On a class of rings

by

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I. In the present paper we use the term *ring* for any additive Abelian group closed with respect to the product operation such that the two-sided distributive law holds (see e. g. [1], [3]). When the associative law for products also holds, we call the ring an *associative ring.*

A ring R is called a *τ-ring* if there exists an element $\tau \in R$ such that for every $a, b, c \in R$ the following equalities hold:

(i) $$a(bc) = \big(a(\tau c)\big)b,$$

(ii) $$\tau(\tau a) = a.$$

The aim of our note is to give a complete representation of τ-rings.

First of all we shall prove that conditions (i) and (ii) are mutually independent.

Let N be the ring of all integers with the usual addition and with trivial multiplication: $ab = 0$ $(a, b \in N)$. It is easy to verify that condition (i) is satisfied for every $a, b, c, \tau \in N$, but N does not satisfy condition (ii).

Further let Q be the ring of all quaternions. Obviously, equality (ii) is satisfied for $\tau = 1$.

Now we shall show that there is no element $\tau \in Q$ satisfying (i) for every $a, b, c \in Q$. Suppose the contrary. Setting $a = b = c = 1$ in (i), we get the equality $\tau = 1$. Consequently, in virtue of (i), we have the equality

$$a(bc) = a(cb) \qquad (a, b, c \in Q).$$

Hence follows the commutative law $bc = cb$ $(b, c \in Q)$, which is impossible. Thus (i) is not true for Q.

II. The following statements are a direct consequence of the definition of τ-rings.

(a) *τ is not a left divisor of zero in R.*

In fact, the equality $\tau a = 0$ implies the following one: $\tau(\tau a) = 0$, whence, by (ii), $a = 0$.

(b) *Every element $a \in R$ can be represented by the product $a = \tau a'$, where a' is uniquely determined by a.*

Putting $a' = \tau a$, we have, according to (ii), $a = \tau a'$. Further, from the equalities $a = \tau a'$, $a = \tau a_1'$ it follows that $\tau(a' - a_1') = 0$, whence, by (a), $a' = a_1'$.

(c) *τ is a right unit element of R.*

Setting $a = b = \tau$ in (i) we get the equality $\tau(\tau c) = (\tau(\tau c))\tau$. Hence, in virtue of (ii), $c = c\tau$ for every $c \in R$.

We remark that except τ there is no right unit element of R. In fact, if $a = a\xi$ for every $a \in R$, then, according to (ii) and (c),

$$\xi = \tau(\tau\xi) = \tau\tau = \tau .$$

Consequently, we have the following assertion:

(d) *τ is uniquely determined by conditions* (i) *and* (ii).

(e) *For every $a, b \in R$ the equality*

$$\tau(ab) = ba$$

holds.

In fact, substituing $a = \tau$, $b = a$ and $c = b$ in (i) and taking into account (ii), we have the equality

$$\tau(ab) = (\tau(\tau b))a = ba .$$

(f) *For every $a, b, c \in R$ the equality*

$$(ab)c = a(c(\tau b))$$

is true.

From (i) we obtain the equality

$$c(ab) = (c(\tau b))a .$$

Consequently,

$$\tau(c(ab)) = \tau((c(\tau b))a) .$$

Hence, using (e), we obtain our assertion.

The generalization of the above formula is given by the following one:

(g) *For every system $a_1, a_2, \ldots, a_n \in R$ $(n \geqslant 3)$ the equality*

$$(\ldots((a_1a_2)a_3)\ldots a_{n-1})a_n = a_1(\ldots((a_n(\tau a_{n-1}))(\tau a_{n-2}))\ldots(\tau a_2))$$

holds.

We shall prove our formula by induction with respect to n. For $n = 3$ our formula is identical to (f). Now let us suppose that it holds for every n-tuple. Consequently, for the n-tuple $a_1a_2, a_3, \ldots, a_{n+1}$ we have the equality

$$((\ldots((a_1a_2)a_3)\ldots a_{n-1})a_n)a_{n+1} = (a_1a_2)(\ldots((a_{n+1}(\tau a_n))(\tau a_{n-1}))\ldots(\tau a_3)) .$$

Substituing in (f) $a = a_1$, $b = a_2$ and

$$c = (\ldots((a_{n+1}(\tau a_n))(\tau a_{n-1}))\ldots(\tau a_3))$$

we obtain our formula for every $(n+1)$-tuple $a_1, a_2, \ldots, a_{n+1} \in R$. Formula (g) is thus proved.

From (e) and (g) the following statement follows:

(h) *For every system $a_1, a_2, \ldots, a_n \in R$ $(n \geqslant 3)$ we have the equality*

$$\tau((\ldots((a_1a_2)a_3)\ldots a_{n-1})a_n) = (\ldots((a_n(\tau a_{n-1}))(\tau a_{n-2}))\ldots(\tau a_2))a_1 .$$

III. Let us consider an associative ring R_0 with the unit element. Further, let us suppose that R_0 is a ring with *involution*, i. e. for every $a \in R_0$ an element a^*, called the *adjoint* of a, is defined such that the conditions

$$(a+b)^* = a^* + b^*, \quad (a^*)^* = a, \quad (a \circ b)^* = b^* \circ a^*$$

are satisfied, where $\circ$ denotes the product operation in R_0.

An element $a \in R_0$ is called *selfadjoint* or *Hermitian* if $a = a^*$. Obviously, the unit element of R_0 is Hermitian (see [4], chap. II and V).

In the sequel we shall denote by $\mathcal{K}(R_0)$ the set R_0 with the usual addition and multiplication defined as follows:

$$(1) \qquad ab = b^* \circ a .$$

If R_0 is the ring of real square matrices of fixed order, then $\mathcal{K}(R_0)$ coincides with the ring of *cracovians* introduced by T. Banachiewicz (see [2]), who has applied it widely to problems in astronomy.

In general, we shall call $\mathcal{K}(R_0)$ the *cracovian ring* generated by R_0.

THEOREM 1. *If R_0 is an associative ring with involution and having the unit element, then $\mathcal{K}(R_0)$ is a τ-ring.*

Proof. To prove our theorem, it suffices to show that in $\mathcal{K}(R_0)$ the two-sided distributive law and equalities (i) and (ii) are true.

Using (1), we have the equalities

$$a(b+c) = (b+c)^* \circ a = b^* \circ a + c^* \circ a = ab + ac ,$$

$$(b+c)a = a^* \circ (b+c) = a^* \circ b + a^* \circ c = ba + ca .$$

Moreover, denoting by e the unit element of R_0, we have the equalities

$$e(ea) = (ea)^* \circ e = (ea)^* = (a^* \circ e)^* = a\,,$$
$$a(bc) = a(c^* \circ b) = (c^* \circ b)^* \circ a = (b^* \circ c) \circ a = b^* \circ (c \circ a) = (c \circ a)b$$
$$= \big((c \circ e) \circ a\big)b = \big(a(c^* \circ e)\big)b = \big(a(ec)\big)b\,.$$

Consequently, equalities (i) and (ii) are satisfied for $\tau = e$. The theorem is thus proved.

Now we shall give the complete representation of τ-rings. Namely, we shall prove the following

THEOREM 2. *Every τ-ring is equal to the cracovian ring generated by an associative ring with involution and having the unit element.*

Proof. Let R be a τ-ring and let R_0 denote the set R with the usual addition and $\circ$-multiplication defined by the formula

$$a \circ b = b(\tau a)\,. \tag{2}$$

Now we shall prove that R_0 is an associative ring having the unit element with involution

$$a^* = \tau a\,. \tag{3}$$

Using (2), we obtain the distributive laws:

$$a \circ (b+c) = (b+c)(\tau a) = b(\tau a) + c(\tau a) = a \circ b + a \circ c\,,$$
$$(b+c) \circ a = a\big(\tau(b+c)\big) = a(\tau b) + a(\tau c) = b \circ a + c \circ a\,.$$

Further, according to (e), we have the equality

$$(a \circ b) \circ c = \big(b(\tau a)\big) \circ c = c\Big(\tau\big(b(\tau a)\big)\Big) = c\big((\tau a)b\big)\,.$$

Hence, using (i), we obtain the associative law

$$(a \circ b) \circ c = \big(c(\tau b)\big)(\tau a) = (b \circ c)(\tau a) = a \circ (b \circ c)\,.$$

The element τ is the unit element of R_0. In fact, by (ii) and (c),

$$a \circ \tau = \tau(\tau a) = a\,, \qquad \tau \circ a = a(\tau\tau) = a\,.$$

From (3) and (e) follow the equalities

$$(a+b)^* = \tau(a+b) = \tau a + \tau b = a^* + b^*,$$
$$(a^*)^* = \tau(\tau a) = a\,,$$
$$(a \circ b)^* = \tau(a \circ b) = \tau\big(b(\tau a)\big) = (\tau a)b = b^* \circ a^*\,.$$

Thus R_0 is an associative ring with involution and having the unit element. Finally, we have the equality

$$b^* \circ a = (\tau b) \circ a = a\big(\tau(\tau b)\big) = ab\,,$$

which implies $R = \mathcal{K}(R_0)$. The theorem is thus proved.

References

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