On the Gentzen Theorem

by

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Recently Kanger [4] has published a proof of the completeness theorem for some modification of the Gentzen formalism (1). In consequence he has given — for that modified formalism — the first model proof of the Gentzen Hauptsatz (2). The essential point of this modification is that all the applied rules of inference are equivalence rules, i.e. they always transform premises into equivalent conclusions. The idea of Kanger's proof is similar to that of Beth (1).

Another interesting modification of the Gentzen formalism has recently been given by Craig [2].

This paper contains a simplification of the proof of Kanger. More exactly, that simplification is the subject of the second section of our paper. To outline the main ideas of the proof, we restrict ourselves to finite sequents only. The general case can be obtained in the analogous way.

The most essential point of Gentzen's idea is that theorems of the predicate calculus can be proved in his formalism by passing from shorter formulas to longer ones only. This aim may be also realized, without the Gentzen notion of a sequent and the sign $\Rightarrow$ (3). Some modification of the Gentzen formalism of the kind mentioned is the subject of the first section of our paper. It is a continuation of Kanger's idea. With every formula $a$ of the first order predicate calculus, we associate uniquely its diagram, i.e. a system of finite sequences of formulas shorter than $a$, such that it gives either a proof of $a$ in the formalism under consideration or a countermodel for that formula. The idea of diagram is, indeed, the same as that of trees used by Beth.

1. Diagrams of formulas. Let $\mathcal{L}$ be a formalized language of the first order predicate calculus. We suppose that:

1° the signs of bounded individual variables are different from the signs of free individual variables, the former being denoted by $\xi, \eta, ...$, the latter by $x, y, ...$.

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(2) See e.g. Schütte [6].
We have the identities
\[
\begin{align*}
(a \land \beta)_m &= a_m \land \beta_m , \\
(a \lor \beta)_m &= a_m \lor \beta_m , \\
(a \to \beta)_m &= a_m \to \beta_m , \\
(\forall \alpha)(a(\xi))_m &= \bigcup_{\xi \in J} a_m(\xi) , \\
(\exists \alpha)(a(\xi))_m &= \bigcup_{\xi \in J} a_m(\xi) ,
\end{align*}
\]
which may be assumed as the inductive definition of \(a_m\).

A formula \(a\) is said to be valid in a realization \(\mathfrak{R}\) in a set \(J \neq 0\) if \(a_m\) is identically equal to the unit element \(\top\) of \(B\). It is said to be valid in a set \(J \neq 0\) if it is valid in every realization \(\mathfrak{R}\) in \(J\). Hence, \(a\) is not valid in the set \(J \neq 0\) if there exists a realization \(\mathfrak{R}\) in \(J\), such that \(a_m\) assumes the zero element \(\bot\) of \(B\) at least once as its value. \(a\) is said to be valid if it is valid in any set \(J \neq 0\).

The letters \(\Gamma\) and \(A\) (with indices if necessary) will always denote finite sequences of formulas, the empty sequence included. If \(\Gamma\) is a sequence \(a_1, \ldots, a_n\) and \(\Gamma'\) is a sequence \(b_1, \ldots, b_m\) \((m \geq 0, n \geq 0)\), then \(\Gamma, a, \Gamma'\) is the sequence \(a_1, \ldots, a_n, a, b_1, \ldots, b_m\). The meaning of the symbol \(\Gamma, a, \Gamma', \beta\) is similar.

A formula \(a\) is said to be indecomposable if either it is elementary, or it is the negation of an elementary formula.

A sequence \(\Gamma\) of formulas is said to be indecomposable if it is formed exclusively of indecomposable formulas. In particular, the empty sequence is indecomposable.

A sequence \(\Gamma\) is said to be fundamental if it contains simultaneously a formula \(a\) and its negation \((-a)\).

By a scheme we shall understand a pair \((\Gamma, \Gamma')\) or a triple \((\Gamma, \Gamma, \Gamma')\) (of non-empty sequences of formulas) written usually in the forms
\[
\frac{\Gamma}{\Gamma'} \quad \frac{\Gamma'}{\Gamma} \quad \frac{\Gamma'}{\Gamma, \Gamma'}
\]
respectively. \(\Gamma\) is called the conclusion of the scheme, and \(\Gamma', \Gamma\) are called its premises. More exactly, if the scheme is of the second form \((3)\), then \(\Gamma'\) and \(\Gamma\) are called the left and the right premise respectively.

If the scheme is of the first form \((3)\), then \(\Gamma'\) is called simultaneously the left and the right premise.

In the sequel we shall consider only the following schemes where \(\Gamma\) denotes an indecomposable sequence (maybe empty):
\[
\begin{align*}
\text{(D)} & & \frac{\Gamma, (a \lor \beta), \Gamma'}{\Gamma, a, \beta, \Gamma'} , \\
\text{(-D)} & & \frac{\Gamma, (-a), \Gamma'}{\Gamma, (a \lor \beta), \Gamma'} , \\
\text{(-D)} & & \frac{\Gamma, (-a), \Gamma'}{\Gamma, (a \lor \beta), \Gamma'}.
\end{align*}
\]
1) \( \Omega_0 \) is the sequence composed only of the formula \( \omega \).

2) \( \Omega_N \) and \( \Omega_\lambda \) are defined if and only if \( \Omega_k \) is neither fundamental nor indecomposable. Then \( \Omega_k \) can be assumed to be the conclusion of exactly one of the schemes \((D), (\neg D), (C), (\neg C), (I), (\neg I), (N), (E), (\neg E), (U), (\neg U)\). Define \( \Omega_N \) and \( \Omega_\lambda \) as the left and the right premise of that scheme, respectively. If that scheme is \((E)\), we assume additionally that the term \( t \) mentioned in \((E)\) is the first term in the sequence \((1)\) such that \( \alpha(t) \) does not appear in any sequence \( \Omega_j \) with \( j < k \). If that scheme is \((U)\), we assume additionally that the variable \( x \) mentioned in \((U)\) is the first free individual variable in the sequence \((1)\) of all terms, such that \( x \) does not appear in any formula in \( \Omega_k \).

Observe that the diagram \((\Omega)\) of \( \omega \) is uniquely determined by \( \omega \). The diagram is said to be finite (infinite) if the set of all sequences \( i \) for which \( \Omega_k \) is defined is finite (infinite).

\( \Omega_k \) is said to be an end sequence of the diagram of \( \omega \) if \( \Omega_k \) is either fundamental or indecomposable, i.e. if \( \Omega_N \) and \( \Omega_\lambda \) are not defined.

Theorem 1. (i) If the diagram of a formula \( \omega \) is finite and all end sequences are fundamental, then \( \omega \) is valid.

(ii) In the opposite case, \( \omega \) is not valid in an enumerable set.

Proof of (i). If \( \Gamma \) is a non-empty sequence of formulas, let \( \alpha \) be the disjunction of all formulas in \( \Gamma \). Observe that

\[
\Gamma, a(\xi), \Gamma', a(\xi) \]

where \( a \) is a free individual variable which does not appear in any formula in the conclusion

\[
\Gamma', a(\xi), \Gamma'
\]

and

\[
\Gamma', (\bigcup a(\xi)), \Gamma'
\]

are valid.

(6) if \( \Gamma \) is a fundamental sequence, then \( a \) is valid.

\[
\begin{align*}
\Gamma, & a(\xi), \Gamma' \\
\Gamma, & (\bigcup a(\xi)), \Gamma'
\end{align*}
\]

is any of the schemes \((D), (\neg D), (C), (\neg C), (I), (\neg I), (N), (E), (\neg E), (U), (\neg U)\), then

\[
\begin{align*}
\Gamma, & a(\xi), \Gamma' \\
\Gamma, & (\bigcup a(\xi)), \Gamma'
\end{align*}
\]

is a rule of inference, i.e. the following property holds: if \( a_0 \) is valid (if \( a_0 \) and \( a_1 \) are valid) in a realization \( \mathfrak{M} \) in a set \( J \neq 0 \), then so is \( a_0 \).

This remark, together with the definition of the diagram \((\Omega)\) of a formula \( \omega \), implies the following property of the diagram:

(7) if \( a_0 \) and \( a_\lambda \) are defined and are valid, then \( a_0 \) is valid.

Hence we infer that if the diagram of \( \omega \) is finite and all end sequences are fundamental, then, for every \( \Omega_k \) in the diagram, \( a_0 \) is valid. In particular \( \omega = a_0 \) is valid.
Proof of (ii). If the hypothesis of (i) is not satisfied, then there exists a sequence \( j \) for which one of the following conditions holds:\(^{(*)}\):

(A) \( j \) is finite and \( \Omega_j \) is an end sequence which is not fundamental,

(B) \( j \) is infinite and, for every finite \( i \leq j \), \( \Omega_i \) is in the diagram of \( \omega \).

Let \( F_a \) be the set of all indecomposable formulas appearing in at least one \( \Omega_i, i \leq j \). Observe that if \( i \leq i' \leq j \) and an indecomposable formula appears in \( \Omega_i \), then it appears also in \( \Omega_{i'} \). Since no \( \Omega_i (i \leq j) \) is fundamental, for every elementary formula

\[
\varphi(\tau_{e_1}, \ldots, \tau_{e_{m}})
\]

at most one of the formulas \( \varphi(\tau_{e_1}, \ldots, \tau_{e_{m}}), \neg \varphi(\tau_{e_1}, \ldots, \tau_{e_{m}}) \) is in \( F_a \).

Let \( J \) be the set of all terms (i.e., the set of all \( \tau_{e_1}, \tau_{e_2}, \ldots \) in (1)).

Let \( \mathcal{M} \) be any realization of \( \mathcal{L} \) in the set \( J \) such that

\[
\varphi^\mathcal{M}(\tau_{e_1}, \ldots, \tau_{e_{m}}) = \varphi(\tau_{e_1}, \ldots, \tau_{e_{m}}),
\]

i.e., such that, for every \( m \)-argument functor \( \varphi \), the value of the corresponding function \( \varphi^\mathcal{M} \) (from \( J^m \) into \( J \)) at the point \( (\tau_{e_1}, \ldots, \tau_{e_{m}}) \) is the term \( \varphi(\tau_{e_1}, \ldots, \tau_{e_{m}}) \).

Every realization \( \mathcal{M} \) of \( \mathcal{L} \) in the set \( J \) of all terms with the above-mentioned property \( (*) \) is called canonical.

We shall precede the proof of (ii) by some lemmas on canonical realizations.

Let \( \mathcal{M} \) be any canonical realization of \( \mathcal{L} \) in the set \( J \) of all terms. According to the general definition \( (\mathrm{p. 55}) \) if \( a = a(x_1, \ldots, x_n) \) is a formula with free individual variables \( x_1, \ldots, x_n \), then \( a^\mathcal{M} \) is a function from \( J^n \) into \( B \), i.e., for every substitution of some terms \( \tau_{e_1}, \ldots, \tau_{e_n} \) for \( x_1, \ldots, x_n \) respectively the value of \( a^\mathcal{M} \) at \( (\tau_{e_1}, \ldots, \tau_{e_n}) \) is a fixed element of \( B \).

Specially interesting is the case where that substitution is the identity, i.e., \( \tau_{e_i} = x_i \) for \( i = 1, \ldots, n \). The value of \( a^\mathcal{M} \) at that special point of \( J^n \) will be denoted by \( \alpha_0^\mathcal{M} \).

By definition,

\[ \beta \]

if \( \beta \) is obtained from \( a \) by the substitution of \( \tau_{e_1}, \ldots, \tau_{e_n} \) for \( x_1, \ldots, x_n \), then the value of \( a^\mathcal{M} \) at the point \( x_1 = \tau_{e_1}, \ldots, x_n = \tau_{e_n} \) is equal to \( \beta^\mathcal{M} \).

It follows from the definition of \( F_a \) (see (A) or (B)) that

(C) If \( \alpha_0^\mathcal{M} = \top \), then the set \( F_a \) contains at least one indecomposable formula \( a \) such that \( \alpha_0^\mathcal{M} = \top \).

Indeed, let \( F \) be the set of all formulas appearing in at least one \( \Omega_i, i \leq j \). Let \( a \) be the shortest formula in \( F \) such that \( \alpha_0^\mathcal{M} = \top \). Then the formula \( a \) is indecomposable.

For suppose that \( a \) is decomposable. Then there exists a sequence \( \iota \preceq j \) (\( \iota \neq j \)) such that \( \Omega_i \) is of the form

\[
\Gamma, \alpha, \Gamma'
\]

with indecomposable \( \Gamma' \).

If \( \alpha \) is of the form

\[
\neg (\bigcup_i \beta(\xi))
\]

then the formula \( \gamma \) of the form \( \bigcap_i \neg \beta(\xi) \) is also in \( F \), has the same length as \( a \) and \( \gamma^\mathcal{M} = \top \).

If \( \alpha \) is of the form

\[
\neg (\bigcap_i \beta(\xi))
\]

then the formula \( \gamma \) of the form \( \bigcup_i \neg \beta(\xi) \) is also in \( F \), has the same length as \( a \) and \( \gamma^\mathcal{M} = \top \).

If \( \alpha \) is of the form

\[
(\beta \cup \gamma)
\]

then the formulas \( \neg \beta \) and \( \gamma \) are also in \( F \), their lengths are not greater than the length of \( a \) and either \( \neg \beta^\mathcal{M} = \top \) or \( \gamma^\mathcal{M} = \top \).

Therefore we may assume in our consideration that \( \alpha \) is neither of the form (9) nor of the forms (10), (11).

Let \( i' \) be the sequence obtained from \( i \) by adding either 0 or 1 at the end, such that \( i' \preceq j \).

If \( \alpha \) has one of the forms

\[
(\beta \cup \gamma), (\neg (\beta \cup \gamma)), (\beta \cap \gamma), (\neg (\beta \cap \gamma)), (\neg (\beta \leftrightarrow \beta)), (\neg (\neg \beta)), (\bigcup_i \beta(\xi))
\]

then it immediately follows from the definition of the schemes (D),

\[
(\neg D), (C), (\neg C), (\neg I), (N), (U)
\]

and from the definition of \( \Omega' \) that \( \Omega' \) is of the form

\[
\Gamma', \alpha_0, \Gamma''
\]

where \( \alpha_0 \) is shorter that \( a \) and \( a^\mathcal{M}_0 = \top \). This contradicts our hypothesis concerning the length of \( \alpha \).

It remains only to consider the case, where \( \alpha \) is of the form

\[
(\bigcup_i \beta(\xi))
\]

In that case \( j \) is infinite. Then, by the last of identities (2), there exists a term \( \tau \) such that the formula

\[
\gamma = \beta(\tau)
\]
satisfies the condition

\[ \gamma^a_{\mathcal{N}} = \vee. \]

It follows from the definition of the diagram (see the additional assumption to the scheme (E) in \( \mathcal{N} \)) that there exists a finite sequence \( i' \) such that \( i \leq i' < j \) and \( \gamma \) appears in \( \mathcal{L}_{i'}. \) Since \( \gamma \) is shorter than \( \alpha, \) this contradicts the hypothesis concerning the length of \( \alpha. \) This proves Lemma (O).

To prove (ii) let \( \mathcal{N} \) be the following canonical realization in the set \( J \) of all terms: the functions \( \phi_x \) are defined by \( (\ast); \) if \( \varphi \) is an \( n \)-argument predicate, then the value \( \phi_x(t_1, \ldots, t_n) \) of \( \phi_x \) at the point \( (t_1, \ldots, t_n) \) \( \in J^n \) is \( \vee \) if the formula \( \varphi(t_1, \ldots, t_n) \) belongs to \( P_a; \) and it is \( \wedge \) in the opposite case. Then, for every formula \( \alpha \) in \( P_a, \)

\[ \alpha^a_{\mathcal{N}} = \wedge. \]

This implies, by (O), that

\[ \alpha^a_{\mathcal{N}} = \wedge, \]

i. e. that \( \alpha \) is not valid in the realization \( \mathcal{N}. \) This completes the proof of Theorem 1.

As we have observed in the proof of (i), if

\[ \frac{\Gamma}{\mathcal{F}} \text{ or } \frac{\Gamma}{\mathcal{F}^a; \mathcal{F}^a} \]

is any of the schemes (D), (D), (C), (C), (I), (I), (N), (E), (E), (U), (U), then

\[ \frac{\Gamma^a}{\mathcal{F}} \text{ and } \frac{\mathcal{F}^a; \mathcal{F}^a}{\mathcal{F}} \]

may be treated as the corresponding rules of inference, denoted, respectively by \((D^a), (D^a), (C^a), (C^a), (I^a), (I^a), (N^a), (E^a), (E^a), (U^a), (U^a)).\)

The following statement results immediately from Theorem 1:

**Corollary 1.** The set composed of all formulas which can be obtained from fundamental sequences by means of the rules of inferences \((D^a), (D^a), (C^a), (C^a), (I^a), (I^a), (N^a), (E^a), (E^a), (U^a), (U^a)\) coincides with the set of all valid formulas. This set coincides with the set of formulas \( \alpha \) having a finite diagram whose all end sequences are fundamental. The diagram of any such formula \( \alpha \) determines a formalized proof of \( \alpha \) in the formalism under consideration.

2. **Diagrams of sequents.** We shall assume here the terminology and the notation of the previous section.

Consider a formalized language \( \mathcal{L} \) of the first order predicate calculus satisfying conditions \( i^0 \) and \( \mathcal{L}^0 \) mentioned in section 1. Besides finite sequences of formulas in \( \mathcal{L} \) we shall consider sequents, i. e. expressions of the form

\[ \Gamma \Rightarrow \Delta \]

where \( \Gamma \) and \( \Delta \) are finite sequences of formulas (which may be empty). The sequence \( \Gamma \) will be called the antecedent and the sequence \( \Delta \) the succedent of that sequent. Sequents will be denoted by \( \Sigma, \Pi \) with indices if necessary.

We recall the definition of validity of sequents. Let \( \mathcal{N} \) be a realization of \( \mathcal{L} \) in a set \( J \neq 0. \) Let \( \gamma_{\mathcal{N}} \) be the conjunction of all formulas in \( \Gamma \) if \( \Gamma \) is a non-empty sequence of formulas, and a fixed formula of the form \( \{a \wedge -a\} \) in the opposite case. Let \( \delta_{\mathcal{N}} \) be the disjunction of all formulas in \( \Delta \) if \( \delta \) is non-empty, and a fixed formula of the form \( \{a \wedge -a\} \) in the opposite case. In particular, if \( \Gamma (\Delta) \) is composed of only one formula \( \beta \), then the conjunction \( \gamma_{\mathcal{N}} \) (the disjunction \( \delta_{\mathcal{N}} \)) reduces to this formula \( \beta. \) Interpreting the sign \( \Rightarrow \) as the Boolean operation of the co-difference \( (\ast) \) in the two-element Boolean algebra \( B \) and \( \gamma_{\mathcal{N}}, \delta_{\mathcal{N}} \) as the function \( \gamma_{\mathcal{N}}M \) and \( \delta_{\mathcal{N}}M \) respectively, we can treat the sequent \( \Sigma \) of the form \( (\ast) \) as the function \( \Sigma_{\mathcal{N}} = (\gamma_{\mathcal{N}} \delta_{\mathcal{N}})_{\mathcal{N}}. \) A sequent \( \Sigma \) is said to be valid in the realization \( \mathcal{N} \) in the set \( J \) if \( \Sigma_{\mathcal{N}} \) is identically equal to the unit element \( \vee \) of \( B. \) It is said to be valid in a set \( J \neq 0 \) if \( \Sigma_{\mathcal{N}} \) is in every realization \( \mathcal{N} \) in \( J. \) \( \Sigma \) is said to be valid if it is valid in every set \( J \neq 0. \)

A sequent will be called **elementary** if it is empty or composed of only elementary formulas. A **sequent** will be called **elementary** if its antecedent and succedent are elementary sequences. A **sequent** will be called **fundamental** provided its antecedent and succedent contain simultaneously a formula \( a. \)

In the sequel we shall consider some schemes for sequents, i. e. some pairs \((\Sigma, \Sigma^a)\) or triples \((\Sigma, \Sigma^a, \Sigma^b)\) of sequents written in the form

\[ \frac{\Sigma}{\Sigma^a} \text{ or } \frac{\Sigma}{\Sigma^a; \Sigma^b} \]

respectively.

\( \Sigma \) will be called the **conclusion** of the scheme and \( \Sigma^a, \Sigma^b \) the left and the right **premise**, respectively (in the case of \( \Sigma^a, \Sigma^b \) will be called simultaneously the left and the right **premise**).

We shall examine the following schemes (\( \Gamma, \Delta \) denote here elementary sequences only):

\[
\begin{align*}
\frac{\Gamma; (a \wedge \beta), \Gamma' \Rightarrow \Delta'}{\Gamma, \alpha, \Gamma' \Rightarrow \Delta'} & \quad (\text{DA}) \\
\frac{\Gamma' \Rightarrow \Delta'; \Gamma, \beta, \Gamma' \Rightarrow \Delta'}{\Gamma' \Rightarrow \Delta'; \alpha, \beta, \Gamma' \Rightarrow \Delta'} & \quad (\text{DS})
\end{align*}
\]

(1) For any elements \( a, b \) of a Boolean algebra \( A, \) we define the co-difference \( a \Rightarrow b \) as \( \neg a \cup b. \)
\( \Pi_i \) is said to be an end sequent of the diagram of \( \Pi \) if it is either fundamental or elementary.

**Theorem 2.** (i) If the diagram of a sequent \( \Pi \) is finite and all end sequents are fundamental, then \( \Pi \) is valid.

(ii) In the opposite case, \( \Pi \) is not valid in an enumerable set.

The statement (i) follows immediately from the definition of diagram and from the facts that every fundamental sequent is valid and that the validity of premises of the considered schemes in a realization \( \mathcal{R} \) in a set \( J \neq 0 \) implies the validity of the conclusion.

The statement (ii) can be established in an analogous way to that used in the proof of Theorem 1 (ii). In fact, if the hypothesis of (i) is not satisfied, then there exists a sequence \( j \) for which one of the following conditions holds:

\( (A^i) \) j is finite and \( \Pi_j \) is an end sequent which is not fundamental,

\( (B^i) \) j is infinite and, for every finite \( i \leq j \), \( \Pi_i \) is in the diagram of \( \Pi \).

Let \( A_{\alpha} \), \( (S_b) \) be the set of all elementary formulas which appear in all antecedents of \( \Pi_i \) (\( i \leq j \)) (in all antecedents of \( \Pi_i \) (\( i \leq j \))). Let \( E_\alpha = A_{\alpha} + S_b \). Notice that \( \Pi \) also occurs in \( E_i \). More exactly, if it occurs in the antecedent (in the succedent of \( \Pi_i \)), then it also occurs in \( \Pi_i \).

Since no \( \Pi \) (\( i \leq j \)) is fundamental, we infer that no elementary formula \( \psi(\tau_1, ..., \tau_n) \) appears simultaneously in the antecedent and in the succedent of the same sequent \( \Pi_i \) for any finite \( i \leq j \).

Let \( \mathcal{R} \) be any canonical realization of \( \Sigma \) in the set \( J \) of all terms. For every sequent \( \Sigma \) of the form (12), let us set

\[ \Sigma_{\mathcal{R}} = (\gamma \tau \rightarrow \delta)_{\mathcal{R}}, \]

where the meaning of \( \alpha_{\mathcal{R}} \) for every formula \( \alpha \) of \( \Sigma \) is the same as in the proof of Theorem 1 (ii).

It follows from the definition of \( E_b \) (see \( (A^i) \) and \( (B^i) \)) that the following statement holds:

\( (C^i) \) For every canonical realization \( \mathcal{R} \), if \( \Pi_{\alpha_{\mathcal{R}}} = \top \), then the set \( E_b \) contains at least one elementary formula \( \alpha \) such that \( \alpha \in A_{\alpha} \) and \( \alpha_{\mathcal{R}} = \bot \), or \( \alpha \in A_{\alpha} \) and \( \alpha_{\mathcal{R}} = \top \).

To prove (ii) let \( \mathcal{R} \) be the following canonical realization of \( \Sigma \); the functions \( \varphi_{\mathcal{R}} \) are defined by (i); if \( \varphi \) is an m-argument predicate, then the value \( \varphi(\tau_1, ..., \tau_n) \) of \( \varphi \) at the point \( (\tau_1, ..., \tau_n) \) is \( \top \) if the formula \( \psi(\tau_1, ..., \tau_n) \) belongs to \( A_{\alpha} \), and \( \bot \) in the opposite case. Consequently,

\[ \alpha_{\mathcal{R}} = \top \text{ for every } \alpha \in A_{\alpha}, \alpha_{\mathcal{R}} = \bot \text{ for every } \alpha \in S_b. \]
Hence, by (C'),
\[ \Pi_2 = \wedge, \]
i.e., \( \Pi \) is not valid in the realization \( \mathfrak{R} \). Thus Theorem 2 is proved.
If
\[ \sum_{\Sigma_1} \text{or} \sum_{\Sigma_1} \sum_1 \]
is any of the schemes (DA), (DS), (CA), (CS), (LA), (IS), (NA), (NS), (EA), (ES), (UA), (US) then
\[ \sum_{\Sigma} \text{and} \sum_{\Sigma_1} \sum_{\Sigma_1} \sum_1 \]
may be treated as the corresponding rules of inferences denoted respectively by
(13) (DA*), (DS*), (CA*), (CS*), (LA*), (IS*), (NA*), (NS*), (EA*), (ES*), (UA*), (US*).

The following statement immediately results from Theorem 2:

**Corollary 2.** The smallest set containing all fundamental sequents and closed with respect to the rules of inference (13) coincides with the set of all valid sequents. That set coincides with the set of sequents \( \Pi \) having a finite diagram, all end sequents of which are fundamental. The diagram of any such sequent \( \Pi \) determines a formalized proof of \( \Pi \) in the formalism under consideration.

As an immediate consequence of Corollary 1 and Corollary 2 we obtain the following

**Corollary 3.** A formula \( \alpha \) is derivable in the formalism considered in Corollary 1 if and only if the sequent \( \Gamma = \alpha \), where \( \Gamma \) is the empty sequence of formulas, is derivable in the formalism considered in Corollary 2.

The following rule of inference
\[ \Gamma = \alpha; \sigma; \sigma'; \sigma'' = \beta' \]
\[ \Gamma'; \Gamma'' = \beta; \beta'' \]
is called cut in the Gentzen formalism.
From Corollary 2 we immediately obtain

**THE GENTZEN THEOREM.** A sequent \( \Pi \) is derivable in the formalism considered in Corollary 2 if and only if it is derivable in the formalism extended by adding the cut to the set (13) of rules of inference.