On the Gentzen Theorem

by

H. Rasiowa and **R. Sikorski** (Warszawa)

Recently Kanger [4] has published a proof of the completeness theorem for some modification of the Gentzen formalism (1). In consequence he has given — for that modified formalism — the first model proof of the Gentzen Hauptsatz (1). The essential point of this modification is that all the applied rules of inference are equivalence rules, i. e. they always transform premises into equivalent conclusions. The idea of Kanger's proof is similar to that of Beth [1].

Another interesting modification of the Gentzen formalism has recently been given by Craig [2].

This paper contains a simplification of the proof of Kanger. More exactly, that simplification is the subject of the second section of our paper. To outline the main ideas of the proof, we restrict ourselves to finite sequents only. The general case can be obtained in the analogous way.

The most essential point of Gentzen's idea is that theorems of the predicate calculus can be proved in his formalism by passing from shorter formulas to longer ones only. This aim may be also realized, without the Gentzen notion of a sequent and the sign $\Rightarrow$ (2). Some modification of the Gentzen formalism of the kind mentioned is the subject of the first section of our paper. It is a continuation of Kanger's idea. With every formula α of the first order predicate calculus, we associate uniquely its diagram, i. e. a system of finite sequences of formulas shorter than α, such that it gives either a proof of α in the formalism under consideration or a countermodel for that formula. The idea of diagram is, indeed, the same as that of trees used by Beth.

1. Diagrams of formulas. Let $\mathcal{L}$ be a formalized language of the first order predicate calculus. We suppose that:

1° the signs of bounded individual variables are different from the signs of free individual variables, the former being denoted by $\xi, \eta, \ldots$, the latter by $x, y, \ldots$;

(1) Cf. Gentzen [3] or Kleene [5].

(2) See e. g. Schütte [6].

2° the set of all terms is denumerable, viz.

(1) $$\tau_1, \tau_2, \ldots$$

is a sequence (fixed in this paper) containing every term exactly once.

The set of all formulas in $\mathcal{L}$ is defined in the usual way, i. e. as the smallest set containing all elementary formulas and closed with respect to logical operations. More exactly, every expression of the form

$$\varrho(\tau_{i_1}, \ldots, \tau_{i_m})$$

where ϱ is an m-argument predicate is an *elementary formula* or a formula of order 1. If α is a formula of order n, then the negation $(-\alpha)$ is a formula of order $n+1$. If α, β are formulas of orders $\leqslant n$ and at least one of them is of order n, then the disjunction $(\alpha \cup \beta)$, the conjunction $(\alpha \cap \beta)$ and the implication $(\alpha \to \beta)$ are formulas of order $n+1$. If $\alpha(x)$ is a formula of an order n with a free individual variable x and the sign ξ does not appear in α, then its particularization $\big(\bigcup_\xi \alpha(\xi)\big)$ and its generalization $\big(\bigcap_\xi \alpha(\xi)\big)$ are formulas of order $n+1$.

We say that a formula α is *shorter* than a formula β provided the order of α is less than the order of β.

In the sequel $\alpha, \beta, \ldots, \alpha(x), \beta(x), \ldots$ etc. will denote exclusively some formulas in $\mathcal{L}$.

We recall the definition of *validity* of formulas. By a *realization* of the language $\mathcal{L}$ in a non-empty set J we understand a mapping $\mathfrak{M}$ which with every n-argument functor φ $(n = 0, 1, 2, \ldots)$ of $\mathcal{L}$ associates an n-argument function $\varphi_\mathfrak{M}$ from the Cartesian product $J^n = J \times \ldots \times J$ (n times) into J, and which with every m-argument predicate ϱ $(m = 1, 2, \ldots)$ associates an m-argument function $\varrho_\mathfrak{M}$ from the Cartesian product J^m into the two-element Boolean algebra B. Let α be any formula of $\mathcal{L}$ with n free individual variables $x_1, \ldots, x_n$. Interpreting

(a) all predicates ϱ in $\mathcal{L}$ as the corresponding functions $\varrho_\mathfrak{M}$;

(b) all functors φ in $\mathcal{L}$ as the corresponding functions $\varphi_\mathfrak{M}$;

(c) all individual variables as variables running trough J;

(d) the logical connectives $\cup$, $\cap$, $\to$, $-$ as the signs of corresponding Boolean operations in B;

(e) the quantifiers $\bigcup_\xi$, $\bigcap_\xi$ as the signs of infinite Boolean joins and meets $\bigcup_{\xi \in J}$, $\bigcap_{\xi \in J}$ respectively,

we can consider $\alpha(x_1, \ldots, x_n)$ as a function $\alpha_\mathfrak{M}(x_1, \ldots, x_n)$ from J^n into B.

We have the identities

(2) $$\begin{aligned} &(\alpha \cup \beta)_\mathfrak{M} = \alpha_\mathfrak{M} \cup \beta_\mathfrak{M}, && (-\alpha)_\mathfrak{M} = -\alpha_\mathfrak{M}, \\ &(\alpha \cap \beta)_\mathfrak{M} = \alpha_\mathfrak{M} \cap \beta_\mathfrak{M}, && \Big(\bigcap_\xi \alpha(\xi)\Big)_\mathfrak{M} = \bigcup_{\xi \in J} \alpha_\mathfrak{M}(\xi), \\ &(\alpha \to \beta)_\mathfrak{M} = \alpha_\mathfrak{M} \to \beta_\mathfrak{M}, && \Big(\bigcup_\xi \alpha(\xi)\Big)_\mathfrak{M} = \bigcup_{\xi \in J} \alpha_\mathfrak{M}(\xi), \end{aligned}$$

which may be assumed as the inductive definition of $\alpha_\mathfrak{M}$.

A formula α is said to be *valid in a realization* $\mathfrak{M}$ in a set $J \neq 0$ if $\alpha_\mathfrak{M}$ is identically equal to the unit element $\bigvee$ of B. It is said to be *valid in a set* $J \neq 0$ if it is valid in every realization $\mathfrak{M}$ in J. Hence, α is not valid in the set $J \neq 0$ if there exists a realization $\mathfrak{M}$ in J, such that $\alpha_\mathfrak{M}$ assumes the zero element $\bigwedge$ of B at least once as its value. α is said to be *valid* if it is valid in any set $J \neq 0$.

The letters Γ and Δ (with indices if necessary) will always denote finite sequences of formulas, the empty sequence included. If Γ is a sequence $\alpha_1, \ldots, \alpha_m$ and Γ' is a sequence $\beta_1, \ldots, \beta_n$ $(m \geqslant 0,\ n \geqslant 0)$, then Γ, α, Γ' is the sequence $\alpha_1, \ldots, \alpha_m, \alpha, \beta_1, \ldots, \beta_n$. The meaning of the symbol $\Gamma, \alpha, \Gamma', \beta$ is similar.

A *formula* α is said to be *indecomposable* if either it is elementary, or it is the negation of an elementary formula.

A sequence Γ of formulas is said to be *indecomposable* if it is formed exclusively of indecomposables formulas. In particular, the empty sequence is indecomposable.

A sequence Γ is said to be *fundamental* if it contains simultaneously a formula α and its negation $(-\alpha)$.

By a *scheme* we shall understand a pair $\{\Gamma, \Gamma^0\}$ or a triple $\{\Gamma, \Gamma^0, \Gamma^1\}$ (of non-empty sequences of formulas) written usually in the forms

(3) $$\frac{\Gamma}{\Gamma^0} \quad \text{and} \quad \frac{\Gamma}{\Gamma^0; \Gamma^1}$$

respectively. Γ is called the *conclusion* of the scheme, and Γ^0, Γ^1 are called its *premises*. More exactly, if the scheme is of the second form (3), then Γ^0 and Γ^1 are called the *left* and the *right premise* respectively. If the scheme is of the first form (3), then Γ^0 is called simultaneously the *left* and the *right premise*.

In the sequel we shall consider only the following schemes where Γ denotes an indecomposable sequence (maybe empty):

(**D**) $$\frac{\Gamma, (\alpha \cup \beta), \Gamma'}{\Gamma, \alpha, \beta, \Gamma'}$$

(**−D**) $$\frac{\Gamma, -(\alpha \cup \beta), \Gamma'}{\Gamma, (-\alpha), \Gamma' ; \Gamma, (-\beta), \Gamma'}$$

(C) $$\frac{\Gamma, (\alpha \cap \beta), \Gamma'}{\Gamma, \alpha, \Gamma' \; ; \; \Gamma, \beta, \Gamma'}$$

(−C) $$\frac{\Gamma, (-(\alpha \cap \beta)), \Gamma'}{\Gamma, (-\alpha), (-\beta), \Gamma'}$$

(I) $$\frac{\Gamma, (\alpha \to \beta), \Gamma'}{\Gamma, (-\alpha), \beta, \Gamma'}$$

(−I) $$\frac{\Gamma, (-(\alpha \to \beta)), \Gamma'}{\Gamma, \alpha, \Gamma' \; ; \; \Gamma, (-\beta), \Gamma'}$$

(−N) $$\frac{\Gamma, (-(-\alpha)), \Gamma'}{\Gamma, \alpha, \Gamma' ;}$$

(E) $$\frac{\Gamma, \left(\bigcup_\xi \alpha(\xi)\right), \Gamma'}{\Gamma, \alpha(\tau), \Gamma', \left(\bigcup_\xi \alpha(\xi)\right)}$$ where τ is a term

(−E) $$\frac{\Gamma, \left(-\left(\bigcup_\xi \alpha(\xi)\right)\right), \Gamma'}{\Gamma, \left(\bigcap_\xi -\alpha(\xi)\right), \Gamma'}$$

(U) $$\frac{\Gamma, \left(\bigcap_\xi \alpha(\xi)\right), \Gamma'}{\Gamma, \alpha(x), \Gamma'}$$ where x is a free individual variable which does not appear in any formula in the conclusion

(−U) $$\frac{\Gamma, \left(-\left(\bigcap_\xi \alpha(\xi)\right)\right), \Gamma'}{\Gamma, \left(\bigcup_\xi \left(-\alpha(\xi)\right)\right), \Gamma'}.$$

The letters $\boldsymbol{i}$, $\boldsymbol{j}$ will denote exclusively finite sequences

(4) $$i_1, \ldots, i_n$$

or infinite sequences

(5) $$i_1, i_2, \ldots$$

of numbers 0 and 1.

We shall write $\boldsymbol{j} \leqslant \boldsymbol{i}$ if $\boldsymbol{j}$ is an initial (proper or non-proper) segment of $\boldsymbol{i}$. If $\boldsymbol{i}$ is the sequence (4), then n is called the *length* of $\boldsymbol{i}$. If $\boldsymbol{i}$ is the sequence (4), then $\boldsymbol{i},0$ and $\boldsymbol{i},1$ denote the sequence $i_1, \ldots, i_n, 0$ and $i_1, \ldots, i_n, 1$ respectively. The empty sequence (the case of $n = 0$ in (4)) is admitted and will be denoted by $\boldsymbol{O}$. The length of $\boldsymbol{O}$ is 0 and $\boldsymbol{O} \leqslant \boldsymbol{i}$ for every $\boldsymbol{i}$.

By a *diagram of a formula* ω in $\mathfrak{L}$ we shall understand a mapping which, with some finite sequences $\boldsymbol{i}$, associates some non-empty finite sequences $\Omega_{\boldsymbol{i}}$ of formulas, and which is defined by induction as follows:

1) $\Omega_{\boldsymbol{O}}$ is the sequence composed only of the formula ω;

2) $\Omega_{\boldsymbol{i},0}$ and $\Omega_{\boldsymbol{i},1}$ are defined if and only if $\Omega_{\boldsymbol{i}}$ is neither fundamental nor indecomposable. Then $\Omega_{\boldsymbol{i}}$ can be assumed to be the conclusion of exactly one of the schemes (**D**), (**−D**), (**C**), (**−C**), (**I**), (**−I**), (**−N**), (**E**), (**−E**), (**U**), (**−U**). Define $\Omega_{\boldsymbol{i},0}$ and $\Omega_{\boldsymbol{i},1}$ as the left and the right premise of that scheme, respectively. If that scheme is (**E**), we assume additionally that the term τ mentioned in (**E**) is the first term in the sequence (1) such that $\alpha(\tau)$ does not appear in any sequence $\Omega_{\boldsymbol{j}}$ with $\boldsymbol{j} \leqslant \boldsymbol{i}$. If that scheme is (**U**), we assume additionally that the variable x mentioned in (**U**) is the first free individual variable in the sequence (1) of all terms, such that x does not appear in any formula in $\Omega_{\boldsymbol{j}}$.

Observe that the diagram $\{\Omega_{\boldsymbol{i}}\}$ of ω is uniquely determined by ω. The diagram is said to be *finite (infinite)* if the set of all sequences $\boldsymbol{i}$ for which $\Omega_{\boldsymbol{i}}$ is defined is finite (infinite).

$\Omega_{\boldsymbol{i}}$ is said to be an *end sequence* of the diagram of ω if $\Omega_{\boldsymbol{i}}$ is either fundamental or indecomposable, i. e. if $\Omega_{\boldsymbol{i},0}$ and $\Omega_{\boldsymbol{i},1}$ are not defined.

THEOREM 1. (i) *If the diagram of a formula ω is finite and all end sequences are fundamental, then ω is valid.*

(ii) *In the opposite case, ω is not valid in an enumerable set.*

Proof of (i). If Γ is a non-empty sequence of formulas, let α_Γ be the disjunction of all formulas in Γ. Observe that

(6) *if Γ is a fundamental sequence, then α_Γ is valid.*

If

$$\frac{\Gamma}{\Gamma^0} \quad \left(\frac{\Gamma}{\Gamma^0 ; \Gamma^1}\right)$$

is any of the schemes (**D**), (**−D**), (**C**), (**−C**), (**I**), (**−I**), (**−N**), (**E**), (**−E**), (**U**), (**−U**), then

$$\frac{\Gamma^0}{\Gamma} \quad \left(\frac{\Gamma^0 ; \Gamma^1}{\Gamma}\right)$$

is a rule of inference, i. e. the following property holds: if α_{Γ^0} is valid (if α_{Γ^0} and α_{Γ^1} are valid) in a realization $\mathfrak{M}$ in a set $J \neq 0$, then so is α_Γ.

This remark, together with the definition of the diagram $\{\Omega_{\boldsymbol{i}}\}$ of a formula ω, implies the following property of the diagram:

(7) *if $\alpha_{\Omega_{\boldsymbol{i},0}}$ and $\alpha_{\Omega_{\boldsymbol{i},1}}$ are defined and are valid, then $\alpha_{\Omega_{\boldsymbol{i}}}$ is valid.*

Hence we infer that if the diagram of ω is finite and all end sequences are fundamental, then, for every $\Omega_{\boldsymbol{i}}$ in the diagram, $\alpha_{\Omega_{\boldsymbol{i}}}$ is valid. In particular $\omega = \alpha_{\Omega_{\boldsymbol{O}}}$ is valid.

Proof of (ii). If the hypothesis of (i) is not satisfied, then there exists a sequence $\boldsymbol{j}$ for which one of the following conditions holds [3]:

(A) $\boldsymbol{j}$ *is finite and* $\Omega_{\boldsymbol{j}}$ *is an end sequence which is not fundamental,*

(B) $\boldsymbol{j}$ *is infinite and, for every finite* $\boldsymbol{i} \leqslant \boldsymbol{j}$, $\Omega_{\boldsymbol{i}}$ *is in the diagram of* ω.

Let F_0 be the set of all indecomposable formulas appearing in at least one $\Omega_{\boldsymbol{i}}$, $\boldsymbol{i} \leqslant \boldsymbol{j}$. Observe that if $\boldsymbol{i} \leqslant \boldsymbol{i}' \leqslant \boldsymbol{j}$ and an indecomposable formula appears in $\Omega_{\boldsymbol{i}}$, then it appears also in $\Omega_{\boldsymbol{i}'}$. Since no $\Omega_{\boldsymbol{i}}$ ($\boldsymbol{i} \leqslant \boldsymbol{j}$) is fundamental, for every elementary formula

$$\varrho(\tau_{k_1}, \ldots, \tau_{k_m})$$

at most one of the formulas $\varrho(\tau_{k_1}, \ldots, \tau_{k_m})$, $\big(-\varrho(\tau_{k_1}, \ldots, \tau_{k_m})\big)$ is in F_0.

Let J be the set of all terms (i. e. the set of all $\tau_1, \tau_2, \ldots$ in (1)).

Let $\mathfrak{M}$ be any realization of $\mathcal{L}$ in the set J such that

$$(*) \qquad \varphi_{\mathfrak{M}}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1, \ldots, \tau_m),$$

i. e. such that, for every m-argument functor φ, the value of the corresponding function $\varphi_{\mathfrak{M}}$ (from J^m into J) at the point $(\tau_1, \ldots, \tau_m)$ is the term $\varphi(\tau_1, \ldots, \tau_m)$.

Every realization $\mathfrak{M}$ of $\mathcal{L}$ in the set J of all terms with the above-mentioned property $(*)$ is called *canonical.*

We shall precede the proof of (ii) by some lemmas on canonical realizations.

Let $\mathfrak{M}$ be any canonical realization of $\mathcal{L}$ in the set J of all terms. According to the general definition (p. 58) if $\alpha = \alpha(x_1, \ldots, x_n)$ is a formula with free individual variables $x_1, \ldots, x_n$, then $\alpha_{\mathfrak{M}}$ is a function from J^m into B, i. e. for every substitution of some terms $\tau_{k_1}, \ldots, \tau_{k_n}$ for $x_1, \ldots, x_n$ respectively the value of $\alpha_{\mathfrak{M}}$ at $(x_1, \ldots, x_n)$ is a fixed element of B. Specially interesting is the case where that substitution is the identity, i. e. $\tau_{k_i} = x_i$ for $i = 1, \ldots, n$. The value of $\alpha_{\mathfrak{M}}$ at that special point of J^m will be denoted by $\alpha^*_{\mathfrak{M}}$.

By definition,

(8) *if* β *is obtained from* α *by the substitution of* $\tau_{k_1}, \ldots, \tau_{k_n}$ *for* $x_1, \ldots, x_n$, *then the value of* $\alpha_{\mathfrak{M}}$ *at the point* $x_1 = \tau_{k_1}, \ldots, x_n = \tau_{k_n}$ *is equal to* $\beta^*_{\mathfrak{M}}$.

It follows from the definition of F_0 (see (A) or (B)) that

(C) *If* $\omega^*_{\mathfrak{M}} = \vee$, *then the set* F_0 *contains at least one indecomposable formula* α *such that* $\alpha^*_{\mathfrak{M}} = \vee$.

Indeed, let F be the set of all formulas appearing in at least one $\Omega_{\boldsymbol{i}}$, $\boldsymbol{i} \leqslant \boldsymbol{j}$. Let α be the shortest formula in F such that $\alpha^*_{\mathfrak{M}} = \vee$. Then the formula α is indecomposable.

(3) This follows e. g. from the compactness of the Cantor discontinuum.

For suppose that α is decomposable. Then there exists a sequence $\boldsymbol{i} \leqslant \boldsymbol{j}$ ($\boldsymbol{i} \neq \boldsymbol{j}$) such that $\Omega_{\boldsymbol{i}}$ is of the form

$$\Gamma,\ \alpha,\ \Gamma'$$

with indecomposable Γ.

If α is of the form

$$(9) \qquad \Big(-\big(\bigcup_{\xi} \beta(\xi)\big)\Big),$$

then the formula γ of the form $\Big(\bigcap_{\xi}\big(-\beta(\xi)\big)\Big)$ is also in F, has the same length as α and $\gamma^*_{\mathfrak{M}} = \vee$.

If α is of the form

$$(10) \qquad \Big(-\big(\bigcap_{\xi} \beta(\xi)\big)\Big),$$

then the formula γ of the form $\Big(\bigcup_{\xi}\big(-\beta(\xi)\big)\Big)$ is also in F, has the same length as α, and $\gamma^*_{\mathfrak{M}} = \vee$.

If α is of the form

$$(11) \qquad (\beta \to \gamma),$$

then the formulas $(-\beta)$ and γ are also in F, their lengths are not greater than the length of α and either $(-\beta)^*_{\mathfrak{M}} = \vee$ or $\gamma^*_{\mathfrak{M}} = \vee$.

Therefore we may assume in our consideration that α is neither of the form (9) nor of the forms (10), (11).

Let $\boldsymbol{i}'$ be the sequence obtained from $\boldsymbol{i}$ by adding either 0 or 1 at the end, such that $\boldsymbol{i}' \leqslant \boldsymbol{j}$.

If α has one of the forms

$$(\beta \cup \gamma),\ \big(-(\beta \cup \beta)\big),\ (\beta \cap \gamma),\ \big(-(\beta \cap \gamma)\big),\ \big(-(\beta \to \beta)\big),\ \big(-(-\beta)\big),\ \big(\bigcap_{\xi} \beta(\xi)\big),$$

then it immediately follows from the definition of the schemes **(D)**, **(−D)**, **(C)**, **(−C)**, **(−I)**, **(−N)**, **(U)**, and from the definition of $\Omega_{\boldsymbol{i}'}$ that $\Omega_{\boldsymbol{i}'}$ is of the form

$$\Gamma'',\ \alpha_0,\ \Gamma'''$$

where α_0 is shorter that α and $\alpha^*_{0\mathfrak{M}} = \vee$. This contradicts our hypothesis concerning the length of α.

It remains only to consider the case, where α is of the form

$$\big(\bigcup_{\xi} \beta(\xi)\big).$$

In that case $\boldsymbol{j}$ is infinite. Then, by the last of identities (2), there exists a term τ such that the formula

$$\gamma = \beta(\tau)$$

satisfies the condition

$$\gamma^*_{\mathfrak{M}} = \vee\,.$$

It follows from the definition of the diagram (see the additional assumption to the scheme (E) in 2)) that there exists a finite sequence i'' such that $i \leqslant i'' \leqslant j$ and γ appears in $\Omega_{i''}$. Since γ is shorter than α, this contradicts the hypothesis concerning the length of α. This proves Lemma (C).

To prove (ii) let $\mathfrak{M}$ be the following canonical realization in the set J of all terms: the functions $\varphi_{\mathfrak{M}}$ are defined by (*); if ϱ is an m-argument predicate, then the value $\varrho_{\mathfrak{M}}(\tau_{k_1}, \ldots, \tau_{k_m})$ of $\varrho_{\mathfrak{M}}$ at the point $(\tau_{k_1}, \ldots, \tau_{k_m}) \in J^m$ is $\vee$ if the formula $\big(-\varrho(\tau_{k_1}, \ldots, \tau_{k_m})\big)$ belongs to F_0; and it is $\wedge$ in the opposite case. Then, for every formula α in F_0,

$$\alpha^*_{\mathfrak{M}} = \wedge\,.$$

This implies, by (C), that

$$\omega^*_{\mathfrak{M}} = \wedge\,,$$

i. e. that ω is not valid in the realization $\mathfrak{M}$. This completes the proof of Theorem 1.

As we have observed in the proof of (i), if

$$\frac{\Gamma}{\Gamma^0} \quad \text{or} \quad \frac{\Gamma}{\Gamma^0\,;\,\Gamma^1}$$

is any of the schemes (**D**), (**−D**), (**C**), (**−C**), (**I**), (**−I**), (**−N**), (**E**), (**−E**), (**U**), (**−U**), then

$$\frac{\Gamma^0}{\Gamma} \quad \text{and} \quad \frac{\Gamma^0\,;\,\Gamma^1}{\Gamma}$$

may be treated as the corresponding rules of inferences, denoted respectively by (**D***), (**−D***), (**C***), (**−C***), (**I***), (**−I***), (**−N***), (**E***), (**−E***), (**U***), (**−U***).

The following statement results immediately from Theorem 1:

COROLLARY 1. *The set composed of all formulas which can be obtained from fundamental sequences by means of the rules of inferences* (**D***), (**−D***), (**C***), (**−C***), (**I***), (**−I***), (**−N***), (**E***), (**−E***), (**U***), (**−U***) *coincides with the set of all valid formulas. This set coincides with the set of formulas ω having a finite diagram whose all end sequences are fundamental. The diagram of any such formula ω determines a formalized proof of ω in the formalism under consideration.*

2. Diagrams of sequents. We shall assume here the terminology and the notation of the previous section.

Consider a formalized language $\mathcal{L}$ of the first order predicate calculus satisfying conditions 1° and 2° mentioned in section 1. Besides finite sequences of formulas in $\mathcal{L}$ we shall consider *sequents*, i. e. expressions of the form

$$\Gamma \Rightarrow \Delta \tag{12}$$

where Γ and Δ are finite sequences of formulas (which may be empty). The sequence Γ will be called the *antecedent* and the sequence Δ the *succedent* of that sequent. Sequents will be denoted by Σ, Π with indices if necessary.

We recall the definition of validity of sequents. Let $\mathfrak{M}$ be a realization of $\mathcal{L}$ in a set $J \neq 0$. Let γ_Γ be the conjunction of all formulas in Γ if Γ is a non-empty sequence of formulas, and a fixed formula of the form $\big(\alpha \cup (-\alpha)\big)$ in the opposite case. Let δ_Δ be the disjunction of all formulas in Δ if δ is non-empty, and a fixed formula of the form $\big(\alpha \cap (-\alpha)\big)$ in the opposite case. In particular, if Γ (Δ) is composed of only one formula β, then the conjunction γ_Γ (the disjunction δ_Δ) reduces to this formula β. Interpreting the sign $\Rightarrow$ as the Boolean operation of the codifference ([4]) in the two-element Boolean algebra B and γ_Γ, δ_Δ as the function $(\gamma_\Gamma)_{\mathfrak{M}}$ and $(\delta_\Delta)_{\mathfrak{M}}$ respectively, we can treat the sequent Σ of the form (12) as the function $\Sigma_{\mathfrak{M}} = (\gamma_\Gamma \rightarrow \delta_\Delta)_{\mathfrak{M}}$. A *sequent* Σ is said to be *valid in the realization* $\mathfrak{M}$ in the set J if $\Sigma_{\mathfrak{M}}$ is identically equal to the unit element $\vee$ of B. It is said to be *valid in a set* $J \neq 0$ if it is valid in every realization $\mathfrak{M}$ in J. Σ is said to be *valid* if it is valid in every set $J \neq 0$.

A *sequence* will be called *elementary* if it is empty or composed only of elementary formulas. A *sequent* will be called *elementary* if its antecedent and succedent are elementary sequences.

A *sequent* will be called *fundamental* provided its antecedent and succedent contain simultaneously a formula α.

In the sequel we shall consider some schemes for sequents, i. e. some pairs $\{\Sigma, \Sigma^0\}$ or triples $\{\Sigma, \Sigma^0, \Sigma^1\}$ of sequents written in the form

$$\frac{\Sigma}{\Sigma^0} \quad \text{or} \quad \frac{\Sigma}{\Sigma^0\,;\,\Sigma^1} \quad \text{respectively.}$$

Σ will be called the *conclusion* of the scheme and Σ^0, Σ^1 the *left* and the *right premise*, respectively (in the case of $\frac{\Sigma}{\Sigma^0}$, Σ^0 will be called simultaneously the *left* and the *right premise*).

We shall examine the following schemes (Γ, Δ denote here elementary sequences only):

$$(\mathbf{DA}) \quad \frac{\Gamma, (\alpha \cup \beta), \Gamma' \Rightarrow \Delta'}{\Gamma, \alpha, \Gamma' \Rightarrow \Delta' \;;\; \Gamma, \beta, \Gamma' \Rightarrow \Delta'} \qquad (\mathbf{DS}) \quad \frac{\Gamma' \Rightarrow \Delta, (\alpha \cup \beta), \Delta'}{\Gamma' \Rightarrow \Delta, \alpha, \beta, \Delta'}$$

([4]) For any elements a, b of a Boolean algebra A, we define the codifference $a \rightarrow b$ as $-a \cup b$.

$$\text{(CA)}\quad \frac{\Gamma, (\alpha \cap \beta), \Gamma' \Rightarrow \Delta'}{\Gamma, \alpha, \beta, \Gamma' \Rightarrow \Delta'} \qquad \text{(CS)}\quad \frac{\Gamma' \Rightarrow \Delta, (\alpha \cap \beta), \Delta'}{\Gamma' \Rightarrow \Delta, \alpha, \Delta' \ ; \ \Gamma' \Rightarrow \Delta, \beta, \Delta'}$$

$$\text{(IA)}\quad \frac{\Gamma, (\alpha \to \beta), \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \alpha \ ; \ \Gamma, \beta, \Gamma' \Rightarrow \Delta'} \qquad \text{(IS)}\quad \frac{\Gamma' \Rightarrow \Delta, (\alpha \to \beta), \Delta'}{\Gamma', \alpha \Rightarrow \Delta, \beta, \Delta'}$$

$$\text{(NA)}\quad \frac{\Gamma, (-\alpha), \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \alpha} \qquad \text{(NS)}\quad \frac{\Gamma' \Rightarrow \Delta, (-\alpha), \Delta'}{\Gamma', \alpha \Rightarrow \Delta, \Delta'}$$

$$\text{(EA)}\quad \frac{\Gamma, \big(\bigcup_{\xi} \alpha(\xi)\big), \Gamma' \Rightarrow \Delta'}{\Gamma, \alpha(x), \Gamma' \Rightarrow \Delta'} \qquad \text{(ES)}\quad \frac{\Gamma' \Rightarrow \Delta, \big(\bigcup_{\xi} \alpha(\xi)\big), \Delta'}{\Gamma' \Rightarrow \Delta, \alpha(\tau), \Delta', \big(\bigcup_{\xi} \alpha(\xi)\big)}$$

$$\text{(UA)}\quad \frac{\Gamma, \big(\bigcap_{\xi} \alpha(\xi)\big), \Gamma' \Rightarrow \Delta'}{\Gamma, \alpha(\tau), \Gamma', \big(\bigcap_{\xi} \alpha(\xi)\big) \Rightarrow \Delta'} \qquad \text{(US)}\quad \frac{\Gamma' \Rightarrow \Delta, \big(\bigcap_{\xi} \alpha(\xi)\big), \Delta'}{\Gamma' \Rightarrow \Delta, \alpha(x), \Delta'}$$

where τ appearing in **(ES)** and **(UA)** is a term, and x occurring in the schemes **(EA)** and **(US)** is a free individual variable which does not appear in any formula of the conclusion of the scheme under consideration.

By a *diagram of a sequent Π* we shall mean a mapping which with some finite sequences $\boldsymbol{i}$ of numbers 0 and 1 associates some sequents $\Pi_{\boldsymbol{i}}$, and which is defined by induction as follows:

1) Π_O is identical with the sequent Π;

2) $\Pi_{\boldsymbol{i},0}$ and $\Pi_{\boldsymbol{i},1}$ are defined if and only if $\Pi_{\boldsymbol{i}}$ is neither fundamental nor elementary. Suppose that the length of $\boldsymbol{i}$ is even. If the antecedent of $\Pi_{\boldsymbol{i}}$ is not elementary, then $\Pi_{\boldsymbol{i}}$ can be treated as the conclusion of exactly one of the schemes for antecedents, i. e. **(DA)**, **(CA)**, **(IA)**, **(NA)**, **(EA)**, **(UA)**. Then $\Pi_{\boldsymbol{i},0}$ and $\Pi_{\boldsymbol{i},1}$ are the left and the right premise of that scheme respectively. If the antecedent of $\Pi_{\boldsymbol{i}}$ is elementary, then $\Pi_{\boldsymbol{i},0}$ and $\Pi_{\boldsymbol{i},1}$ are equal to $\Pi_{\boldsymbol{i}}$. Suppose now that the length of $\boldsymbol{i}$ is odd. If the succedent of $\Pi_{\boldsymbol{i}}$ is not elementary, then $\Pi_{\boldsymbol{i}}$ can be considered as the conclusion of exactly one scheme for succedents, i. e. **(DS)**, **(CS)**, **(IS)**, **(NS)**, **(ES)**, **(US)**. Then $\Pi_{\boldsymbol{i},0}$ and $\Pi_{\boldsymbol{i},1}$ are the left and the right permise of that scheme respectively. If the succedent of $\Pi_{\boldsymbol{i}}$ is elementary, then $\Pi_{\boldsymbol{i},0}$ and $\Pi_{\boldsymbol{i},1}$ are equal to $\Pi_{\boldsymbol{i}}$.

Moreover, we assume that if the scheme **(ES)** (the scheme **(UA)**) has been applied, then the term τ appearing in **(ES)** in **(UA)**) is the first term in the sequence (1) of all terms such that $\alpha(\tau)$ does not appear in the succedent (in the antecedent) of a sequent $\Pi_{\boldsymbol{j}}$ with $\boldsymbol{j} \leqslant \boldsymbol{i}$. If the scheme **(EA)** or **(US)** has been applied, then the variable x mentioned in the scheme **(EA)** or **(US)** is the first free individual variable in sequence (1) such that x does not occurs in any formula in $\Pi_{\boldsymbol{i}}$.

Observe that the diagram $\{\Pi_{\boldsymbol{i}}\}$ of Π is uniquely determined by Π.

$\Pi_{\boldsymbol{i}}$ is said to be an *end sequent* of the diagram of Π if it is either fundamental or elementary.

THEOREM 2. (i) *If the diagram of a sequent Π is finite and all end sequents are fundamental, then Π is valid.*

(ii) *In the opposite case, Π is not valid in an enumerable set.*

The statement (i) follows immediately from the definition of diagram and from the facts that every fundamental sequent is valid and that the validity of premises of the considered schemes in a realization $\mathfrak{M}$ in a set $J \neq 0$ implies the validity of the conclusion.

The statement (ii) can be established in an analogous way to that used in the proof of Theorem 1 (ii). In fact, if the hypothesis of (i) is not satisfied, then there exists a sequence $\boldsymbol{j}$ for which one of the following conditions holds:

(A′) *$\boldsymbol{j}$ is finite and $\Pi_{\boldsymbol{j}}$ is an end sequent which is not fundamental,*

(B′) *$\boldsymbol{j}$ is infinite and, for every finite $\boldsymbol{i} \leqslant \boldsymbol{j}$, $\Pi_{\boldsymbol{i}}$ is in the diagram of Π.*

Let A_0, (S_0) be the set of all elementary formulas which appear in all antecedents of $\Pi_{\boldsymbol{i}}$ $(\boldsymbol{i} \leqslant \boldsymbol{j})$ (in all succedents of $\Pi_{\boldsymbol{i}}$ $(\boldsymbol{i} \leqslant \boldsymbol{j})$). Let $E_0 = A_0 + S_0$. Notice that if $\boldsymbol{i} \leqslant \boldsymbol{i}' \leqslant \boldsymbol{j}$ and an elementary formula occurs in $\Pi_{\boldsymbol{i}}$, then it also occurs in $\Pi_{\boldsymbol{i}'}$. More exactly, if it occurs in the antecedent (in the succedent of $\Pi_{\boldsymbol{i}}$), then it also occurs in the antecedent (in the succedent) of $\Pi_{\boldsymbol{i}'}$.

Since no $\Pi_{\boldsymbol{i}}$ $(\boldsymbol{i} \leqslant \boldsymbol{j})$ is fundamental, we infer that no elementary formula $\varrho(\tau_{k_1}, \ldots, \tau_{k_n})$ appears simultaneously in the antecedent and in the succedent of the same sequent $\Pi_{\boldsymbol{i}}$ for any finite $\boldsymbol{i} \leqslant \boldsymbol{j}$.

Let $\mathfrak{M}$ be any canonical realization of $\mathcal{L}$ in the set J of all terms. For every sequent Σ of the form (12), let us set

$$\Sigma^*_{\mathfrak{M}} = (\gamma_\Gamma \to \delta_\Delta)^*_{\mathfrak{M}},$$

where the meaning of $\alpha^*_{\mathfrak{M}}$ for every formula α of $\mathcal{L}$ is the same as in the proof of Theorem 1 (ii).

It follows from the definition of E_0 (see (A′) and (B′)) that the following statement holds:

(C′) *For every canonical realization $\mathfrak{M}$, if $\Pi^*_{\mathfrak{M}} = \vee$, then the set E_0 contains at least one elementary formula α such that either $\alpha \in A_0$ and $\alpha^*_{\mathfrak{M}} = \wedge$, or $\alpha \in S_0$ and $\alpha^*_{\mathfrak{M}} = \vee$.*

To prove (ii) let $\mathfrak{M}$ be the following canonical realization of $\mathcal{L}$: the functions $\varphi_{\mathfrak{M}}$ are defined by $(*)$; if ϱ is an m-argument predicate, then the value $\varrho_{\mathfrak{M}}(\tau_{k_1}, \ldots, \tau_{k_m})$ of $\varrho_{\mathfrak{M}}$ at the point $(\tau_{k_1}, \ldots, \tau_{k_m}) \in J^m$ is $\vee$ if the formula $\varrho(\tau_{k_1}, \ldots, \tau_{k_m})$ belongs to A_0, and is $\wedge$ in the opposite case. Consequently,

$$\alpha^*_{\mathfrak{M}} = \vee \quad \text{for every} \quad \alpha \text{ in } A_0, \quad \alpha^*_{\mathfrak{M}} = \wedge \quad \text{for every} \quad \alpha \text{ in } S_0.$$

Hence, by (C′),

$$\Pi^*_{\mathfrak{M}} = \wedge\,,$$

i. e. Π is not valid in the realization $\mathfrak{M}$. Thus Theorem 2 is proved.

If

$$\frac{\Sigma}{\Sigma^0} \quad \text{or} \quad \frac{\Sigma}{\Sigma^0;\ \Sigma^1}$$

is any of the schemes **(DA)**, **(DS)**, **(CA)**, **(CS)**, **(IA)**, **(IS)**, **(NA)**, **(NS)**, **(EA)**, **(ES)**, **(UA)**, **(US)** then

$$\frac{\Sigma^0}{\Sigma} \quad \text{and} \quad \frac{\Sigma^0;\ \Sigma^1}{\Sigma}$$

may be treated as the corresponding rules of inferences denoted respectively by

(13) **(DA*)**, **(DS*)**, **(CA*)**, **(CS*)**, **(IA*)**, **(IS*)**, **(NA*)**, **(NS*)**, **(EA*)**, **(ES*)**, **(UA*)**, **(US*)**.

The following statement immediately results from Theorem 2:

COROLLARY 2. *The smallest set containing all fundamental sequents and closed with respect to the rules of inference* (13) *coincides with the set of all valid sequents. That set coincides with the set of sequents Π having a finite diagram, all end sequents of which are fundamental. The diagram of any such sequent Π determines a formalized proof of Π in the formalism under consideration.*

As an immediate consequence of Corollary 1 and Corollary 2 we obtain the following

COROLLARY 3. *A formula a is derivable in the formalism considered in* Corollary 1 *if and only if the sequent $\Gamma \Rightarrow a$, where Γ is the empty sequence of formulas, is derivable in the formalism considered in Corollary* 2.

The following rule of inference

$$\frac{\Gamma' \Rightarrow \Delta', a\ ;\ a, \Gamma'' \Rightarrow \Delta''}{\Gamma', \Gamma'' \Rightarrow \Delta', \Delta''}$$

is called *cut* in the *Gentzen formalism*.

From Corollary 2 we immediately obtain

THE GENTZEN THEOREM. *A sequent Π is derivable in the formalism considered in Corollary* 2 *if and only if it is derivable in the formalism extended by adding the cut to the set* (13) *of rules of inference.*

References

[1] E. W. Beth, *Conséquence sémantique et dérivation formelle*, Communication présentée au Coll. du C. N. R. S. (Paris 1955).

[2] W. Craig, *Linear reasoning. A new form of the Herbrand-Gentzen theorem*, Journal of Symbolic Logic 22 (1957), p. 250-268.

[3] G. Gentzen, *Untersuchungen über das logische Schliessen*, Mathematische Zeitschrift 39 (1934-5), p. 176-210, 405-431.

[4] S. Kanger, *Provability in Logic*, Acta Univ. Stockholmiensis, Stockholm Studies in Philosophy 1 (1957), p. 1-47.

[5] S. C. Kleene, *Introduction to metamathematics*, Amsterdam 1952.

[6] K. Schütte, *Schlußweisen-Kalküle der Prädikatenlogik*, Math. Annalen 122 (1950/51), p. 47-65.

Reçu par la Rédaction le 13. 1. 1959