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An abstract form of the measure theoretic method of exhaustion

by

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In many instances of partially ordered sets X it is very desirable to obtain a least upper bound of some $Z \subset X$, especially in the case where it is unique, as the least upper bound of some countable subset of Z. If this is possible, the order completion of X, when it exists, is in a sense $\aleph_0$-accessible.

Such is the situation in the case where X is the family of all measurable functions on a measure space $(X, \mathfrak{M}, \mu)$, which is of vital importance to the theory of stochastic processes. A similar situation exists for some classes of partially ordered linear spaces of Kantorovitch as well as in many other instances.

In this paper this common situation is described in an abstract form and a theorem is proved which is named the *Principle of Exhaustion*. This theorem can be of some use in dealing with particular problems of the kind.

In particular, from the point of view of the measure theory, this theorem can be regarded as an abstract formulation of the so called *method of exhaustion* (e. g. see [1], p. 76). It may appear surprising that this method, which is a very frequent tool in the measure theory, turns out to be due entirely to partial ordering. On the other hand, if a measure μ defined on a σ-field $\mathfrak{B}$ is regarded as an order-preserving mapping of $\mathfrak{B}$ (partially ordered by inclusion) into the real line (with the usual ordering) (see [2]), then it can be seen that this generalization is quite natural. It is very interesting that the Radon-Nikodym theorem can be obtained directly by the application of the Principle of Exhaustion as it is stated in [1]. It should be noticed, however, that there is a proof of the Radon-Nikodym theorem, where the Principle of Exhaustion is omitted (see e. g. [3]).

Let X be an arbitrary abstract set. X is called *partially ordered*, if there is a relation $x \leqslant y$ in X which is transitive and semi-antisymmetric, i. e. such that $x \leqslant y$ and $y \leqslant x$ implies $x = y$.

An arbitrary $Z \subset X$ is said to be a *net* of elements of X, if for each pair $x, y \in Z$ there is an element $z \in Z$ such that $x, y \leqslant z$.

A net $Z \subset X$ is *bounded*, if there is an $x_0 \in X$ with $Z \leqslant x_0$ (where $Z \leqslant x_0$ means: $x \in Z$ implies $x \leqslant x_0$).

Let $Z \subset X$ be an arbitrary net. We call $x_0 \in X$ the *limit* of Z and we write $x_0 = \lim Z$, if

a. $Z \leqslant x_0$,

b. If $Z \leqslant x_0'$, then $x_0 \leqslant x_0'$.

It is easy to see that $\lim Z$ is simply the least upper bound of Z.

Let $\aleph$ be an arbitrary fixed cardinal number. A net $Z \subset X$ is said to be an $\aleph$-*net*, if $\overline{\overline{Z}} = \aleph$. We say that X is $\aleph$-*complete*, if each bounded $\aleph$-net $Z \subset X$ has the limit in X. X is said to be *complete*, if it is $\aleph$-complete for each $\aleph \leqslant \overline{\overline{X}}$. A complete partially ordered set X is said to be $\aleph$-*accessible*, if each bounded net $Z \subset X$ contains an $\aleph$-subnet $U \subset Z$ with $\lim U = \lim Z$.

Let X and Y be partially ordered sets and let f be a mapping of X into Y. We say that f is *monotone*, if $x \leqslant x'$ implies $fx \leqslant fx'$, we say that f is *strictly monotone*, if f is monotone and $x \leqslant x'$ and $fx = fx'$ implies $x = x'$, and we say that f is $\aleph$-*continuous*, if for each $\aleph$-net $Z \subset X$ we have $\lim fZ = f(\lim Z)$, provided $\lim Z$ exists; f is *continuous*, if it is $\aleph$-continuous for each $\aleph \leqslant \overline{\overline{X}}$.

In the case where f is monotone the condition of $\aleph_0$-continuity is equivalent to the following one: for each countable non-decreasing sequence (x_n) (i. e. such that $x_n \leqslant x_{n+1}$ for $n = 1, 2, \ldots$), $\lim fx_n = f(\lim x_n)$, provided $\lim x_n$ exists.

In fact, let Z be an arbitrary $\aleph_0$-net in X and let $\lim Z$ exist. We can set $Z = (x_n)$, where (x_n) is a countable sequence. Let $\bar{x}_1 = x_1$, $\bar{x}_2 = x_{n_2}$, where $\bar{x}_1 \leqslant x_{n_2}$, $x_2 \leqslant x_{n_2}$ and $\bar{x}_m = x_{n_m}$, where $\bar{x}_{m-1} \leqslant x_{n_m}$, $x_m \leqslant x_{n_m}$. Clearly $(\bar{x}_m)$ is non-decreasing and $\lim \bar{x}_n = \lim Z$, $\lim f\bar{x}_n = \lim fZ$. Therefore $\lim fZ = \lim f\bar{x}_n = f \lim \bar{x}_n = f \lim Z$ and hence f is $\aleph_0$-continuous.

THEOREM 1 (the Principle of Exhaustion). *Let f be a strictly monotone continuous mapping of a partially ordered $\aleph$-complete set X into a partially ordered complete and $\aleph$-accessible set Y. Then*

(a) *X is complete and $\aleph$-accessible,*

(b) *If $Z \subset X$ is a net with all the $\aleph$-subnets bounded and, if fZ is bounded, then so is Z,*

(c) *f is continuous.*

At first we give some applications of Theorem 1.

A. Let $(T, \mathfrak{B}, \mu)$ be a measure space, where T is an abstract set, $\mathfrak{B}$ a σ-field of subsets of T and μ a σ-measure defined on $\mathfrak{B}$. We denote by S the set of all classes of $\mathfrak{B}$-measurable functions, where $x = y$, if

$x(t) = y(t)$ almost everywhere. It is well-known that with the usual partial ordering S is an $\aleph_0$-complete partially ordered set.

Similarly

$$L_a \overset{\text{def}}{=} \left\{x \in S \colon \int \left|a\big(x(t)\big)\right| d\mu(t) < +\infty\right\},$$

where a is a strictly monotone mapping of R into itself, is an $\aleph_0$-complete subset of S.

We prove the following

THEOREM 2. (i) *If μ is σ-finite, then S is complete, $\aleph_0$-accessible.*

(ii) *If there is a decomposition of T into $\aleph$ mutually almost disjoint sets from $\mathfrak{B}$ with finite positive measures, then S is complete, $\aleph$-accessible and is not $\aleph'$-accessible for $\aleph' < \aleph$* ([1]).

(iii) *If a is a strictly monotone mapping of R into itself, then the set $L_a \subset S$ is complete, $\aleph_0$-accessible.*

From (i) immediately follows the Theorem on the existence of a separable stochastic process (see [1], Theorem 2.4).

Further, we obtain the completeness of the Kantorovitch linear lattices of measurable functions on a given measure space (see [4]).

Let $Y = R =$ the real line, $X = S$. If μ is σ-finite, then there is an integrable function $x_0 \geqslant 0$ such that $x_0 x = x_0 y$ holds in the case of $x = y$ only, and we have

$$f_{x_0}(x) \overset{\text{def}}{=} \int x_0(t) \frac{x(t)}{1 + |x(t)|} d\mu(t) < +\infty$$

for each $x \in S$ (if μ is finite, we can set $x_0(t) \equiv 1$). Clearly R is complete, $\aleph_0$-accessible. Hence, by virtue of the theorem, if μ is σ-finite, then S is complete, $\aleph_0$-accessible and (i) holds.

In the case where μ is not σ-finite let $\mathfrak{F}$ consist of mutually almost disjoint sets of finite measure with $T = \bigcup_{A \in \mathfrak{F}} A$ and let $\aleph = \overline{\overline{\mathfrak{F}}}$.

We denote by $R^{\aleph}$ the Cartesian product of $\aleph$ copies of the real line R, i. e. the set of all real-valued functions $\{t(A)\}$ defined on the set $\mathfrak{F}$. We introduce in $R^{\aleph}$ a relation of partial ordering setting $t_1 \leqslant t_2$, if, for each $A \in \mathfrak{F}$, $t_1(A) \leqslant t_2(A)$. Since R is complete, $\aleph_0$-accessible, $R^{\aleph}$ is complete, $\aleph\aleph_0 = \aleph$-accessible. We define a mapping g of S into $R^{\aleph}$ as follows:

$$g(x) = \left\{\int \chi_A(t) \frac{x(t)}{1 + |x(t)|} d\mu(t)\right\},$$

where χ_A is the characteristic function of $A \in \mathfrak{F}$. Each component

$$(*) \qquad \int \chi_A(t) \frac{x(t)}{1 + |x(t)|} d\mu(t)$$

([1]) From (ii) it follows that there exists exactly one cardinal number $\aleph$ that satisfies the conditions of (ii).

of $g(x)$ is monotone and, in virtue of preceding example, it is continuous. Thus, $g(x)$ is monotone and continuous. Further, in virtue of $(*)$, g is strictly monotone and by the Principle of Exhaustion we infer that S is complete, $\aleph$-accessible. Clearly, if $\mu(A)>0$ for each $A\in\mathfrak{F}$, then the net generated by $\{\chi_A\colon A\in\mathfrak{F}\}$ is not $\aleph'$-accessible for $\aleph'<\aleph$ and thus (ii) holds.

To prove (iii) let in the sequel a denote a strictly monotone mapping of R into itself.

The mapping

$$f_a(x)\overset{\text{def}}{=\!=}\int a\big(x(t)\big)\,d\mu(t)$$

of L_a into R is strictly monotone and $\aleph$-continuous. Hence, in virtue of the Principle of Exhaustion, L_a is complete $\aleph_0$-accessible and (iii) holds.

Thus the proof of Theorem 2 is complete.

B. Now we will show that the Principle of Exhaustion, as formulated in our Theorem, is essentially the only tool which is used in [1] to prove the Radon-Nikodym theorem.

In order to obtain the Radon-Nikodym theorem it is sufficient to show that, if ν and μ are finite non-negative σ-measures on a measure space $(T,\mathfrak{B})$ and if ν is absolutely continuous with respect to μ, then there exists a finite valued $\mathfrak{B}$-measurable function f_0 such that $\nu(E)=\int\limits_E f_0\,d\mu$.

Let $\mathfrak{M}$ be the family of all measurable finite-valued functions f with $\int\limits_E f\,d\mu\leqslant\nu(E)$ for $E\in\mathfrak{B}$. Clearly for $f_1,f_2\in\mathfrak{M}$ there are $E_1,E_2\in\mathfrak{B}$ such that $E_1\cup E_2=T$ and

$$(f_1\vee f_2)t=\begin{cases}f_1(t) & \text{for}\quad t\in E_1,\\ f_2(t) & \text{for}\quad t\in E_2.\end{cases}$$

Hence, $\mathfrak{M}$ is a net with respect to the usual partial ordering.

If f_0 is a μ-essentially maximal function in $\mathfrak{M}$, then it satisfies the equality $\nu(E)=\int\limits_E f_0\,d\mu$. Indeed, if for a certain $E_0\in\mathfrak{B}$, $\nu(E_0)>\int\limits_{E_0}f_0\,d\mu$, then there is a number $\varepsilon>0$ with $\nu(E_0)-\int\limits_{E_0}f_0\,d\mu>\varepsilon\mu(E_0)$. By virtue of the Hahn decomposition there is a maximal positive set A_0 [2] of $\eta(E)=\nu(E)-\int\limits_E f_0\,d\mu-\varepsilon\mu(E)$.

We have $f_0+\varepsilon\chi_{A_0}\in\mathfrak{K}$ and $f_0+\varepsilon\chi_{A_0}$ non $\leqslant f_0$ a. e. μ.

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[2] A maximal positive set (i. e. a maximal set having all subsets with non-negative measures) can be obtained likewise by taking the upper bounds of "maximal" families of mutually disjoint sets with positive (or negative) measures. It can be checked directly that such families are often countable and so the Principle of Exhaustion, as formulated in Theorem 1, is superfluous.

But $\mathfrak{M}$ is $\aleph_0$-complete and $\varphi(f)=\int f\,d\mu$ is μ essentially strictly monotone, and by the Principle of Exhaustion there is an $f_0\in\mathfrak{M}$ which is μ essentially maximal and the Radon-Nikodym theorem is proved.

Before the proof of Theorem 1 we introduce some useful definitions and lemmas.

Let X be a partially ordered set and let Z be an arbitrary subset of X. We call Z *$\aleph$-closed* in X, if, for each $\aleph$-net, $U\subset Z$, $\lim U\in Z$, provided it exists in X. The meet of all $\aleph$-closed sets containing the given set $Z\subset X$ is said to be the *$\aleph$-closure of* Z and is denoted by $\bar{Z}$.

Let f be a mapping of the set X into a partially ordered set Y and let $Z^*\subset X$ be an extension of Z. We say that Z^* is *admissible* with respect to f if, for each $y\in Y$, $fZ\leqslant y$ implies $fZ^*\leqslant y$.

LEMMA 1. (a) *If X is $\aleph$-complete and Z is a net, then $\bar{Z}$ is a net also.*

(b) *If f is an $\aleph$-continuous mapping of X into a partially ordered set Y, then for each subset $Z\subset X$, $\bar{Z}\subset Z$ is admissible with respect to f.*

Proof. Let U be an arbitrary subset of X. We set $U^0=$ the set of all limits $\lim W$ of the $\aleph$-nets $W\subset U$.

Clearly $U\subset U^0$ and

(1) If X is $\aleph$-complete and U is a net, then U^0 is also a net.

(2) If f is an $\aleph$-continuous mapping of X into a partially ordered set Y, then $U^0\subset U$ is admissible with respect to f.

In fact, let U be a net and let $x_1,x_2\in U^0$. Then, there are $\aleph$-nets $U_1,U_2\subset X$ with $\lim U_1=x_1$, $\lim U_2=x_2$. Since U is a net, there is an $\aleph$-net $U_3\subset U$ with $U_1\cup U_2\subset U_3$. If X is $\aleph$-complete and $\lim U_3=x_3$, then $x_1\leqslant x_3$ and $x_2\leqslant x_3$, and so U^0 is a net and (1) holds.

If $fU\leqslant y$ and $\bar{x}\in U^0$, then there is an $\aleph$-net $U'\subset U$ with $\lim U'=\bar{x}$. Since f is $\aleph$ continuous, it follows that $f(\lim U')=f\bar{x}\leqslant y$, and so U^0 is admissible with respect to f and (2) holds.

Now, let Z be an arbitrary subset of X and let $(Z_\xi)_{\xi<a}$ be a transfinite sequence, where $Z_1=Z$, $Z_{\xi<1}=Z_\xi^0$ and $Z_\eta=\bigcup\limits_{\xi<\eta}Z_\xi$ if η is a limit number. It is clear that for a sufficiently large ordinal number ξ_0 we have $Z_{\xi_0+1}=Z_{\xi_0}$ and then $\bar{Z}=\bigcup\limits_{\xi<\xi_0}Z_\xi$. If Z_ξ is a net, then, by (1), $Z_{\xi+1}$ is also a net and if Z_ξ are nets for $\xi<\eta$, where η is a limit number, then Z_ξ is also a net. Hence if Z is a net, then so is $\bar{Z}$ and (a) holds. Let f satisfy the assumptions of (b). By virtue of (2) if $Z_\xi\supset Z$ is admissible with respect to f, then so is $Z_{\xi+1}\supset Z$.

If $Z_\xi\supset Z$ are admissible for $\xi<\eta$, where η is a limit number, then $Z_\eta\supset Z$ is also admissible. Hence (b) holds and thus Lemma 1 is proved.

LEMMA 2. *Let f be a strictly monotone mapping of a partially ordered set X into a partially ordered set Y and let Z be a net in X. If fx_0 is the least upper bound of fZ and $x_0 \in Z$, then x_0 is the least upper bound of Z.*

Proof. Let $x \in Z$. Since Z is a net, there is $z_x \in Z$ such that $x_0 \leqslant z_x$, $x \leqslant z_x$. Since fx_0 is the least upper bound of fZ, we have $fz_x \leqslant fx_0$ and since f is monotone, $fx_0 \leqslant fz_x$. Hence $fx_0 = fz_x$. But f is strictly monotone and therefore $z_x = x_0$. Hence $x \leqslant x_0$ and this completes the proof.

LEMMA 3. *An $\aleph$-continuous, mapping f of an $\aleph$-accessible complete partially ordered set X into an $\aleph$-complete partially ordered set Y is continuous.*

Proof. Let U be an arbitrary bounded net in X.

By the $\aleph$-accessibility of X, there is an $\aleph$-net $U_0 \subset U$ with $\lim U_0 = \lim U$. By the $\aleph$-continuity of f we have $f(\lim U) = f(\lim U_0) = \lim(fU_0) \leqslant \lim(fU)$. But $U \leqslant \lim U$ and $fU \leqslant f(\lim U)$, $\lim fU \leqslant f(\lim U)$. Hence $f(\lim U) = \lim(fU)$ and f is continuous.

Proof of Theorem 1. Suppose that each $\aleph$-subset of Z is bounded in X and let fZ be bounded in Y. Since Y is complete and $\aleph$-accessible, there is a subset $Z_0 \subset Z$ with $\overline{\overline{Z}}_0 \leqslant \aleph$ such that $y_0 \overset{\mathrm{df}}{=} \lim fZ = \lim fZ_0$. But Z is a net and hence there is an $\aleph$-net Z_1, with $Z_0 \subset Z_1 \subset Z$. Clearly $y_0 = \lim fZ_1$. Since X is $\aleph$-complete, there is $x_0 = \lim Z_1$ and by the $\aleph$-continuity of f we obtain $fx_0 = y_0$. By Lemma 1, $y_0 = \lim f\bar{Z}$ and since $x_0 \in \bar{Z}$ we can apply Lemma 2. Hence, x_0 is the least upper bound of $\bar{Z}$ and therefore also the least upper bound of Z. Thus X is complete and $\aleph$-accessible. The continuity of f follows immediately from Lemma 3 and thus Theorem 1 is proved.

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A problem on Baire classes

by

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Introduction. In what follows all sets are subsets of $[0,1]$, all functions are defined thereover and have real values.

E. Marczewski introduced the following definition:

A function $f(x)$ is *almost continuous with respect to a closed set F at a point x_0 in F*, if a function $g(x)$ exists, such that $g(x) = f(x)$ a. e. and if $\lim\limits_{\substack{x\to x_0\\ x\in F}} g(x) = f(x_0)$.

He raised the following problem:

P. Let $f(x)$ be such that for every non-empty closed set F there is x_0 in F such that $f(x)$ is almost continuous w. r. t. F at x_0. Is then $f(x)$ a. e. equal to a function of Baire class $\leqslant 1$?

A modified form of the problem has been generalized by T. Traczyk to one concerning a Boolean algebra on which a certain topology is defined. He proved that under conditions of P a set X of zero measure exists, such that $f(x)$ is of Baire class $\leqslant 1$ w. r. t. $[0,1]-X$.

Independently the present author has generalized in a different direction and has concluded that — under conditions of P — a function $g(x)$ of Baire class $\leqslant 1$ w. r. t. $[0,1]$ exists, such that $f(x) = g(x)$ a. e. This conclusion is a special case of theorem II of this paper — which does not seem to be covered by Traczyk's work.

Though theorem II provides a more general result than is asked for in Professor Marczewski's question, I wish to emphasize that it was his problem that gave me the original incentive.

I must express my gratitude to Mr. H. Kestelman for a very valuable suggestion which shortened the argument leading to theorem I considerably.

Lastly I wish to thank the referee of Fundamenta Mathematicae for useful suggestions shortening the text.

Statement of the theorems. Before stating theorems I and II let us introduce the following definition:

Given a closed set F, and a point x_0 in F, a function $f(x)$ *has property $D(\alpha)$ with respect to F at x_0*, if for any $\varepsilon > 0$ there is an open neighbourhood