal to a function of Baire class $\leqslant 1$?

A modified form of the problem has been generalized by T. Traczyk to one concerning a Boolean algebra on which a certain topology is defined. He proved that under conditions of P a set X of zero measure exists, such that $f(x)$ is of Baire class $\leqslant 1$ w. r. t. $[0,1]-X$.

Independently the present author has generalized in a different direction and has concluded that — under conditions of P — a function $g(x)$ of Baire class $\leqslant 1$ w. r. t. $[0,1]$ exists, such that $f(x) = g(x)$ a. e. This conclusion is a special case of theorem II of this paper — which does not seem to be covered by Traczyk's work.

Though theorem II provides a more general result than is asked for in Professor Marczewski's question, I wish to emphasize that it was his problem that gave me the original incentive.

I must express my gratitude to Mr. H. Kestelman for a very valuable suggestion which shortened the argument leading to theorem I considerably.

Lastly I wish to thank the referee of Fundamenta Mathematicae for useful suggestions shortening the text.

Statement of the theorems. Before stating theorems I and II let us introduce the following definition:

Given a closed set F, and a point x_0 in F, a function $f(x)$ *has property $D(a)$ with respect to F at x_0*, if for any $\varepsilon > 0$ there is an open neighbourhood

I of x_0 and a function $g(x)$ of Baire class $\leqslant \alpha$, such that $|f(x)-g(x)| < \varepsilon$ a. e. over $F \cap I$.

THEOREM I. *If $\alpha > 0$, and if a function $f(x)$ is such that for each non-empty closed set F, $f(x)$ has property $D(\alpha)$ w. r. t. F at least at one point x_0 in F, then for any $\varepsilon > 0$ there exists a function $\varphi(x)$ of Baire class $\leqslant \alpha$, such that $|f(x)-\varphi(x)| < \varepsilon$ a. e.*

THEOREM II. *Under conditions of theorem* I *there exists a function $g(x)$ of Baire class $\leqslant \alpha$, such that $f(x) = g(x)$ a. e.*

Proof of the theorems. Notation. Denote by $\mathcal{B}$ $(\leqslant \alpha)$ the *aggregate* of functions of Baire class $\leqslant \alpha$, by $\mathcal{G}(\alpha)$ the *α-additive class* of Borel sets.

Proof of theorem I. Let us recall first the following known facts:

(1) A theorem due to Lebesgue [1]: *If $[0,1]$ is an enumberable union $\bigcup_n S_n$ of disjoint sets S_n such that, for each n, $S_n \in \mathcal{G}(\alpha)$, and there is a function $g_n(x)$ in $\mathcal{B}$ $(\leqslant \alpha)$, such that $g_n(x) = g(x)$ over S_n, then $g(x) \in \mathcal{B}(\leqslant \alpha)$.*

(2) Romanowski's theorem [2]:

Suppose that $\mathcal{E}$ is a non-empty aggregate of open subintervals of $[0,1]$ *with the following properties:*

(i) *If $(a,b) \in \mathcal{E}$ and $(b,c) \in \mathcal{E}$, then $(a,c) \in \mathcal{E}$.*

(ii) *If $(a,b) \in \mathcal{E}$ and $a \leqslant \alpha < \beta \leqslant b$, then $(\alpha,\beta) \in \mathcal{E}$.*

(iii) *If $(a+1/n, b-1/n) \in \mathcal{E}$ for all sufficiently large n, then $(a,b) \in \mathcal{E}$.*

(iv) *If P is a non-empty perfect set in* $[0,1]$ *and every open interval of* $[0,1]$ *contiguous to P belongs to $\mathcal{E}$, then there is an open interval which belongs to $\mathcal{E}$ and includes a point of P.*

Then $(0,1) \in \mathcal{E}$.

Having fixed $\alpha > 0$ and $\varepsilon > 0$, denote by $\mathcal{E}$ the aggregate of open intervals (a,b) for which there exists a function $\varphi(x)$ in $\mathcal{B}$ $(\leqslant \alpha)$ such that $|f(x)-\varphi(x)| < \varepsilon$ a. e. over (a,b). To prove theorem I we shall show that $\mathcal{E}$ satisfies conditions of (2).

Since $[0,1]$ is closed, by conditions of theorem I, $\exists x_0$ in $[0,1]$ such that $f(x)$ has property $D(\alpha)$ w. r. t. $[0,1]$ at x_0, i. e. an interval I open w. r. t. $[0,1]$ and a function $g(x)$ in $\mathcal{B}$ $(\leqslant \alpha)$ exist, such that $|f(x)-g(x)| < \varepsilon$ a. e. over I. Thus $I \in \mathcal{E}$, i. e. $\mathcal{E}$ is not empty.

Now let $(a,b) \in \mathcal{E}$ and $(b,c) \in \mathcal{E}$. Then there are functions $g_1(x)$ and $g_2(x)$ in $\mathcal{B}$ $(\leqslant \alpha)$ such that $|g_1(x)-f(x)| < \varepsilon$ a. e. over (a,b) and $|g_2(x)-f(x)| < \varepsilon$ a. e. over (b,c). Then put:

$$g(x) = \begin{cases} g_1(x) & \text{over } (b,c), \\ g_2(x) & \text{over } (a,b], \\ 0 & \text{over } [0,1]-(a,c). \end{cases}$$

The sets $(a,b]$, (b,c) and $[0,1]-(a,c)$ are disjoint. Their union is $[0,1]$. They are all in $\mathcal{G}(\alpha)$. Also over each of these sets $g(x)$ equals some function in $\mathcal{B}$ $(\leqslant \alpha)$. Thus by (1), $g(x) \in \mathcal{B}(\leqslant \alpha)$.

Also by the above: $|f(x)-g(x)| < \varepsilon$ a. e. over (a,b) and a. e. over (b,c). Hence: $|f(x)-g(x)| < \varepsilon$ a. e. over (a,c). Condition (i) is thus satisfied.

Satisfaction of condition (ii) is obvious.

Next assume that $0 \leqslant a < b \leqslant 1$ and that for all sufficiently large n: $(a+1/n, b-1/n) \in \mathcal{E}$. Let k be the smallest value of n for which $a+1/n < b-1/n$. Then, by definition of $\mathcal{E}$, there is a sequence of functions $\{g_n(x)\}$, such that, for each $n \geqslant k$, $g_n(x) \in \mathcal{B}$ $(\leqslant \alpha)$ and $|g_n(x)-f(x)| < \varepsilon$ a. e. over $(a+1/n, b-1/n)$. Put: $(a+1/k, b-1/k) = J_k$ and for $n > k$:

$$\big(a+1/(n+1),\, b-1/(n+1)\big) - (a+1/n,\, b-1/n) = J_n.$$

Let: $[0,1]-(a,b) = K$. Put further:

$$g(x) = \begin{cases} g_n(x) & \text{over } J_n \quad \text{for each} \quad n \geqslant k, \\ 0 & \text{over } K. \end{cases}$$

Now each J_n belongs to $\mathcal{G}(\alpha)$ and so does K. Over each of these sets $g(x)$ equals some function in $\mathcal{B}$ $(\leqslant \alpha)$. Also $[0,1] = K \cup \{\bigcup_k^\infty J_n\}$ and the members of the union are disjoint. Hence by (1): $g(x) \in \mathcal{B}$ $(\leqslant \alpha)$. Further, by the above, $|g(x)-f(x)| < \varepsilon$ a. e. over each J_n. Thus $|g(x)-f(x)| < \varepsilon$ a. e. over $\bigcup_k^\infty J_n$ and $\bigcup_k^\infty J_n = (a,b)$. Condition (iii) is thus satisfied.

Now let P be a non-empty perfect set such that all intervals contiguous to P belong to $\mathcal{E}$. Since P is closed, by condition of theorem I, there are x_0 in P, an interval I, open w. r. t. $[0,1]$ including x_0, and a function $g_0(x)$ in $\mathcal{B}$ $(\leqslant \alpha)$ such that $|f(x)-g(x)| < \varepsilon$ a. e. over $I \cap P$. The intervals contiguous to P form an enumerable aggregate $\{I_n\}$. Since each $I_n \in \mathcal{E}$, a sequence of functions $\{g_n(x)\}$ exists such that for each n: $g_n(x) \in \mathcal{B}$ $(\leqslant \alpha)$ and $|g_n(x)-f(x)| < \varepsilon$ a. e. over I_n.

Next put:

$$g(x) = \begin{cases} g_0(x) & \text{over } I \cap P, \\ g_n(x) & \text{over } I_n \text{ for each } n, \\ 0 & \text{over } [0,1]-\{(I \cap P) \cup (\bigcup_n I_n)\} \\ & \qquad = [0,1]-\{I \cup (\bigcup_n I_n)\} = K \text{ (say)}. \end{cases}$$

Now I and each I_n are open w. r. t. $[0,1]$. Hence K is closed. Thus K, $I \cap P$ and each I_n belong to $\mathcal{G}(\alpha)$. They form a disjoint enumerable union which equals $[0,1]$. Also, over each of these sets $g(x)$ equals some function in $\mathcal{B}(\leqslant\alpha)$. Hence, by (1), $g(x)\in\mathcal{B}(\leqslant\alpha)$.

Next, $I=(I\cap P)\cup\{\bigcup_n (I\cap I_n)\}$. This union is enumerable and $|f(x)-g(x)|<\varepsilon$ a. e. over each member. Thus $|f(x)-g(x)|<\varepsilon$ a. e. over I. Hence, and from the above: $I\in\mathcal{E}$.

We see therefore that $\not\exists x_0$ in P and I in $\mathcal{E}$ such that $x_0\in I$. Hence, condition (iv) is satisfied.

We conclude: $(0,1)\in\mathcal{E}$, i. e. there exists a function $\varphi(x)$ in $\mathcal{B}(\leqslant\alpha)$ and such that $|f(x)-\varphi(x)|<\varepsilon$ a. e. over $(0,1)$, and hence a. e. over $[0,1]$. Theorem I is thus proved.

Theorem I leads to the obvious but important

COROLLARY I. *Under conditions of theorem* I *a sequence of functions* $\{f_n(x)\}$ *exists, such that for each* n:

(i) $f_n(x)\in\mathcal{B}(\leqslant\alpha)$, (ii) $|f_n(x)-f(x)|<2^{-n}$ *a. e.*

Proof of theorem II. We recall the following known facts:

(a) If $g(x)\in\mathcal{B}(\leqslant\alpha)$ and $f(x)\in\mathcal{B}(\leqslant\alpha)$ then: $[g(x)-f(x)]\in\mathcal{B}(\leqslant\alpha)$.

(b) If $g_r(x)\in\mathcal{B}(\leqslant\alpha)$ for each $r\leqslant n$, then: $(\sum_{r=1}^{n} g_r(x))\in\mathcal{B}(\leqslant\alpha)$.

(c) If $g_n(x)\to g(x)$ uniformly over $[0,1]$ as $n\to\infty$ and if $g_n(x)\in\mathcal{B}(\leqslant\alpha)$ for each n, then: $g(x)\in\mathcal{B}(\leqslant\alpha)$.

(d) Let $[g(x)]_a^b$ denote $g(x)$ truncated by bounds a and b. Then if $g(x)$ belongs to $\mathcal{B}(\leqslant\alpha)$ so does $[g(x)]_a^b$.

Under conditions of theorem I corollary I holds. By that corollary a sequence $\{f_n(x)\}$ exists such that for each n: $f_n(x)\in\mathcal{B}(\leqslant\alpha)$ and $|f_n(x)-f(x)|<2^{-n}$ a. e., i. e.: $\mu\{E(|f_n(x)-f(x)|\geqslant 2^{-n})\}=0$. Then putting $X=\bigcup_{n=1}^{\infty}E(|f_n(x)-f(x)|\geqslant 2^{-n})$ and $T=[0,1]-X$ we have: $\mu(X)=0$, $\mu(T)=1$ and for each x in T and each n: $|f_n(x)-f(x)|<2^{-n}$.

Next for all x put $g_1(x)=f_1(x)$ and for $n>1$

$$g_n(x)=\sum_{k=2}^{n}[f_k(x)]-f_{k-1}(x)]_{-3\cdot 2^{-n}}^{3\cdot 2^{-n}}+f_1(x)\,.$$

Now, since $f_k(x)$ $(k=1,2,\ldots)$ belongs to $\mathcal{B}(\leqslant\alpha)$ by (a), (b) and (d) $g_n(x)$ does for each n.

We observe next that by Weierstrass' M-test $g_n(x)$ converges uniformly over $[0,1]$ to some limit $g(x)$, which by (c) belongs to $\mathcal{B}(\leqslant\alpha)$.

Now — as stated above — over T:

$$|f_n(x)-f(x)|<2^{-n} \quad \text{for any } n.$$

Thus for all x in T

$$|f_n(x)-f_{n-1}(x)|<3\cdot 2^{-n}$$

and hence $f_n(x)=g_n(x)$.

Hence over T: $f(x)=\lim_{n\to\infty}f_n(x)=g(x)$. This completes the proof of theorem II.

Notes to theorems I and II. Note I. In problem P each non-empty closed set F includes a point x_0 such that a function $g(x)$ exists which is a. e. equal to $f(x)$ and for which: $\lim_{\substack{x\to x_0\\ x\in F}} g(x)=f(x_0)$. This means that for any $\varepsilon>0$ there is an interval I open w. r. t. $[0,1]$ including x_0 and such that $|g(x)-f(x_0)|<\varepsilon$ for all x in $I\cap F$. In this case, since $f(x)=g(x)$ a. e., $|f(x)-f(x_0)|<\varepsilon$ a. e. over $I\cap F$. Now $f(x_0)$ is a constant and belongs therefore to $\mathcal{B}(\leqslant 1)$. Thus $f(x)$ has property $D(1)$ over F at x_0. The affirmative answer to the question is therefore a special case of theorem II.

Note II. By considering the characteristic function of a nowhere dense, closed set of positive measure, the reader can easily verify that neither theorem is true for $\alpha=0$.

Note III. Let us introduce one more difinition: A function $f(x)$ has property M if for any $\varepsilon>0$ and any closed set F there is x_0 in F, an open neighbourhood I of x_0 and a measurable function $g(x)$ such that $f(x)=g(x)$ over $I\cap F$.

Since any measurable function $g(x)$ is a. e. equal to a member of $\mathcal{B}(\leqslant 2)$, if $f(x)$ has property M it satisfies conditions of theorem I for $\alpha=2$, and is then — by theorem II — a. e. equal to a member of $\mathcal{B}(\leqslant 2)$. The same conclusion is obviously reached for $f(x)$ if it satisfies conditions of theorem I for any α.

Note IV. All results of this paper, except those of note III, remain valid if the class of sets of measure zero is replaced by any σ-ideal of subsets of $[0,1]$, i. e. a class of such subsets closed under enumerable union and the passage from a set to any of its subsets.

References

[1] Ch. J. de la Vallée Poussin, *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*, Paris 1916.

[2] H. Kestelman, *The modern theories of integration.*

DEPARTMENT OF MATHEMATICS,
THE UNIVERSITY,
READING, ENGLAND

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