Neutral ideals and congruences

by

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Introduction. This is an attempt to find whether there is a (1-1)-correspondence between the congruence relations and neutral ideals of a lattice (1) in general, and of a modular lattice in particular (Problem 73 of [1], p. 161). This problem, though not solved fully in the course of this paper, has led to certain results which are interesting in themselves.

By defining a neutral ideal in a particular way (as in [1], p. 80), it is shown that the neutral ideals of a lattice \( L \) correspond (1-1) to a sublattice of the lattice of congruences on \( L \). Hence to every neutral ideal of \( L \) corresponds a lattice congruence \( \theta \) on \( L \). It is therefore natural to ask whether every congruence \( \theta \) on \( L \) has a neutral ideal as the ideal of elements congruent to zero. As is shown in Section 4 below, the answer to this question is in the negative.

In Section 2, I discuss the properties of neutral ideals. In Section 3, I prove the correspondence between neutral ideals and congruences; finally, in Section 4, I give counter examples to show the inconsistency of certain questions which arise naturally.

1. Preliminaries. The symbols \( \leq, \leq, + \) will denote inclusion, non-inclusion, sum (least upper bound) and product (greatest lower bound) in any lattice \( L \); while the symbols \( \subseteq, \cup, \cap, \cap \) will refer to set-inclusion union (set sum), intersection (set product), membership and non-membership, respectively. Small letters \( a, b, ... \) will denote elements of the lattice and capital letters \( A, B, ... \) will stand for ideals of the lattice.

All lattices which are considered during the course of this paper have \( 0 \), the null element of the lattice.

A non-null subset \( \mathcal{I} \) of elements of a lattice \( L \) is called an ideal if and only if

(i) \( x \in \mathcal{I}, y \in \mathcal{I} \Rightarrow x + y \in \mathcal{I} \)

and

(ii) \( x \in \mathcal{I} \) and \( t \leq x \Rightarrow t \in \mathcal{I} \) (Stone [6]).

(1) For general information regarding lattices, see Birkhoff [1].
An ideal \( \mathcal{D} \) of a lattice \( L \) is called neutral if and only if
\[
 t \leq (x + a)(y + b), y \in \mathcal{D} \Rightarrow t \leq x + a \cdot b
\]
for some \( z \in \mathcal{D} \).

An ideal \( \mathcal{D} \) of a lattice \( L \) is called \( m \)-neutral if and only if \( \mathcal{D} \) as an element of the lattice of ideals of \( L \) distributes finite products.

A binary relation \( \theta \) on \( L \) is said to be an equivalence relation if it satisfies

(i) \( x = x(\theta) \) (reflexive),

(ii) \( x = y(\theta) \Rightarrow y = x(\theta) \) (symmetric),

(iii) \( x = y(\theta), \ y = z(\theta) \Rightarrow x = z(\theta) \) (transitive).

If it further satisfies the substitution property

(iv) \( x = x(\theta), \ y = y(\theta) \Rightarrow x + y = x(\theta) + y(\theta) \),

then it is called an additive congruence.

An equivalence relation \( \theta \) which has the substitution property

(v) \( x = x(\theta), \ y = y(\theta) \Rightarrow x \cdot y = x(\theta) \cdot y(\theta) \)

is called a multiplicative congruence.

If the binary relation \( \theta \) satisfies the conditions (i)-(v) then it is said to be a lattice congruence or merely a congruence on \( L \). The congruences on a lattice \( L \) form a complete lattice (see [1], p. 24).

The sum and product of an arbitrary family of congruences are defined as follows:

\[
a = b \bigcup \{b_i \}
\]

if there exists a finite sequence \( a = a_0, a_1, \ldots, a_n = b \) such that \( a_{i-1} = a_i(\theta_i) \) for some \( \theta_i, j = (1, 2, \ldots, n) \), and

\[
a = b \bigcap \{b_i \}
\]

if \( a = b(\theta_i) \) for every \( i \).

2. The purpose of this section is to show that the neutral ideals of a modular lattice form a \( L \)-distributive lattice. Certain preliminary results are needed.

**Lemma 1.** The principal ideal generated by a single element \( c \in \) of a lattice \( L \) is neutral if and only if \( c \) distributes finite products.

**Proof.** Let the principal ideal \( I \) generated by \( c \) be neutral; then
\[
t \leq (x + a)(y + b), y \in I \Rightarrow t \leq x + a \cdot b
\]
for some \( \in I \). But since \( a, y \in I \), we can take \( x = e \) and \( y = c \). Therefore
\[
t \leq (t + a)(c + b) \Rightarrow t \leq x + a \cdot b
\]
for some \( \in I \).

Now, any \( x \in I \) is less than \( c \) and hence \( t \leq x + a \cdot b \). Therefore \( t \leq (c + a)(c + b) = t \leq c + a \cdot b \). Hence \( (c + a)(c + b) \leq \leq a \cdot b \). But \((c + a)(c + b) \geq c + a \cdot b \) for all \( a, b, c \in L \). Therefore \((c + a)(c + b) = c + a \cdot b \). Hence \( c \) distributes finite products.

Conversely, let \( c \) distribute finite products. Then
\[
(c + a)(c + b) \leq c + a \cdot b
\]
and when
\[
t \leq (x + a)(y + b) = t \leq (c + a)(c + b)
\]
since \( I \) is the principal ideal generated by \( c \). Therefore \( t \leq (x + a)(y + b) \Rightarrow t \leq (c + a)(c + b) \leq t \leq c + a \cdot b \); that is, \( t \leq x + a \cdot b \) for \( x \in I \). Hence \( I \) is a neutral ideal of \( L \).

**Corollary.** When the lattice \( L \) is modular, if \( c \) distributes finite products, it distributes finite sums also (see [1], p. 78) and hence is a neutral element (see [3]).

Hence in the case of a modular lattice \( L \) the lemma reads as follows:

The principal ideal generated by an element \( c \in L \) is neutral if and only if \( c \) is neutral.

**Lemma 2.** The sum of any family of neutral ideals of a lattice \( L \) is a neutral ideal.

**Proof.** Let \( N = \sum N_i \), each \( N_i \) being a neutral ideal. Elements \( x, y \in N \) are given by
\[
x = x_1 + x_2 + \ldots + x_n, \quad y = y_1 + y_2 + \ldots + y_r,
\]
where \( x \), \( y \) being finite and \( x_i \in N_i \) and \( y_i \in N_i \) (vide [1], p. 140).

Since \( 0 \) belongs to every ideal, we can add as many zeros as are necessary and write
\[
x = x_1 + x_2 + \ldots + x_n, \quad y = y_1 + y_2 + \ldots + y_r,
\]
such that \( x_i, y_i \in N_i \) for each \( i \). Therefore
\[
t \leq (x + a)(y + b), x, y \in N \Rightarrow t \leq x + a \cdot b
\]

for some \( x \in I \). But since \( x, y \in I \), we can take \( x = c \) and \( y = c \). Therefore
\[
t \leq (t + a)(c + b) \Rightarrow t \leq x + a \cdot b
\]
for some \( x \in I \).

Therefore \( N \) is a neutral ideal.
It is easily seen that 0 is a neutral ideal. Therefore the neutral ideals on any lattice $L$ form a partially ordered set with 0, and every non-void subset of $L$ has a least upper bound (by Lemma 2). Hence the neutral ideals of any lattice $L$ form a complete lattice (vide [1], p. 49).

**Lemma 3.** The product of two and hence also of a finite number of neutral ideals of a modular lattice $L$ is a neutral ideal.

**Proof.** Let $X$ and $Y$ be neutral ideals of $L$. Let $x, y \in X \cap Y = \{x, y \in X\}$ and $x, y \in Y$. Now

$$x \leq (x + a)(y + b) \Rightarrow x \leq x + a \cdot b \quad (x \in X);$$

also

$$x \leq (x + a)(y + b) \Rightarrow x \leq x + a \cdot b \quad (x \in Y).$$

Therefore

$$t \leq (x + a)(y + b) \Rightarrow t \leq (x + a \cdot b)(x + a \cdot b) \quad (\in X \cap Y)$$

$$= t \leq a \cdot b + x_1(a_2 + a \cdot b) \quad (L \text{ is modular})$$

$$= t \leq a \cdot b + x_1(\{x_2 + a \cdot b\} \quad (0, x_1 \in Y \text{ for some } x_2 \in Y)$$

$$= t \leq a \cdot b + x_1 \cdot y_2 \cdot z_2 \cdot a \cdot b \quad (L \text{ is modular})$$

$$= t \leq x_1 \cdot y_2 \cdot z_2 \cdot a \cdot b \quad (x_2 \in X \cap Y)$$

$$= t \leq x + a \cdot b \quad \text{for some } x \in X \cap Y.$$}

Therefore $X \cap Y$ is a neutral ideal. This proof can be extended to a finite number of neutral ideals.

**Theorem 1.** The neutral ideals of a modular lattice satisfy the infinite distributive law: $N \cap (\sum N_i) = \sum (N \cap N_i)$.

**Proof.** Now $N \cap (\sum N_i) \supseteq \sum (N \cap N_i)$. Therefore it is enough if we prove that $N \cap (\sum N_i) \subseteq \sum (N \cap N_i)$.

Let $a \in N \cap (\sum N_i)$; that is, $a \in N$ and $a \in \sum N_i$, $a \leq x_1 + x_2 + \ldots + x_n$ for finite $\nu$ and $a \in N_i$. Therefore

$$a = (0 + a)(x_1 + x_2 + \ldots + x_n) \quad (0, x_1 \in N_i)$$

$$= a \leq y_1 + a \cdot (x_2 + \ldots + x_n) \quad (0, x_1 \in N_i)$$

$$= a \leq a \cdot y_1 + (0 \cdot a)(x_2 + \ldots + x_n) \quad (L \text{ is modular})$$

$$= a \leq a \cdot y_1 + a \cdot (x_2 + \ldots + x_n) \quad (y \in N_i)$$

$$= a \leq a \cdot (a \cdot y_1 + a \cdot (x_2 + \ldots + x_n) \quad (L \text{ is modular})$$

$$= a \leq a \cdot (a \cdot y_1 + a \cdot (x_2 + \ldots + x_n)) \quad (L \text{ is modular})$$

$$= a \leq a \cdot y_1 + a \cdot y_2 + a \cdot (x_2 + \ldots + x_n) \quad (y \in N_i)$$

$$= a \leq (a \cdot y_1 + a \cdot y_2 + a \cdot (x_2 + \ldots + x_n)) \quad (L \text{ is modular})$$

$$= a \leq (a \cdot y_1 + a \cdot y_2 + a \cdot (x_2 + \ldots + x_n))$$

Therefore, $I = a \cap N$ is a neutral ideal.

**Corollary 1.** The neutral ideals of a modular lattice form a complete $\sum$-distributive lattice. Hence it is pseudocomplemented (for definition, see [1], p. 147).

**Corollary 2.** When the lattice is distributive, all ideals are neutral. Therefore the ideals of a distributive lattice form a pseudocomplemented distributive lattice.

**Lemma 4.** An ideal $I$ of a lattice $L$ is neutral if and only if $I$ is $m$-neutral.

**Proof.** Let $I$ be neutral; that is,

$$t \leq (x + a)(y + b)(x, y \in I) \Rightarrow t \leq a \cdot b \quad (a \in I).$$

Consider two ideals $A, B$ of $L$ such that $a \in A$ and $b \in B$: then $a \cdot b \in A \cap B$. Hence

$$t \in (I \cup A) \cap (I \cup B)$$

$$\Rightarrow t \leq (x + a)(y + b)$$

$$\Rightarrow t \leq a \cdot b \quad (a \in I)$$

$$\Rightarrow t \in I \cup (A \cap B).$$

Therefore $(I \cup A) \cap (I \cup B) = I \cup (A \cap B)$.

Thus $I$ distributes finite products and hence is $m$-neutral.

Conversely, let $I$ be $m$-neutral; that is,

$$(I \cup A) \cap (I \cup B) = I \cup (A \cap B)$$

for any two ideals $A, B$ of $L$. Therefore

$$(I \cup A) \cap (I \cup B) \subseteq I \cup (A \cap B).$$

Choose $A, B$ as the principal ideals generated by $a, b$ respectively. Let $x, y \in I$. Then

$$t \leq (x + a)(y + b)$$

$$\Rightarrow t \in (I \cup A) \cap (I \cup B)$$

$$\Rightarrow t \in I \cup (A \cap B)$$

$$\Rightarrow t \leq a \cdot b \quad (a \in I)$$

$$\Rightarrow t \leq a \cdot b \quad (a \in I).$$

Therefore, $I$ is a neutral ideal.

**Corollary.** When the lattice $L$ is modular, an $m$-neutral ideal is neutral as an element of the lattice of ideals of $L$. Hence an ideal of a modular lattice $L$ is neutral if and only if it is neutral as an element of the lattice of ideals of $L$. 

Therefore $N \cap (\sum N_i) \subseteq \sum (N \cap N_i)$ and hence

$$(N \cap (\sum N_i) = \sum (N \cap N_i).$$
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(v) a multiplicative congruence:

\[ x = y \Rightarrow x \cdot z = y \cdot z. \]

By (iv), \[ x = y \Rightarrow x + a = y + a \quad \text{for} \quad a \in I. \]

So \[ x \cdot z + a \cdot b \leq (x + a)(z + b) \]

for all \( x \in L \) and \( a, b \in I \), or \[ x \cdot z + a \cdot b \leq (y + a)(z + b) \]

for some \( a \in I \).

Similarly,

\[ y \cdot z + a \cdot b \leq (y + a)(z + b) \leq (x + a)(z + b) \leq x \cdot z + d \quad \text{for some} \quad d \in I. \]

Therefore,

\[ x \cdot z + a \cdot b + c + d \leq y \cdot z + a \cdot b + c + d \]

and

\[ x \cdot z + a \cdot b + c + d \geq y \cdot z + a \cdot b + c + d, \]

that is,

\[ x \cdot z + a \cdot b + c + d = y \cdot z + a \cdot b + c + d \quad \text{and} \quad a \cdot b + c + d \in I. \]

Hence \( x \cdot z = y \cdot z \).

Thus, \( x = y \) if and only if \( x + a = y + a \) for some \( a \in I \). Let \( t \leq (a + x)(b + y) \), \( a, b \in I \).

Then

\[ t = t \cdot (a + x)(b + y), \]

\[ t = t \cdot x \cdot y \cdot (t) = t + x \cdot y = x \cdot y \cdot (t) \quad \text{since} \quad t \text{ is a lattice congruence.} \]

Therefore \( t + x \cdot y + c = x \cdot y + c \) for some \( c \in I \); that is, \( t \leq x \cdot y + c \). Hence

\[ t \leq (a + x)(b + y) \quad (a, b \in I) \]

\[ \Rightarrow t \leq c + x \cdot y \quad \text{for some} \quad c \in I. \]

Therefore \( I \) is a neutral ideal.

Corollary. Any ideal of a distributive lattice \( L \) is a zero class under some suitable congruence relation on \( L \).

Definition. The congruence modulo a neutral ideal is called a neutral congruence.

It is easily seen that there is a \((1,1)\)-correspondence between neutral ideals and neutral congruences. If a non-neutral congruence \( \Phi \) has a neutral ideal \( N \) as its zero class and if \( \Theta \) is the neutral congruence corresponding to the neutral ideal \( N \), then \( \Theta \subseteq \Phi \).
For if \( a = b(\theta) \) then \( a + t = b + t \) for some \( t \in N_1 \); also \( t = 0(\Phi) \). Therefore \( a + t = b + t \Rightarrow a = b(\Phi) \). Hence \( \theta \subseteq \Phi \).

**Theorem 2.** The neutral congruences on a lattice \( L \) form a complete \( \Sigma \)-distributive lattice.

**Proof.** Let \( \theta_1, \theta_2 \) be an arbitrary family of neutral congruences on \( L \). Let \( \theta = \sum \theta_i \). Let \( N_1, N_2, \ldots \) be the corresponding neutral ideals and let \( N = \sum N_i \). Let \( \Phi \) be the neutral congruence corresponding to the neutral ideal \( N \). Then \( \theta = \Phi \); for, \( a = b(\theta) \) implies that there exists a finite sequence \( a = a_0, a_1, \ldots, a_n = b \) such that \( a_{i-1} = a_i(\theta_i) = a_{i-1} + t_i \) for some \( t_i \in N_i \). Therefore,

\[
\begin{align*}
  a_0 + t_0 &= a_0 + t_0, \\
  a_1 + t_1 &= a_1 + t_1, \\
  & \quad \quad \quad \quad \quad \vdots \\
  a_{i-1} + t_{i-1} &= a_{i-1} + t_{i-1}, \\
  a_{i} + t_{i} &= a_{i} + t_{i}.
\end{align*}
\]

That is,

\[
\begin{align*}
  a_0 + t_0 + \ldots + t_i &= a_0 + t_0 + \ldots + t_i, \\
  a_1 + t_1 + \ldots + t_i &= a_1 + t_1 + \ldots + t_i, \\
  & \quad \quad \quad \quad \quad \vdots \\
  a_{n-1} + t_{n-1} &= a_{n-1} + t_{n-1}, \\
  a_n + t_n &= a_n + t_n.
\end{align*}
\]

Therefore \( a = b(\Phi) \) and hence \( \theta \subseteq \Phi \).

Next let

\[
\begin{align*}
  a &= b(\Phi) \; \Rightarrow a + t_k &= b + t_k \; \forall \; t_k \in N_i \; \text{for some} \; i \; \text{and finite} \; k. \\
  a + t_k + \ldots + t_{k+1} &= a + t_k + \ldots + t_{k+1} \\
  a + t_k + \ldots + t_{k+1} &= a + t_k + \ldots + t_{k+1}(\theta_i) \\
  & \quad \quad \quad \quad \quad \vdots \\
  a + t_n &= a + t_n.
\end{align*}
\]

Therefore, \( a + t_k + \ldots + t_n = a(\theta) \).

Similarly \( b + t_k + \ldots + t_n = b(\theta) \). But \( a + t_k + \ldots + t_n = b + t_k + \ldots + t_n \).

Hence \( a = b(\theta) \). Therefore \( \Phi \subseteq \theta \). Hence \( \theta = \Phi \).

Therefore, an arbitrary sum of a number of neutral congruences is neutral and the identity congruence is also a neutral congruence. Therefore the neutral congruences on any lattice \( L \) form a complete lattice (see [1], p. 49). Further, the congruences on any lattice \( L \) and in particular the neutral congruences, satisfy the infinite distributive law \( \theta \cap (\sum \theta_i) = \sum (\theta \cap \theta_i) \) ([1], p. 34). Therefore the neutral congruences on any lattice form a complete \( \Sigma \)-distributive lattice \( L \).

**Corollary 1.** The distributive lattice \( L \) is a sublattice of the lattice of all congruence relations on \( L \) (see [4]).

**Corollary 2.** The number of neutral elements of \( L \) is at most equal to the number of neutral congruences on \( L \).

**Lemma 6.** The product of a finite number of neutral congruences on a modular lattice \( L \) is a neutral congruence.

**Proof.** Let \( \theta_1, \theta_2 \) be two neutral congruences on \( L \). Let \( N_1, N_2 \) be the corresponding neutral ideals. Then \( N_1 \cap N_2 \) is a neutral ideal (Lemma 3). Let \( \theta \) be the neutral congruence corresponding to \( N_1 \cap N_2 \). Then \( \theta = \theta_1 \cap \theta_2 \).

For let \( a = b(\theta) \Rightarrow a + x = b + x \) for \( a \in N_1 \cap N_2 \).

Therefore \( a \in N_1 \) and \( a \in N_2 \). Hence \( a = b(\theta_1) \) and \( a = b(\theta_2) \). Hence \( \theta \subseteq \theta_1 \cap \theta_2 \).

Next let \( a = b(\theta_1) \) and \( a = b(\theta_2) \). Thus \( a + x = b + x \) for some \( a \in N_1 \) and \( a + y = b + y \) for some \( y \in N_2 \). Therefore

\[
\begin{align*}
  (a + x)(a + y) &= (b + x)(b + y) \\
  a + x \cdot (a + y) &= b + x \cdot (b + y) \quad \text{\( L \) is modular).}
\end{align*}
\]

Therefore \( a + x \cdot y \leq a + x \cdot (a + y) \leq b + x \cdot (b + y) \leq b + x \cdot (y + b) \leq b + x \cdot (y + b) \quad \text{\( L \) is modular).}

Now

\[
\begin{align*}
  a + x \cdot y &\leq b + x \cdot (a + y) \\
  &\leq b + x \cdot (b + y) \\
  &\leq b + x \cdot (y + b) \\
  &\leq b + x \cdot (y + b) \quad \text{\( L \) is modular).}
\end{align*}
\]

That is, \( a + x \leq b + x \) for \( a, x \in N \).

Similarly \( b + x \leq a + x \) for \( a, x \in N \). Hence \( a + x + a + x + b + x + a + x \in N \) and \( x + a + x + b + x + a + x \in N \). Therefore \( a = b(\theta) \); that is, \( \theta_1 \cap \theta_2 \subseteq \theta \).

Hence \( \theta = \theta_1 \cap \theta_2 \).

Therefore the product of two and hence of a finite number of neutral congruences of a modular lattice \( L \) is a neutral congruence.

**Remark.** When \( L \) is a simple modular lattice with \((0, 1) \), no element of \( L \) is neutral (except \((0, 0) \) and \((1, 1) \).

**Proof.** Let \( L \) be a simple lattice. Then there are only two congruences and both of them are neutral. Therefore \( L \) (the lattice of congruences on \( L \)) is simple.

Further, the number of neutral elements of \( L \leq \) the number of number of neutral congruences on \( L \), i.e. 2 in this case (Corollary 2 of Theorem 2). But any lattice has at least two neutral elements, viz., the zero and one of the lattice. Therefore \( L \) has no neutral elements excepting 0 and 1.
The converse is not necessarily true; that is, when \( L \) has only 0 and 1 as neutral elements, \( L \) need not be simple. This is shown in figure 2.

4. In this section we give certain examples to prove certain negative statements.

Example 1. The kernel of every congruence on a lattice \( L \) (even when \( L \) is modular) need not necessarily be neutral.

\[ \begin{array}{c}
\text{Fig. 2} \\
\end{array} \]

Let \( \theta \) be a congruence on the lattice of fig. 3. \( I \) is the zero class of the congruence \( \theta \) on \( L \). \( I \) is not a neutral ideal, for though \( f \subseteq (a + d)(a + e) \)

\[ f \subseteq a + d \cdot e \quad \text{for all} \quad a \in L. \]

Example 2. If \( \theta_1 \) and \( \theta_4 \) are two congruences on a lattice \( L \) such that the kernel of \( \theta_1 \) and the kernel of \( \theta_4 \) are neutral ideals; then the kernel of \( \theta_1 \cup \theta_4 \) need not be equal to (the kernel of \( \theta_1 \)) \( \cup \) (the kernel of \( \theta_4 \)).

\[ \begin{array}{c}
\text{Fig. 4} \\
\end{array} \]

For, \( \theta_1 \) is a congruence on \( L \) with \( I_1 \) as its kernel and \( \theta_4 \) a second congruence with \( I_4 \) as kernel. But the kernel of \( \theta_1 \cup \theta_4 \) is equal to the whole of \( L \), whereas the (kernel of \( \theta_1 \)) \( \cup \) (kernel of \( \theta_4 \)) = \( I_4 \).

Example 3. If \( \theta_1 \) and \( \theta_4 \) are two congruences on a lattice \( L \) such that the kernel of \( \theta_1 \) and the kernel of \( \theta_4 \) are neutral ideals, the kernel of \( \theta_1 \cup \theta_4 \) need not be neutral.

\[ \begin{array}{c}
\text{Fig. 5} \\
\end{array} \]

\( I_1 \) and \( I_4 \) are neutral ideals on \( L \) and are the kernels of two congruences \( \theta_1 \) and \( \theta_4 \) on \( L \). But the kernel \( I_1 \cup I_4 \) of the congruence \( \theta_1 \cup \theta_4 \) is not a neutral ideal.

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References


