On a class of rings

by

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I. In the present paper we use the term ring for any additive Abelian group closed with respect to the product operation such that the two-sided distributive law holds (see e.g. [1], [3]). When the associative law for products also holds, we call the ring an associative ring.

A ring $R$ is called a $\tau$-ring if there exists an element $\tau \in R$ such that for every $a, b, c \in R$ the following equalities hold:

(i) $a(bc) = (a(c))b$,

(ii) $\tau(ra) = a$.

The aim of our note is to give a complete representation of $\tau$-rings.

First of all we shall prove that conditions (i) and (ii) are mutually independent.

Let $N$ be the ring of all integers with the usual addition and with trivial multiplication: $ab = 0$ $(a, b \in N)$. It is easy to verify that condition (i) is satisfied for every $a, b, c, \tau \in N$, but $N$ does not satisfy condition (ii).

Further let $Q$ be the ring of all quaternions. Obviously, equality (ii) is satisfied for $\tau = 1$.

Now we shall show that there is no element $\tau \in Q$ satisfying (i) for every $a, b, c \in Q$. Suppose the contrary. Setting $a = b = c = 1$ in (i), we get the equality $\tau = 1$. Consequently, in virtue of (i), we have the equality

$a(bc) = a(cb)$ $(a, b, c \in Q)$.

Hence follows the commutative law $bc = cb$ $(b, c \in Q)$, which is impossible. Thus (i) is not true for $Q$.

II. The following statements are a direct consequence of the definition of $\tau$-rings.

(a) $\tau$ is not a left divisor of zero in $R$.

In fact, the equality $r\tau = 0$ implies the following one: $r\tau a = 0$, whence, by (ii), $a = 0$. 
(b) Every element \( a \in R \) can be represented by the product \( a = ra' \), where \( a' \) is uniquely determined by \( a \).

Putting \( a' = ra \), we have, according to (ii), \( a = ra' \). Further, from the equalities \( a = ra' \), \( a = ra \); it follows that \( r(a'-a) = 0 \), whence, by (ii), \( a' = a \).

(c) \( r \) is a right unit element of \( R \).

Setting \( a = b = r \) in (i) we get the equality \( r(\tau c) = (\tau c)r \). Hence, in virtue of (ii), \( c = cr \) for every \( c \in R \).

We remark that except \( r \) there is no right unit element of \( R \). In fact, if \( a = a' \) for every \( a \in R \), then, according to (ii) and (c),

\[
\tau \xi = \tau \tau \xi = \tau \tau = \tau .
\]

Consequently, we have the following assertion:

(i) \( \tau \) is uniquely determined by conditions (i) and (ii).

(e) For every \( a, b \in R \) the equality

\[
\tau(ab) = ba
\]

holds.

In fact, substituting \( a = r, b = a \) and \( c = b \) in (i) and taking into account (ii), we have the equality

\[
\tau(ab) = \tau(rb)a = ba .
\]

(f) For every \( a, b, c \in R \) the equality

\[
(ab)c = a(bc)
\]

is true.

From (i) we obtain the equality

\[
\tau(ab) = \tau(\tau(b))a .
\]

Consequently,

\[
\tau(\tau(c)) = \tau(\tau(b))a .
\]

Hence, using (e), we obtain our assertion.

The generalization of the above formula is given by the following one:

(g) For every system \( a_1, a_2, \ldots, a_n \in R \) (\( n \geq 3 \)) the equality

\[
\tau(...(a_n a_{n-1}) a_{n-2} \ldots a_1(\tau a_{n-1})\tau a_{n-2} ... \tau a_1)...(\tau a_n) = a
\]

holds.

We shall prove our formula by induction with respect to \( n \). For \( n = 3 \) our formula is identical to (f). Now let us suppose that it holds for every \( n \)-tuple. Consequently, for the \( n \)-tuple \( a_1, a_2, a_3, \ldots, a_{n+1} \) we have the equality

\[
\tau(...(a_n a_{n-1}) a_{n-2} \ldots a_1(\tau a_{n-1})\tau a_{n-2} ... \tau a_1)...(\tau a_n) = a
\]

Substituting in (f) \( a = a_1, b = a_2 \) and

\[
\tau(...(a_n a_{n-1})\tau a_{n-2} ... \tau a_1)...(\tau a_n) = a
\]

we obtain our formula for every \((n+1)\)-tuple \( a_1, a_2, \ldots, a_{n+1} \in R \). Formula (g) is thus proved.

From (e) and (g) the following statement follows:

(h) For every system \( a_1, a_2, \ldots, a_n \in R \) (\( n \geq 3 \)) we have the equality

\[
\tau(...(a_n a_{n-1}) a_{n-2} \ldots a_1(\tau a_{n-1})\tau a_{n-2} ... \tau a_1)...(\tau a_n) = a_1
\]

III. Let us consider an associative ring \( R_o \) with the unit element.

Further, let us suppose that \( R_o \) is a ring with involution, i.e., for every \( a \in R_o \) an element \( a^* \), called the adjoint of \( a \), is defined such that the conditions

\[
(a + b)^* = a^* + b^*, \quad (a^*)^* = a, \quad (a^*)^* = b^* \quad a^*
\]

are satisfied, where \( \cdot \) denotes the product operation in \( R_o \).

An element \( a \in R_o \) is called self-adjoint or Hermitian if \( a = a^* \).

Obviously, the unit element of \( R_o \) is Hermitian (see [4], chap. II and V).

In the sequel we shall denote by \( \mathcal{K}(R_o) \) the set \( R_o \) with the usual addition and multiplication defined as follows:

\[
ab = b^* \cdot a .
\]

If \( R_o \) is the ring of real square matrices of fixed order, then \( \mathcal{K}(R_o) \) coincides with the ring of 

\[
\tau \text{ r and o (see [2]), who has applied it widely to problems in astronomy.}
\]

In general, we shall call \( \mathcal{K}(R_o) \) the crassovian ring generated by \( R_o \).

THEOREM 1. If \( R_o \) is an associative ring with involution and having the unit element, then \( \mathcal{K}(R_o) \) is a \( \tau \)-ring.

Proof. To prove our theorem, it suffices to show that in \( \mathcal{K}(R_o) \) the two-sided distributive law and equalities (i) and (ii) are true.

Using (1), we have the equalities

\[
a(b + c) = (b + c)^* = a^* = c^* \cdot a = ab + ac ,
\]

\[
(b + c)^* a = a^* \cdot (b + c) = a^* \cdot b + a^* \cdot c = ba + ca .
\]
Moreover, denoting by $e$ the unit element of $R_e$, we have the equalities

\[ e(a) = (ea)^* = e = (ea)^* = (e^* + e)^* = a, \]
\[ a(bc) = a(e^* + b) = (e^* + b)^* = a = (b^* + e)^* = b^* + (e^* + e)^* = (b^* + e)^* = (b + e)^* = (b + e) = [(c + e) + a]b = (a + e)c, \]
\[ = [(c + e) + a]b = (a + e)c. \]

Consequently, equalities (i) and (ii) are satisfied for $\tau = e$. The theorem is thus proved.

Now we shall give the complete representation of $\tau$-rings. Namely, we shall prove the following theorem:

**THEOREM 3.** Every $\tau$-ring is equal to the cracovian ring generated by an associative ring with involution and having the unit element.

**Proof.** Let $R$ be a $\tau$-ring and let $R_e$ denote the set $R$ with the usual addition and $*$-multiplication defined by the formula

\[ a \ast b = b(\tau a). \]

Now we shall prove that $R_e$ is an associative ring having the unit element with involution

\[ a^* = \tau a. \]

Using (2), we obtain the distributive laws:

\[ a \ast (b + c) = (b + c) \ast (\tau a) = b \ast (\tau a) + c \ast (\tau a) = a \ast b + a \ast c, \]
\[ (b + c) \ast a = a(\tau (b + c)) = a(\tau b + a(\tau c)) = b + a + c \ast a. \]

Further, according to (c), we have the equality

\[ (a \ast b) \ast c = (b(\tau a)) \ast c = c(\tau (b(\tau a))) = c(\tau a b). \]

Hence, using (i), we obtain the associative law

\[ (a \ast b) \ast c = c(\tau b)(\tau a) = (b \ast c)(\tau a) = a \ast (b \ast c). \]

The element $\tau$ is the unit element of $R_e$. In fact, by (ii) and (c),

\[ a \ast \tau = \tau (\tau a) = a, \quad \tau \ast a = a(\tau a) = a. \]

From (3) and (e) follow the equalities

\[ (a + b)^* = \tau (a + b) = \tau a + \tau b = a^* + b^*, \]
\[ (a^*)^* = \tau (\tau a) = a, \]
\[ (a + b)^* = \tau (a + b) = \tau (b(\tau a)) = (\tau a)b = b^* \ast a^*. \]