# On the determining of the form of congruences in abstract algebras with equationally definable constant elements

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As we know all the homomorphic images of any abstract algebra $A$ are determined (up to isomorphisms) by the congruences in $A$. Therefore the knowledge of the form of all congruences in $A$ is very important. For some abstract algebras the form of congruences is already determined, e. g. in any group, ring and boolean algebra. Let $\boldsymbol{G} = \langle G, \cdot, {}^{-1}\rangle$, $\boldsymbol{R} = \langle R, +, -, \cdot\rangle$ and $\boldsymbol{B} = \langle B, \cup, \cap, '\rangle$ be any group, ring and boolean algebra. Every congruence $\sim$ in those algebras has one of the following forms:

1° for all $x$ and $y$ in $G$, $x \sim y$ if and only if $x \cdot y^{-1} \epsilon N$, where $N$ is a normal subgroup of $\boldsymbol{G}$,

2° for all $x$ and $y$ in $R$, $x \sim y$ if and only if $x + (-y) \epsilon I$, where $I$ is an ring-ideal in $\boldsymbol{R}$,

3° for all $x$ and $y$ in $B$, $x \sim y$ if and only if $x \cap y' \cup x' \cap y \epsilon J$, where $J$ is a boolean-ideal in $\boldsymbol{B}$.

The sets $N$, $I$ and $J$ are whole abstraction classes of congruence $\sim$ in $\boldsymbol{G}$, $\boldsymbol{R}$ and $\boldsymbol{B}$ determined respectively by the unit $e = x \cdot x^{-1} = y \cdot y^{-1}$ of $\boldsymbol{G}$, the zero element $0 = x + (-x) = y + (-y)$ in $\boldsymbol{R}$ and by the boolean zero $0^* = x \cap x' \cup x' \cap x = y \cap y' \cup y' \cap y$ in $\boldsymbol{B}$. Hence it follows that every congruence $\sim$ in $\boldsymbol{G}$, $\boldsymbol{R}$ and $\boldsymbol{B}$ has one of the following properties:

$1^{\circ\circ}$ for all $x$, $y$ in $G$, $x \sim y$ if and only if $x \cdot y^{-1} \sim e$,

$2^{\circ\circ}$ for all $x$, $y$ in $R$, $x \sim y$ if and only if $x + (-y) \sim 0$,

$3^{\circ\circ}$ for all $x$, $y$ in $B$, $x \sim y$ if and only if $x \cap y' \cup x' \cap y \sim 0^*$.

The properties $1^{\circ\circ}$, $2^{\circ\circ}$ and $3^{\circ\circ}$ are very similar. We see that the ways of the determining of the form of congruences in $\boldsymbol{G}$, $\boldsymbol{R}$ and $\boldsymbol{B}$ are analogical. J. Łoś has set the following question: Can be determined the form of congruences in every equationally definable class $\mathfrak{A}_0$ of algebras with equationally definable constant elements in an analogical way as in groups? The solution of this problem is negativ (see (5.5)).

The purpose of this paper is to give the sufficient and necessary conditions for the determining of the form of congruences in such class $\mathfrak{A}_0$ in the analogical way as in groups. Those conditions are imposed upon the set $E(\mathfrak{A}_0)$ of all equations which are valid in every algebra of $\mathfrak{A}_0$.

**§ 1. Terms, notations and lemmas.** By a *k-ary operation* on the set $A$ we understand a function $F(x_1, x_2, \ldots, x_k)$ defined on $A$ and with values in $A$. A system $\boldsymbol{A} = \langle A, F_1, F_2, \ldots, F_n\rangle$, where $A$ is a non-empty set and $F_i$ are $k_i$-ary operations on $A$, is called an *algebra of the type* $\Delta = \langle k_1, k_2, \ldots, k_n\rangle$. Two algebras of the same type are called *similar*. In the sequel we shall denote algebras by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ and their sets by $A, B, C, \ldots$ Let $\mathfrak{A}$ be the class of all algebras of the type $\Delta$. An algebra $\boldsymbol{B} \in \mathfrak{A}$ is a *subalgebra of algebra* $\boldsymbol{A} \in \mathfrak{A}$ if $B$ is a subset of $A$ and moreover the operations $F_i$ of $\boldsymbol{B}$ and $\boldsymbol{A}$ are identical on $B$. Let $\boldsymbol{A} \in \mathfrak{A}$ and let $A_0$ be a non-empty subset of $A$. The least subalgebra of $\boldsymbol{A}$ which contains $A_0$ will be denoted by $\{A_0\}_{\boldsymbol{A}}$. If $\{A_0\}_{\boldsymbol{A}} = \boldsymbol{A}$, then $A_0$ is a *set of generators for* $\boldsymbol{A}$ and $\boldsymbol{A}$ is also called *generated by* $A_0$. Let $F$ and $F'$ be $k$-ary operations on the sets $A$ and $B$ respectively. A mapping $h$ is an *homomorphism* of operation $F$ into (onto) operation $F'$ if $h(A) \subset B$ $\big(h(A) = B\big)$ and

$$h\big(F(a_1, a_2, \ldots, a_k)\big) = F'\big(h(a_1), h(a_2), \ldots, h(a_k)\big)$$

for all $a_1, a_2, \ldots, a_k$ in $A$.

A homomorphism one-to-one is called an *isomorphism*. A mapping $h$ of $A$ into (onto) $B$ is an *homomorphism of the algebra* $\boldsymbol{A} = \langle A, F_1, F_2, \ldots, F_n\rangle$ into (onto) algebra $\boldsymbol{B} = \langle B, F_1', F_2', \ldots, F_n'\rangle$ if $h$ is an homomorphism of every operations $F_i$ into (onto) operation $F_i'$, $i = 1, 2, \ldots, n$. A binary relation $\sim$ in the set $A$ is *a congruence of an operation* $F(x_1, x_2, \ldots, x_k)$ defined on $A$ if it is symetric, reflexive and transitive, and if $a_1, a_2, \ldots, a_k$, $a_1', a_2', \ldots, a_k' \in A$, $a_1 \sim a_1'$, $a_2 \sim a_2'$, ..., $a_k \sim a_k'$ implies

$$F(a_1, a_2, \ldots, a_k) \sim F(a_1', a_2', \ldots, a_k').$$

A binary relation $\sim$ defined in set $A$ is a *congruence in an algebra* $\boldsymbol{A} = \langle A, F_1, F_2, \ldots, F_n\rangle$ if $\sim$ is a congruence of every operation $F_i$, $i = 1, 2, \ldots, n$. If $\sim$ is a congruence in $\boldsymbol{A}$, then $A/\sim$ denotes the set of all abstraction classes of $\sim$ in $A$. By $a/\sim$, for $a$ in $A$, is denoted that abstraction class to which $a$ belongs (¹). Puting for all $a_1/\sim, a_2/\sim, \ldots, a_{k_i}/\sim$ in $A/\sim$

$$F_i/\sim(a_1/\sim, a_2/\sim, \ldots, a_{k_i}/\sim) = F_i(a_1, a_2, \ldots, a_{k_i})/\sim$$

(¹) $a/\sim$ is the set of all $b$ in $A$ such that $a \sim b$ and it is called abstraction class of $\sim$ determined by $a$. $A/\sim$ is the set of all abstraction classes $a/\sim$ with $a \in A$.



we obtain a $k_i$-ary operation $F_i/\sim$ in $A/\sim$ and therefore the system $\boldsymbol{A}/\sim = \langle A/\sim, F_1/\sim, F_2/\sim, \ldots, F_n/\sim\rangle$ is an algebra in class $\mathfrak{A}$. $\boldsymbol{A}/\sim$ is the quotient algebra formed by dividing the algebra $\boldsymbol{A}$ by the congruence $\sim$. Let $\mathfrak{A}_0$ be any subclass of the class $\mathfrak{A}$ and let $C$ be an arbitrary set. We shall say that an algebra $\boldsymbol{A} \in \mathfrak{A}$ is a *free algebra for* $\mathfrak{A}_0$ *freely generated by* $C$ if it has the following two properties:

1° $C$ is a set of generators for algebra $\boldsymbol{A}$,

2° every mapping $f$ of the set $C$ into arbitrary algebra $\boldsymbol{B} \in \mathfrak{A}_0$ may be extended to a homomorphism $h$ of $\boldsymbol{A}$ into $\boldsymbol{B}$.

$\boldsymbol{A}$ is $\mathfrak{A}_0$*-free algebra freely generated by* $C$ if $\boldsymbol{A} \in \mathfrak{A}_0$ and $\boldsymbol{A}$ is a free algebra for $\mathfrak{A}_0$ freely generated by $C$. Two $\mathfrak{A}_0$-free algebras freely generated by sets of the same power are isomorphic (see [4], p. 20). There exists (up to isomorphism) at most one $\mathfrak{A}_0$-free algebra freely generated by a set of power $\mathfrak{m}$. However, the $\mathfrak{A}_0$-free algebras do not always exist. Their existence depends on the class $\mathfrak{A}_0$. If $\mathfrak{A}_0 = \mathfrak{A}$, then there exist such algebras, i. e. there exist $\mathfrak{A}$-free algebras freely generated by sets of arbitrary power. $\mathfrak{A}$-free algebras are also called *absolutely free of the type* $\Delta$. For a construction of $\mathfrak{A}$-free algebras see my paper [4], p. 21.

Now using the notion of $\mathfrak{A}$-free algebra we shall give the definitions of $\mathfrak{A}$*-term* and $\mathfrak{A}$*-equation*. Let $X$ be an arbitrary set of power $\aleph_0$ composed by the following different elements: $x_1, x_2, \ldots, x_m, \ldots$, $m < \omega_0$. Moreover let

$$\boldsymbol{W} = \langle W, f_1, f_2, \ldots, f_n\rangle$$

be a fixed $\mathfrak{A}$-free algebra freely generated by $X$. The elements of the algebra $\boldsymbol{W}$ are called $\mathfrak{A}$*-terms*, the elements in $X$ as *variables* are considered. The pairs $\langle\tau, \vartheta\rangle$, where $\tau$ and $\vartheta$ are $\mathfrak{A}$-terms, are called $\mathfrak{A}$*-equations*. In the sequel we shall denote the $\mathfrak{A}$-terms by $\vartheta, \vartheta^1, \vartheta^2, \ldots, \vartheta', \vartheta'^1, \vartheta'^2, \ldots$, $\delta, \delta^1, \delta^2, \ldots, \delta', \delta'^1, \delta'^2, \ldots, \tau, \tau^1, \tau^2, \ldots, \varphi, \varphi^1, \varphi^2, \ldots, \psi, \psi^1, \ldots$ and an $\mathfrak{A}$-equation $\langle\tau, \vartheta\rangle$ by $\ulcorner\tau = \vartheta\urcorner$. The set of all $\mathfrak{A}$-equations will be denoted by $E_{\mathfrak{A}}$. Let $\boldsymbol{A} \in \mathfrak{A}$. An $\mathfrak{A}$-equation $\ulcorner\tau = \vartheta\urcorner$ *is valid in the algebra* $\boldsymbol{A}$ if $h(\tau) = h(\vartheta)$, for every homomorphism $h$ of the algebra $\boldsymbol{W}$ into $\boldsymbol{A}$. By $\mathfrak{A}(E)$, for $E \subset E_{\mathfrak{A}}$, we shall denote the set of all algebras $\boldsymbol{A} \in \mathfrak{A}$ in which every $\mathfrak{A}$-equation in $E$ is valid. By $E_{\mathfrak{A}}(\mathfrak{A}_0)$, for $\mathfrak{A}_0 \subset \mathfrak{A}$, will be denoted the set of all $\mathfrak{A}$-equations which are valid in every algebra belonging to class $\mathfrak{A}_0$. A class $\mathfrak{A}_0 \subset \mathfrak{A}$ is called *equationally definable* if $\mathfrak{A}_0 = \mathfrak{A}\big(E_{\mathfrak{A}}(\mathfrak{A}_0)\big)$. In the sequel the $\mathfrak{A}$-terms and $\mathfrak{A}$-equations will be called briefly terms and equations. Let $\tau \in W$ be a term. The least subset $X_0 \subset X$ such that $\tau \in \{X_0\}_{\boldsymbol{W}}$ is called the *support of* $\tau$, it will be denoted by $s(\tau)$. The *support of an equation* $\ulcorner\tau = \vartheta\urcorner$ is the set $s(\ulcorner\tau = \vartheta\urcorner) = s(\tau) \cup s(\vartheta)$, where $s(\tau)$ and $s(\vartheta)$ are the supports of terms $\tau$ and $\vartheta$ respectively. The elements in $s(\tau)$ and $s(\ulcorner\tau = \vartheta\urcorner)$ are those variables which appear in term $\tau$ and equation

$\ulcorner \tau = \vartheta \urcorner$. The term $\tau$ with support $s(\tau) = (x_{m_1}, x_{m_2}, \ldots, x_{m_l})$, where $m_1 < m_2 < \ldots < m_l$, will be denoted also by $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$. The *endomorphisms* of $\boldsymbol{W}$, i. e. the homomorphisms $\eta(\boldsymbol{W}) \subset \boldsymbol{W}$, are called *substitutions* in $\boldsymbol{W}$. If $\eta$ is a substitution in $\boldsymbol{W}$, then $\eta(\tau)$, for $\tau \in W$, is called *$\eta$-substitution of $\tau$*. $\eta(\tau)$ is a substitution of $\tau$, where the terms $\eta(x)$ are substituted for variables $x$. Let $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$, $\vartheta_1, \vartheta_2, \ldots, \vartheta_l$ be arbitrary terms. We denote by $\tau(\vartheta_1, \vartheta_2, \ldots, \vartheta_l)$ the $\eta$-substitution of term $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$ with $\eta(x_{m_1}) = \vartheta_1, \eta(x_{m_2}) = \vartheta_2, \ldots, \eta(x_{m_l}) = \vartheta_l$. Let $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$ be any $\mathfrak{A}$-term and $\boldsymbol{A}$ an arbitrary algebra in $\mathfrak{A}$. The term $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$ defines in $A$ an $l$-ary operation $\tau_A$ such that for all $a_1, a_2, \ldots, a_l$ in $A$

$$\tau_A(a_1, a_2, \ldots, a_l) = h\big(\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})\big),$$

where $h$ is any homomorphism of $\boldsymbol{W}$ in $\boldsymbol{A}$ with $h(x_{m_i}) = a_i$, for $i = 1, 2, \ldots, l$. $\tau_A$ is called the *analytical operation in $A$ defined by term $\tau$*. The following theorems can be easily proved:

(1.1) *If $\sim$ is a congruence in an algebra $\boldsymbol{A} \in \mathfrak{A}$, then, for every $\mathfrak{A}$-term $\tau$, the relation $\sim$ is a congruence of the operation $\tau_A$.*

(1.2) *If $h$ is a homomorphism of an algebra $\boldsymbol{A} \in \mathfrak{A}$ into an algebra $\boldsymbol{B}$, then, for every $\mathfrak{A}$-term $\tau$, the mapping $h$ is a homomorphism of the operation $\tau_A$ into the operation $\tau_B$.*

Birkhoff [2] has proved the next theorem

(1.3) *If $\mathfrak{A}_0 \subset \mathfrak{A}$ is equationally definable class, then there exist $\mathfrak{A}_0$-free algebras freely generated by the sets of arbitrary power* [2].

The following theorem is known:

(1.4) *$\boldsymbol{A}$ is free algebra for $\mathfrak{A}_0 \subset \mathfrak{A}$ freely generated by a set $C$ if and only if it has the following properties:*

1° *$C$ is a set of generators for $\boldsymbol{A}$,*

2° *for all terms $\tau, \vartheta \in W$ and for every homomorphism $h$ of $\boldsymbol{W}$ into $\boldsymbol{A}$ with $h\big(s(\tau) \cup s(\vartheta)\big) \subset C$ and with $h(x) \neq h(y)$ for $x \neq y$ and $x, y \in s(\tau) \cup s(\vartheta)$, if $h(\tau) = h(\vartheta)$, then the equation $\ulcorner \tau = \vartheta \urcorner$ is valid in every algebra in $\mathfrak{A}_0$* [3].

The theorem (1.4) may be also expressed in the following form:

(1.5) *$\boldsymbol{A}$ is a free algebra for $\mathfrak{A}_0 \subset \mathfrak{A}$ freely generated by a set $C$ if and only if it has the following properties:*

1° *$C$ is a set of generators for $\boldsymbol{A}$,*

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[2] For (1.3) we suppose that $\mathfrak{A}_0$ has an algebra with at least two element. The proof of (1.3) is also contained in [3], p. viii-ix, and [4], p. 37-40.

[3] $h(s(\tau) \cup s(\vartheta))$ is the set of all elements $h(x)$ with $x$ contained in the set $s(\tau) \cup s(\vartheta)$, where $s(\tau)$ and $s(\vartheta)$ are the supports of terms $\tau$ and $\vartheta$. The theorem (1.4) easily results from the form of free algebras in equationally definable classes which is given e. g. in [4], theorem (8.3), p. 38.



2° *for all $\mathfrak{A}$-terms $\tau(x_{m_1}, x_{m_2}, \ldots, x_{m_l})$ and $\vartheta(x_{n_1}, x_{n_2}, \ldots, x_{n_r})$ and for all $c_1, c_2, \ldots, c_l, c_{l+1}, \ldots, c_{l+r}$ in $C$, if*

$$\tau_A(c_1, c_2, \ldots, c_l) = \vartheta_A(c_{l+1}, c_{l+2}, \ldots, c_{l+r}),$$

*then the equation $\ulcorner \tau(x'_1, x'_2, \ldots, x'_l) = \vartheta(x'_{l+1}, x'_{l+2}, \ldots, x'_{l+r}) \urcorner$ belongs to $E_{\mathfrak{A}}(\mathfrak{A}_0)$, where $x'_1, x'_2, \ldots, x'_l, x'_{l+1}, \ldots, x'_{l+r}$ are such variables that $x'_i = x'_j$ if and only if $c_i = c_j$.*

We note yet the following obvious theorem:

(1.6) *Let $\boldsymbol{A} \in \mathfrak{A}$ be an algebra generated by a set $C$. Then for every element $a \in A$ there exists an $\mathfrak{A}$-term $\tau$ with $\overline{\overline{s(\tau)}} = l$ and the different elements $c_1, c_2, \ldots, c_l$ in $C$ such that*

$$a = \tau_A(c_1, c_2, \ldots, c_l).$$

## § 2. The form of congruence generated by a given set.

Let $\boldsymbol{A} = \langle A, F_1, F_2, \ldots, F_n \rangle$ be an algebra in $\mathfrak{A}$ and let $\sim_1$ and $\sim_2$ be the congruences in $\boldsymbol{A}$. We say that the congruence $\sim_1$ is *smaller* than the congruence $\sim_2$: $\sim_1 \leqslant \sim_2$, if $a \sim_1 b$ implies $a \sim_2 b$ for $a$ and $b$ in $A$. Let $U$ be any subset of the set $A \times A$. The least congruence $\sim$ in $\boldsymbol{A}$ such that $\langle a, b \rangle \in U$ implies $a \sim b$ is called *generated by $U$*. The purpose of this paragraph is to give the form of congruence in $\boldsymbol{A}$ generated by $U$. By $D$ will be denoted the set of all pairs $\langle a, a \rangle$ with $a \in A$. Let $p$ be any natural number and moreover let $a^1, a^2, \ldots, a^p$ be arbitrary elements in $A$. We shall say that the sequence $[a^1, a^2, \ldots, a^p]$ is a *$U$-chain in $\boldsymbol{A}$ with the length $p$* if there exist the $\mathfrak{A}$-terms $\tau^i$ with $\overline{\overline{s(\tau^i)}} = m_i$, $i = 1, 2, \ldots, p-1$, $\varphi^i$ with $\overline{\overline{s(\varphi^i)}} = n_i$, $i = 2, 3, \ldots, p$, and there exist the elements $b_j^i, b_j'^i$, $i = 1, 2, \ldots, p-1$, $j = 1, 2, \ldots, m_i$, $v_j^i, v_j'^i$, $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, n_i$, in $A$ such that

(1) $\langle b_j^i, b_j'^i \rangle \in U \cup D$ for $i = 1, 2, \ldots, p-1$, $j = 1, 2, \ldots, m_i$,

(2) $\langle v_j^i, v_j'^i \rangle \in U \cup D$ for $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, n_i$,

(3) $a^1 = \tau_A^1(b_1^1, b_2^1, \ldots, b_{m_1}^1)$,

(4) $a^i = \varphi_A^i(v_1^i, v_2^i, \ldots, v_{n_i}^i) = \tau_A^i(b_1^i, b_2^i, \ldots, b_{m_i}^i)$ for $i = 2, 3, \ldots, p-1$,

(5) $a^p = \varphi_A^p(v_1^p, v_2^p, \ldots, v_{n_p}^p)$,

(6) $\varphi_A^{i+1}(v_1'^{i+1}, v_2'^{i+1}, \ldots, v_{n_{i+1}}'^{i+1}) = \tau_A^i(b_1'^i, b_2'^i, \ldots, b_{m_i}'^i)$ for $i = 1, 2, \ldots, p-1$.

Let $[a^1, a^2, \ldots, a^p]$ be a $U$-chain in $\boldsymbol{A}$. The elements $a^1$ and $a^p$ are called *the first and the last element* of that $U$-chain. Always is $p \geqslant 2$.

It is easy to verify that

(2.1) *$[a, a]$ for all $a$ in $A$, is a $U$-chain in $\boldsymbol{A}$.*

(2.2) *If* $[a^1, a^2, \ldots, a^p]$ *is a* $U$*-chain in* $A$, *then also* $[a^p, a^{p-1}, \ldots, a^1]$ *is a* $U$*-chain in* $A$.

(2.3) *If* $[a^1, a^2, \ldots, a^p]$ *and* $[a^p, a^{p+1}, \ldots, a^q]$ *are the* $U$*-chains in* $A$, *then* $[a^1, a^2, \ldots, a^p, a^{p+1}, \ldots, a^q]$ *is an* $U$*-chain in* $A$.

(2.4) *If* $[a_j^1, a_j^2, \ldots, a_j^p]$ *for* $j = 1, 2, \ldots, k_i$, *are* $U$*-chains in* $A$, *then the sequence*

$$[F_i(a_1^1, a_2^1, \ldots, a_{k_i}^1), F_i(a_1^2, a_2^2, \ldots, a_{k_i}^2), \ldots, F_i(a_1^p, a_2^p, \ldots, a_{k_i}^p)]$$

*is a* $U$*-chain in* $A$.

(2.5) *If* $[a, \ldots, c, \ldots, b]$ *is a* $U$*-chain in* $A$, *then* $[a, a, \ldots, c, c, \ldots, b, b]$ *is also a* $U$*-chain in* $A$.

(2.6) *If* $\langle a, b\rangle \in U$, *then* $[a, b]$ *is a* $U$*-chain in* $A$.

The next theorem gives the form of congruence in $A$ generated by set $U \subset A \times A$.

(2.7) *The relation* $\sim$ *in* $A$ *such that for all* $a$ *and* $b$ *in* $A$ $a \sim b$ *if and only if there exists a* $U$*-chain in* $A$ *with first element* $a$ *and the last element* $b$, *is the congruence in* $A$ *generated by* $U$.

Proof. From (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) it follows that $\sim$ is a congruence in $A$ such that $\langle a, b\rangle \in U$ implies $a \sim b$. Let $\sim_1$ be any congruence in $A$ such that $\langle a, b\rangle \in U$ implies $a \sim_1 b$. Suppose that $a \sim b$. Then there exists a $U$-chain $[a^1, a^2, \ldots, a^p]$ in $A$ with $a = a^1$ and $b = a^p$. From the definition of $U$-chain it follows that $a^1 \sim_1 a^2 \sim_1 \ldots \sim_1 a^p$, or $a \sim_1 b$. Thus $\sim$ is the congruence in $A$ generated by $U$ and the theorem is proved.

(2.8) *If* $\mathbf{B}$ *is a subalgebra of* $A$ *and* $U$ *is a subset of* $B \times B \subset A \times A$ *and* $\sim_1$ *and* $\sim$ *are the congruences in* $\mathbf{B}$ *and* $A$ *generated by* $U$, *then* $\sim_1 \leqslant \sim$.

Proof. This follows easily from (2.7).

**§ 3. $\psi(x, y)$-congruences and $\psi(x, y)$-normal sets.** Let $A = \langle A, F_1, F_2, \ldots, F_n\rangle$ be an algebra in $\mathfrak{A}$ and let $\psi(x, y)$ be arbitrary $\mathfrak{A}$-term with two different variables $x$ and $y$.

(3.1) *A congruence* $\sim$ *in* $A$ *is called a* $\psi(x, y)$*-congruence in* $A$ *if it has the following property*:

(*) *there exists an element* $c$ *in* $A$ *such that for all* $a$ *and* $b$ *in* $A$, $a \sim b$ *if and only if* $\psi_A(a, b) \sim \psi_A(c, c)$.

As a simple consequence of (3.1) we obtain

(3.2) *If* $\sim$ *is a* $\psi(x, y)$*-congruence in* $A$, *then for all* $d \in A$ *and for all* $a$, $b \in A$, $a \sim b$ *if and only if* $\psi_A(a, b) \sim \psi_A(d, d)$.

Proof. By (3.1) the congruence $\sim$ has the property (*). Let $d$ be an arbitrary element in $A$. Since $d \sim d$, we have, by (*), $\psi_A(d, d) \sim \psi_A(c, c)$.



Hence if $\psi_A(a, b) \sim \psi_A(d, d)$, then $\psi_A(a, b) \sim \psi_A(c, c)$ and therefore, by (*), $a \sim b$. Since $\psi_A(d, d) \sim \psi_A(c, c)$ for all $d \in A$, $\psi_A(d, d) \sim \psi_A(b, b)$ for all $d$ and $b$ in $A$. Let $a \sim b$. By the theorem (1.1) the conditions $a \sim b$ and $b \sim b$ imply $\psi_A(a, b) \sim \psi_A(b, b)$. Thus $a \sim b$ implies $\psi_A(a, b) \sim \psi_A(d, d)$ and the theorem is proved.

From (3.1) and (3.2) the following theorem results

(3.3) *A congruence* $\sim$ *in* $A$ *is* $\psi(x, y)$*-congruence in* $A$ *if and only if it has the following property*:

(**) *for all* $d$, $a$ *and* $b$ *in* $A$, $a \sim b$ *if and only if* $\psi_A(a, b) \sim \psi_A(d, d)$.

Proof. If the condition (**) holds, then also the condition (*) holds and therefore by (3.1) $\sim$ is a $\psi(x, y)$-congruence in $A$. If $\sim$ is a $\psi(x, y)$-congruence in $A$ then by (3.2) the congruence $\sim$ has the property (**). This finishes our proof.

Other characterization of $\psi(x, y)$-congruence is given in the next theorem.

(3.4) *A congruence* $\sim$ *in* $A$ *is* $\psi(x, y)$*-congruence in* $A$ *if and only if it has the following property*:

$({}^{*}_{*})$ *there exists an abstraction class* $N$ *of* $\sim$ *such that for all* $a$ *and* $b$ *in* $A$, $a \sim b$ *if and only if* $\psi_A(a, b) \in N$.

Proof. Suppose that $\sim$ is a $\psi(x, y)$-congruence in $A$. By (3.2) $\psi_A(c, c)/\sim\ = \psi_A(d, d)/\sim$ for all $c, d \in A$. Let put $N = \psi_A(c, c)/\sim$. By (3.2) for all $a, b \in A$, $a \sim b$ if and only if $\psi_A(a, b) \sim \psi_A(c, c)$ or $a \sim b$ if and only if $\psi_A(a, b) \in N$. Therefore the congruence $\sim$ has the property $({}^{*}_{*})$ for $N = \psi_A(c, c)/\sim$ and the necessity of $({}^{*}_{*})$ is proved. Now we shall prove the sufficiency of $({}^{*}_{*})$. Suppose that a congruence $\sim$ in $A$ has the property $({}^{*}_{*})$. Always is $d \sim d$. Hence by $({}^{*}_{*})$, $\psi_A(d, d) \in N$ for all $d \in A$. Since $N$ is an abstraction class of $\sim$ and $\psi_A(d, d) \in N$, $N = \psi_A(d, d)/\sim$, for $d \in A$. Therefore, by $({}^{*}_{*})$, $a \sim b$ if and only if $\psi_A(a, b) \sim \psi_A(d, d)$ or the congruence $\sim$ is a $\psi(x, y)$-congruence in $A$ and the theorem is proved. From the proof of the theorem (3.4) it follows that for every $\psi(x, y)$-congruence $\sim$ in $A$ there exists one and only one abstraction class $N$ of $\sim$ which fulfils the condition $({}^{*}_{*})$. That class $N$ is determined by any element $\psi_A(c, c)$, for $c \in A$. Therefore the subsets of $A$ which are the abstraction classes of $\psi(x, y)$-congruences in $A$ determined by the elements $\psi_A(c, c)$, are called $\psi(x, y)$*-normal in* $A$. Thus $N \subset A$ is a $\psi(x, y)$-normal set in $A$ if $N = \psi_A(c, c)/\sim$, where $c \in A$ and $\sim$ is a $\psi(x, y)$-congruence in $A$. Let $N$ be a $\psi(x, y)$-normal set in $A$. Then the relation $\sim$ in $A$ such that $a \sim b$ if and only if $\psi_A(a, b) \in N$, is a $\psi(x, y)$-congruence in $A$. It is called *induced by* $N$. The algebra $A/\sim$ will be denoted also by $A/N$ and the abstraction class $a/\sim$ by $a/N$. The correspondence of the

$\psi(x,y)$-congruences to $\psi(x,y)$-normal sets is one-to-one. From the properties of $\psi(x,y)$-congruence the following theorem results:

(3.5) *A set $N\subset A$ is $\psi(x,y)$-normal in $A$ if and only if it has the following properties:*

(3.5.1) $\psi_A(a,a)\in N$ *for all $a$ in $A$,*

(3.5.2) $\psi_A(a,b)\in N$ *for all $a$ and $b$ in $N$,*

(3.5.3) *for all $b\in N$ and for all $a\in A$, if $\psi_A(a,\psi_A(b,b))\in N$, then $a$ belongs to $N$,*

(3.5.4) *for all $a,b\in A$ if $\psi_A(a,b)\in N$, then $\psi_A(b,a)\in N$,*

(3.5.5) *for all $a,b,c\in A$ if $\psi_A(a,b)\in N$ and $\psi_A(b,c)\in N$, then $\psi_A(a,c)\in N$,*

(3.5.6) *for all $i=1,2,\ldots,n$ and for all $a_1,a_2,\ldots,a_{k_i}$, $b_1,b_2,\ldots,b_{k_i}$ in $A$ if $\psi_A(a_j,b_j)\in N$, for $j=1,2,\ldots,k_i$, then*

$$\psi_A\big(F_i(a_1,a_2,\ldots,a_{k_i}),F_i(b_1,b_2,\ldots,b_{k_i})\big)\in N\,.$$

Proof. If $N$ has those properties, then the relation $\sim$ in $A$ such that $a\sim b$ if and only if $\psi_A(a,b)\in N$, is by (3.5.1), (3.5.4), (3.5.5) and (3.5.6) a congruence in $\boldsymbol{A}$. By (3.5.2) and (3.5.3) $N$ is an abstraction class of $\sim$ and therefore by (3.4) $\sim$ is a $\psi(x,y)$-congruence in $\boldsymbol{A}$. If $N=\psi_A(c,c)/\sim$, where $\sim$ is a $\psi(x,y)$-congruence in $\boldsymbol{A}$, then obviously $N$ has the properties (3.5.1)-(3.5.6). This finishes our proof.

It is easy to prove that the $\psi(x,y)$-normal sets have the following properties which are analogical to properties of normal subgroups in groups:

(3.6) *The intersection of arbitrary family of $\psi(x,y)$-normal sets in $\boldsymbol{A}$ is also a $\psi(x,y)$-normal set in $\boldsymbol{A}$.*

(3.7) *If $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $N$ is a $\psi(x,y)$-normal set in the algebra $\boldsymbol{A}_1=\{N\cup B\}_{\boldsymbol{A}}$, then $N_1=N\cap B$ is $\psi(x,y)$-normal set in the algebra $\boldsymbol{B}$ and the algebras $\boldsymbol{A}_1/N$ and $\boldsymbol{B}/N_1$ are isomorphic.*

(3.8) *If $h$ is a homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ and $N$ is a $\psi(x,y)$-normal set in $\boldsymbol{B}$, then the set $h^{-1}(N)$ is $\psi(x,y)$-normal in $\boldsymbol{A}$.*

(3.9) *Let $N$ be a $\psi(x,y)$-normal set in $\boldsymbol{A}$. Then the natural homomorphism $h(a)=a/N$ of $\boldsymbol{A}$ onto $\boldsymbol{A}/N$ mapps one-to-one the $\psi(x,y)$-normal sets in $\boldsymbol{A}$ containing $N$ onto the $\psi(x,y)$-normal sets in $\boldsymbol{A}/N$.*

(3.10) *Let $N$ be a $\psi(x,y)$-normal set in $\boldsymbol{A}$ and let $h(a)=a/N$ be the natural homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}=\boldsymbol{A}/N$. If $V$ is a $\psi(x,y)$-normal set in $\boldsymbol{B}$, then $M=h^{-1}(V)$ is $\psi(x,y)$-normal in $\boldsymbol{A}$ and the algebras $\boldsymbol{B}/V$ and $\boldsymbol{A}/M$ are isomorphic.*



Let $h$ be a homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ and moreover let $\sim$ be a congruence in $\boldsymbol{B}$. Then the relation $\sim^{-1}$ in $\boldsymbol{A}$ such that

$$a_1\sim^{-1}a_2 \quad \text{if and only if} \quad h(a_1)\sim h(a_2)\,,$$

is a congruence in $\boldsymbol{A}$.

The congruence $\sim^{-1}$ is called the *$h^{-1}$-image of* $\sim$. Now prove the following theorem:

(3.11) *Let $h$ be a homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$ and moreover let $\sim$ be a congruence in $\boldsymbol{B}$. Then $\sim$ is a $\psi(x,y)$-congruence in $\boldsymbol{B}$ if and only if $\sim^{-1}$ is a $\psi(x,y)$-congruence in $\boldsymbol{A}$, where $\sim^{-1}$ is the $h^{-1}$-image of $\sim$.*

Proof. If $\sim$ is a $\psi(x,y)$-congruence in $\boldsymbol{B}$, then from (3.8) it follows that $\sim^{-1}$ is a $\psi(x,y)$-congruence in $\boldsymbol{A}$. Now suppose that $\sim^{-1}$ is a $\psi(x,y)$-congruence in $\boldsymbol{A}$. Hence by (3.1) $\sim^{-1}$ has the following property:

(k) *there exists $c\in A$ such that for all $a_1$ and $a_2$ in $A$*

$$a_1\sim^{-1}a_2 \text{ if and only if } \psi_A(a_1,a_2)\sim^{-1}\psi_A(c,c)\,.$$

We shall prove that for all $b_1$ and $b_2$ in $B$ we have

(m) $$b_1\sim b_2 \text{ if and only if } \psi_B(b_1,b_2)\sim\psi_B\big(h(c),h(c)\big)\,.$$

In fact, let $\psi_B(b_1,b_2)\sim\psi_B\big(h(c),h(c)\big)$ and let $b_1=h(a_1)$, $b_2=h(a_2)$, where $a_1,a_2\in A$. Therefore $\psi_B\big(h(a_1),h(a_2)\big)\sim\psi_B\big(h(c),h(c)\big)$. Hence by (1.2) we obtain $h\big(\psi_A(a_1,a_2)\big)\sim h\big(\psi_A(c,c)\big)$ and thus, by the definition of $\sim^{-1}$, $\psi_A(a_1,a_2)\sim^{-1}\psi_A(c,c)$. Hence by (k) we have $a_1\sim^{-1}a_2$ and therefore, by the definition of $\sim^{-1}$, $h(a_1)\sim h(a_2)$ or $b_1\sim b_2$. Conversely, let $b_1\sim b_2$ and let $b_1=h(a_1)$, $b_2=h(a_2)$, where $a_1,a_2\in A$. Therefore $h(a_1)\sim h(a_2)$. Hence, by the definition of $\sim^{-1}$, we have $a_1\sim^{-1}a_2$ and thus, by (k), we obtain $\psi_A(a_1,a_2)\sim^{-1}\psi_A(c,c)$ or $h\big(\psi_A(a_1,a_2)\big)\sim h\big(\psi_A(c,c)\big)$. Hence, by (1.2), $\psi_B\big(h(a_1),h(a_2)\big)\sim\psi_B\big(h(c),h(c)\big)$ or $\psi_B(b_1,b_2)\sim\psi_B\big(h(c),h(c)\big)$ and the lemma (m) is proved. From (m) and (3.1) it follows that $\sim$ is a $\psi(x,y)$-congruence in $\boldsymbol{B}$ and the theorem (3.11) is proved.

Let $h$ be a homomorphism of an algebra $\boldsymbol{A}$ onto $\boldsymbol{B}$. The relation $\sim_h$ in $A$ such that for all $a$ and $b$ in $A$

$$a\sim_h b \quad \text{if and only if} \quad h(a)=h(b)$$

is a congruence in $\boldsymbol{A}$. It is called *induced by $h$*. Obviously the algebra $\boldsymbol{B}$ is isomorphic to $\boldsymbol{A}/\sim_h$. But it is not always that there exists a $\psi(x,y)$-normal set $N$ in $\boldsymbol{A}$ such that $\boldsymbol{B}$ is isomorphic to $\boldsymbol{A}/N$. We have only the following obvious theorem:

(3.12) *If $h$ is a homomorphism of $A$ onto $B$ and the congruence $\sim_h$ induced by $h$ is a $\psi(x, y)$-congruence in $A$, then the algebra $B$ is isomorphic to $A/N$, where $N$ is the $\psi(x, y)$-normal set in $A$ such that $N = \psi_A(c, c)/\sim_h$ with $c \in A$.*

In the next paragraph we shall give the necessary and sufficient conditions for to be each congruence in $A$ a $\psi(x, y)$-congruence. If $A$ fulfils those conditions, then each homomorphic image of $A$ is isomorphic to $A/N$, where $N$ is some $\psi(x, y)$-normal set in $A$.

**§ 4. On the determining of the form of congruences by terms.** Let $A = \langle A, F_1, F_2, \ldots, F_n \rangle$ be an algebra in $\mathfrak{A}$ and let $\psi(x, y)$ be an arbitrary $\mathfrak{A}$-term with two different variables $x$ and $y$.

(4.1) *We shall say that the term $\psi(x, y)$ determines the form of congruences in $A$ if every congruence in $A$ is a $\psi(x, y)$-congruence in $A$.*

From (3.11) the following theorem results:

THEOREM 1. *If the term $\psi(x, y)$ determines the form of congruences in $A$ and $h$ is a homomorphism of $A$ onto $B$, then the term $\psi(x, y)$ determines the form of congruences in $B$.*

Proof. Let $\sim$ be an arbitrary congruence in $B$ and let $\sim^{-1}$ be the $h^{-1}$-image of $\sim$. $\sim^{-1}$ is a congruence in $A$. Hence by the hypothesis of theorem $\sim^{-1}$ is a $\psi(x, y)$-congruence in $A$. Thus by the theorem (3.11) $\sim$ is a $\psi(x, y)$-congruence in $B$ and the theorem 1 is proved.

If the equation $\ulcorner\psi(x, x) = \psi(y, y)\urcorner$ is valid in $A$, i. e. if for all $a$ and $b$ in $A$ we have $\psi_A(a, a) = \psi_A(b, b)$, then we say that this equation defines a *constant element in $A$*. If the equation $\ulcorner\psi(x, x) = \psi(y, y)\urcorner$ is valid in every algebra $A$ belonging to a class $\mathfrak{A}_0 \subset \mathfrak{A}$, then we say that this equation defines a *constant element in class $\mathfrak{A}_0$*. In order that the form of congruences in $A$ be determined by the term $\psi(x, y)$ it is necessary but not sufficient that the equation $\ulcorner\psi(x, x) = \psi(y, y)\urcorner$ defines a constant element in $A$. This follows from the next theorem:

THEOREM 2. *The term $\psi(x, y)$ determines the form of congruences in algebra $A$ if and only if the following conditions are satisfied:*

(4.2) $\psi_A(a, a) = \psi_A(b, b)$ *for all $a$ and $b$ in $A$,*

(4.3) $a \sim^* b$ *for all $a$ and $b$ in $A$, where $\sim^*$ is the congruence in $A$ generated by the set $\big(\langle\psi_A(a, b), \psi_A(a, a)\rangle\big) \subset A \times A$.*

Proof. Suppose that the conditions (4.2) and (4.3) are satisfied. Let $\sim$ be any congruence in $A$ and let $c \in A$. By the theorem (1.1) the conditions $a \sim b$ and $b \sim b$ imply $\psi_A(a, b) \sim \psi_A(b, b)$ or $a \sim b$ implies $\psi_A(a, b) \sim \psi_A(b, b)$. But, by (4.2), $\psi_A(b, b) = \psi_A(c, c)$. Hence $a \sim b$ implies $\psi_A(a, b) \sim \psi_A(c, c)$. Now let $\psi_A(a, b) \sim \psi_A(c, c)$. Then, by (4.2), $\psi_A(a, b) \sim \psi_A(a, a)$. By (4.3) $a \sim^* b$. Since $\sim^* \leqslant \sim$ and $a \sim^* b$, we have $a \sim b$ or the condition $\psi_A(a, b) \sim \psi_A(c, c)$ implies $a \sim b$. Therefore $\sim$ is a $\psi(x, y)$-congruence in $A$. Thus the sufficency of the conditions (4.2) and (4.3) is proved. Now we shall prove the necessity of those conditions. Suppose that the term $\psi(x, y)$ determines the form of congruences in $A$. The relation of identity $=$ in $A$ is a $\psi(x, y)$-congruence in $A$. Hence it follows (4.2). Let $a, b \in A$ and let $\sim^*$ be the congruence in $A$ generated by the set $\big(\langle\psi_A(a, b), \psi_A(a, a)\rangle\big) \subset A \times A$. By the supposition $\sim^*$ is a $\psi(x, y)$-congruence in $A$. Since $\psi_A(a, b) \sim^* \psi_A(a, a)$, by (3.2) we obtain $a \sim^* b$ and therefore the condition (4.3) is also satisfied. Thus the theorem 2 is proved. From the theorem 2 and the theorem (2.7) the following theorem results.



THEOREM 3. *The term $\psi(x, y)$ determines the form of congruences in $A$ if and only if the following conditions are satisfied:*

(4.4) $\psi_A(a, a) = \psi_A(b, b)$ *for all $a$ and $b$ in $A$,*

(4.5) *for all $a, b \in A$ there exists a $\big(\langle\psi_A(a, b), \psi_A(a, a)\rangle\big)$-chain in $A$ with the first element $a$ and the last element $b$.*

Proof. This follows from the theorems 2 and (2.7).

As a simple consequence of the theorem 3 we obtain the next.

THEOREM 4. *If the term $\psi(x, y)$ determines the form of congruences in every subalgebra of $A$ generated by two different elements, then the term $\psi(x, y)$ determines the form of congruences in $A$.*

Proof. Let $a, b \in A$. By the supposition the term $\psi(x, y)$ determines the form of congruences in $\{(a, b)\}_A$. Hence, by theorem 2, we have $\psi_A(a, a) = \psi_A(b, b)$ and $a \sim_1^* b$, where $\sim_1^*$ is the congruence in $\{(a, b)\}_A$ generated by the set $\big(\langle\psi_A(a, b), \psi_A(a, a)\rangle\big)$. Let $\sim^*$ be the congruence in $A$ generated also by that set. By the theorem (2.8) we have $\sim_1^* \leqslant \sim^*$. Hence since $a \sim_1^* b$, also $a \sim^* b$. Therefore the conditions (4.2) and (4.3) are satisfied by $A$. Thus by the theorem 2 the term $\psi(x, y)$ determines the form of congruences in $A$ and the theorem 4 is proved.

From the theorem 4 the following theorem immediately results:

THEOREM 5. *If $\mathfrak{m} \geqslant 2$ is any cardinal number and the term $\psi(x, y)$ determines the form of congruences in every subalgebra $\{B\}_A$ of $A$ with $2 \leqslant \overline{\overline{B}} \leqslant \mathfrak{m}$, then the term $\psi(x, y)$ determines the form of congruences in $A$.*

Proof. The hypothesis of the theorem 5 implies one of the theorem 4. Hence by theorem 4 we obtain the theorem 5.

If the term $\psi(x, y)$ determines the form of congruences in every algebra in $\mathfrak{A}_0 \subset \mathfrak{A}$, then we say that the *term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$*.

THEOREM 6. *If $A$ is a free algebra for $\mathfrak{A}_0 \subset \mathfrak{A}$ freely generated by a set $C$ with $\overline{\overline{C}} \geqslant 2$ and the term $\psi(x, y)$ determines the form of congruences in $A$, then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$.*

Proof. Let $D$ be any algebra in $\mathfrak{A}_0$. Moreover let $\{B\}_D$ be an arbitrary, subalgebra of $D$ generated by a set $B$ with $\overline{\overline{B}} \leqslant \overline{\overline{C}} = \mathfrak{m}$. Since $A$ is a free algebra for $\mathfrak{A}_0$ freely generated by $C$, $\{B\}_D$ is a homomorphic image of $A$. Hence, by the theorem 1, the term $\psi(x, y)$ determines the form of congruences in $\{B\}_D$. Thus, by the theorem 5, the term $\psi(x, y)$ determines the form of congruences in $D$ and the theorem is proved.

From the theorem 6 results

THEOREM 7. *Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be a class which has an $\mathfrak{A}_0$-free algebra $A$ freely generated by a set $C$ with $\overline{\overline{C}} \geqslant 2$. Then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$ if and only if the term $\psi(x, y)$ determines the form of congruences in $A$.*

Proof. If the term $\psi(x, y)$ determines the form of congruences in $A$ then, by the theorem 6, the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$. Conversely, if the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$, then since $A \in \mathfrak{A}_0$, by the definition, the term $\psi(x, y)$ determines the form of congruences in $A$. This finishes our proof. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be an equationally definable class. Then by the theorem (1.3) there exist $\mathfrak{A}_0$-free algebras freely generated by the sets of arbitrary power. Hence and from the theorem 7 the next theorem results:

THEOREM 8. *Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be an equationally definable class. Then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$ if and only if there exists an $\mathfrak{A}_0$-free algebra $A$ freely generated by $C$ with $\overline{\overline{C}} \geqslant 2$ such that the term $\psi(x, y)$ determines the form of congruences in $A$.*

Proof. This follows from the theorem 7 and (1.3).

By $P$ will be denoted the set of all pairs $\langle \vartheta, \vartheta \rangle$, where $\vartheta$ is an $\mathfrak{A}$-term. Let $\mathfrak{A}_0$ be any subclass of $\mathfrak{A}$.

(4.6) We shall say that *the class $\mathfrak{A}_0$ fulfils the condition* (S) if it has the following properties:

(S$_1$) *the equation $\ulcorner \psi(x, x) = \psi(y, y) \urcorner$ belongs to $E_{\mathfrak{A}}(\mathfrak{A}_0)$;*

(S$_2$) *there exists a natural number $p$ and there exist the terms $\tau^i$ with $\overline{\overline{s(\tau^i)}} = m_i$, $i = 1, 2, \ldots, p-1$, the terms $\varphi^i$ with $\overline{\overline{s(\varphi^i)}} = n_i$, $i = 2, 3, \ldots, p$, the terms $\vartheta^i_j, \vartheta'^i_j$, $i = 1, 2, \ldots, p-1$, $j = 1, 2, \ldots, m_i$, and the terms $\delta^i_j, \delta'^i_j$, $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, n_i$, such that*

1° $\langle \vartheta^i_j, \vartheta'^i_j \rangle \in \big(\langle \psi(x, y), \psi(x, x) \rangle\big) \cup P$, *for* $i = 1, 2, \ldots, p-1$, $j = 1, 2, \ldots, m_i$,

2° $\langle \delta^i_j, \delta'^i_j \rangle \in \big(\langle \psi(x, y), \psi(x, x) \rangle\big) \cup P$, *for* $i = 2, 3, \ldots, p$, $j = 1, 2, \ldots, n_i$,



3° *the following equations belong to $E_{\mathfrak{A}}(\mathfrak{A}_0)$:*

(a$_1$) $$\ulcorner x = \tau^1(\vartheta^1_1, \vartheta^1_2, \ldots, \vartheta^1_{m_1}) \urcorner,$$

(a$_2$) $$\ulcorner \varphi^i(\delta^i_1, \delta^i_2, \ldots, \delta^i_{n_i}) = \tau^i(\vartheta^i_1, \vartheta^i_2, \ldots, \vartheta^i_{m_i}) \urcorner, \text{ for } i = 2, 3, \ldots, p-1,$$

(a$_3$) $$\ulcorner y = \varphi^p(\delta^p_1, \delta^p_2, \ldots, \delta^p_{n_p}) \urcorner,$$

(a$_4$) $$\ulcorner \varphi^{i+1}(\delta'^{i+1}_1, \delta'^{i+1}_2, \ldots, \delta'^{i+1}_{n_{i+1}}) = \tau^i(\vartheta'^i_1, \vartheta'^i_2, \ldots, \vartheta'^i_{m_i}) \urcorner, \text{ for } i = 1, 2, \ldots, p-1.$$

Now we prove the following theorems.

THEOREM 9. *If a class $\mathfrak{A}_0 \subset \mathfrak{A}$ fulfils the condition* (S), *then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$.*

Proof. Let $A$ be any algebra in $\mathfrak{A}_0$. From (S$_1$) it follows that $\psi_A(a, a) = \psi_A(b, b)$, for $a, b \in A$, or the condition (4.4) is fulfilled by $A$. Let now $a$ and $b$ be arbitrary elements in $A$ and let $h$ be any homomorphism of $W$ into $A$ with $h(x) = a$ and $h(y) = b$ ($W$ is the algebra of all $\mathfrak{A}$-terms). From (a$_1$), (a$_2$), (a$_3$) and (a$_4$) it follows that the sequence

$$\big[h(x), h\big(\varphi^2(\delta^2_1, \delta^2_2, \ldots, \delta^2_{n_2})\big), \ldots, h\big(\varphi^{p-1}(\delta^{p-1}_1, \delta^{p-1}_2, \ldots, \delta^{p-1}_{n_{p-1}})\big), h(y)\big]$$

is a $\big(\langle \psi_A(a, b), \psi_A(a, a) \rangle\big)$-chain in $A$ with the first element $a = h(x)$ and the last element $b = h(y)$. Thus the condition (4.5) is fulfilled by $A$. Therefore, by the theorem 3, the term $\psi(x, y)$ determines the form of congruences in $A$ and the theorem 9 is proved.

THEOREM 10. *Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be a class which has an $\mathfrak{A}_0$-free algebra $A$ freely generated by a set $C$ with $\overline{\overline{C}} \geqslant 2$. Then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$ if and only if $\mathfrak{A}_0$ fulfils the condition* (S).

Proof. The suffiency follows from the theorem 9. Suppose that the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}_0$. Since $A \in \mathfrak{A}_0$, the term $\psi(x, y)$ determines the form of congruences in $A$. Let $c$ and $d$ be any different elements in $C$. By the theorem 3 the conditions (4.4) and (4.5) are fulfilled by $A$. Hence by (4.4) we have $\psi_A(c, c) = \psi_A(d, d)$. Since $A$ is $\mathfrak{A}_0$-free algebra freely generated by $C$ and $c, d$ are two different element in $C$, from the theorems (1.4) and (1.5) it follows that the equation $\ulcorner \psi(x, x) = \psi(y, y) \urcorner$ belongs to $E_{\mathfrak{A}}(\mathfrak{A}_0)$ and thus $\mathfrak{A}_0$ has the property (S$_1$). Similarly we prove that $\mathfrak{A}_0$ has the property (S$_2$). In fact, the condition (4.5) is fulfilled by $A$. Therefore there exists a $\big(\langle \psi_A(c, d), \psi_A(c, c) \rangle\big)$-chain in $A$ with the first element $c$ and the last element $d$. From the definition of $U$-chain and from the properties of $\mathfrak{A}_0$-free algebras given in the theorems (1.4) and (1.5) and from the theorem (1.6) it follows that $\mathfrak{A}_0$ has the property (S$_2$) or $\mathfrak{A}_0$ fulfils the condition (S). Thus the theorem 10 is proved.

Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be an equationally definable class. Then by (1.3) there exist the $\mathfrak{A}_0$-free algebras freely generated by the set of arbitrary power. Hence by the theorem 10 we obtain the following

THEOREM 11. *The term $\psi(x, y)$ determines the form of congruences in an equationally definable class $\mathfrak{A}_0 \subset \mathfrak{A}$ if and only if the class $\mathfrak{A}_0$ fulfils the condition* (S).

Proof. This follows from (1.3) and the theorem 10.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be the classes of all algebras of the type $\Delta = \langle k_1, k_2, \ldots, k_n\rangle$ and $\Delta' = \langle k_1, k_2, \ldots, k_n, k_{n+1}, \ldots, k_m\rangle$ where $m \geqslant n$. Let $\boldsymbol{A}' = \langle A', F_1, F_2, \ldots, F_n, F_{n+1}, \ldots, F_m\rangle$ be an algebra in $\mathfrak{A}'$. The algebra $\langle A', F_1, F_2, \ldots, F_n\rangle$ will be denoted by $\boldsymbol{A}'|\Delta$ and it belongs to $\mathfrak{A}$. Let $\mathfrak{A}'_0 \subset \mathfrak{A}'$. The class of all algebras $\boldsymbol{A}'|\Delta$ with $\boldsymbol{A}' \in \mathfrak{A}'_0$ will be denoted by $\mathfrak{A}'_0|\Delta$. Let $\psi(x, y)$ be an $\mathfrak{A}$-term with two different variables, it is also an $\mathfrak{A}'$-term.

Now we prove the following theorem:

THEOREM 12. *If the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}'_0|\Delta$, then the term $\psi(x, y)$ determines the form of congruences in $\mathfrak{A}'_0$.*

Proof. Let $\boldsymbol{A}' \in \mathfrak{A}'_0$ and let $\sim$ be an arbitrary congruences in $\boldsymbol{A}'$. Then $\sim$ is a congruence in $\boldsymbol{A}'|\Delta$. Hence by the hypothesis of our theorem $\sim$ is a $\psi(x, y)$-congruence in $\boldsymbol{A}'|\Delta$. From the definition of $\psi(x, y)$-congruence it follows that $\sim$ is also a $\psi(x, y)$-congruence in $\boldsymbol{A}'$ and the theorem is proved.

**§ 5. Applications.** Now we shall see as the condition (S) is realized in the equationally definable classes mentioned in introduction of this paper.

I. Groups. Let $\mathfrak{A}$ be the class of all algebras of the type $\langle 2, 1\rangle$ with the operation signs $\cdot$ and $^{-1}$. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be the class of all groups.

(5.1) *The term $x \cdot y^{-1}$ determines the form of congruences in the class of all groups $\mathfrak{A}_0$.*

Proof. As we know to $E_{\mathfrak{A}}(\mathfrak{A}_0)$ belong the following equations:

$$(\mathrm{S}_1^*) \qquad \ulcorner x \cdot x^{-1} = y \cdot y^{-1}\urcorner,$$

$$(\mathrm{a}_1^*) \qquad \ulcorner x = (x \cdot y^{-1}) \cdot y\urcorner,$$

$$(\mathrm{a}_3^*) \qquad \ulcorner y = y\urcorner,$$

$$(\mathrm{a}_4^*) \qquad \ulcorner y = (x \cdot x^{-1}) \cdot y\urcorner.$$

Hence it follows that $\mathfrak{A}_0$ fulfils the condition (S) for $p = 2$ and $\tau^1(x, y) = x \cdot y$, $\vartheta_1^1 = x \cdot y^{-1}$, $\vartheta_2^1 = y$, $\vartheta_1'^1 = x \cdot x^{-1}$, $\vartheta_2'^1 = y$ and $\varphi^2 = y$, $\delta_1^1 = y$, $\delta_1'^1 = y$. Thus by the theorem 11 we obtain (5.1). From (5.1) it follows that a subset $N$ of a group $\boldsymbol{G}$ is $x \cdot y^{-1}$-normal in $\boldsymbol{G}$ if and only if it is a normal subgroup of $\boldsymbol{G}$.



II. Rings. Let $\mathfrak{A}'$ be the class of all algebras of the type $\Delta' = \langle 2, 1, 2\rangle$ with the operation signs $+$, $-$ and $\cdot$. Let $\mathfrak{A}'_0 \subset \mathfrak{A}'$ be the class of all rings. As we know the algebras in class $\mathfrak{A}'_0|\langle 2, 1\rangle$ are groups with the operation signs $+$ and $-$. By the theorem (5.1) the term $x + (-y)$ determines the form of congruences in $\mathfrak{A}'_0|\langle 2, 1\rangle$. Hence by the theorem 12 we obtain the following theorem:

(5.2) *The term $x + (-y)$ determines the form of congruences in the class of all rings $\mathfrak{A}'_0$.*

III. Boolean algebras. Let $\mathfrak{A}$ be the class of all algebras of the type $\langle 2, 2, 1\rangle$ with the operation signs $\cup$, $\cap$, $'$. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be the class of all boolean algebras with zero element definable by an equation.

(5.3) *The term $(x \cap y') \cup (x' \cap y)$ determines the form of congruences in the class of all boolean algebras $\mathfrak{A}_0$.*

Proof. To $E_{\mathfrak{A}}(\mathfrak{A}_0)$ belong the following equations:

$$(\mathrm{S}_1') \qquad \ulcorner (x \cap x') \cup (x' \cap x) = (y \cap y') \cup (y' \cap y)\urcorner,$$

$$(\mathrm{a}_1') \qquad \ulcorner x = \{[(x \cap y') \cup (x' \cap y)] \cap y'\} \cup \{[(x \cap y') \cup (x' \cap y)]' \cap y\}\urcorner,$$

$$(\mathrm{a}_3') \qquad \ulcorner y = y\urcorner,$$

$$(\mathrm{a}_4') \qquad \ulcorner y = \{[(x \cap x') \cup (x' \cap x)] \cap y'\} \cup \{[(x \cap x') \cup (x' \cap x)]' \cap y\}\urcorner.$$

Hence it follows that $\mathfrak{A}_0$ fulfils the condition (S) for $p = 2$ and

$$\tau^1(x, y) = (x \cap y') \cup (x' \cap y), \quad \vartheta_1^1 = (x \cap y') \cup (x' \cap y), \ \vartheta_2^1 = y,$$
$$\vartheta_1'^1 = (x \cap x') \cup (x' \cap x), \ \vartheta_2'^1 = y, \quad \varphi^2 = y, \ \delta_1^2 = y, \ \delta_1'^2 = y.$$

Thus by the theorem 11 we obtain (5.3). From (5.3) it follows that a set $N$ is $[(x \cap y') \cup (x' \cap y)]$-normal in an boolean algebra $\boldsymbol{B}$ if and only if $N$ is an ideal in $\boldsymbol{B}$.

Now let $\mathfrak{A}_0 \subset \mathfrak{A}$ be the class of all boolean algebras with the unit element definable by an equation.

(5.4) *The term $(x \cap y) \cup (x' \cap y')$ determines the form of congruences in the class of all boolean algebras $\mathfrak{A}_{\mathfrak{c}}$.*

Proof. To $E_{\mathfrak{A}}(\mathfrak{A}_0)$ belong the following equations:

$$(\mathrm{S}_1'') \qquad \ulcorner (x \cap x) \cup (x' \cap x') = (y \cap y) \cup (y' \cap y')\urcorner,$$

$$(\mathrm{a}_1'') \qquad \ulcorner x = [(x \cap x) \cup (x' \cap x')] \cap x\urcorner,$$

$$(\mathrm{a}_2'') \qquad \ulcorner [(x \cap y) \cup (x' \cap y')] \cap x = [(x \cap y) \cup (x' \cap y')] \cap y\urcorner,$$

$$(\mathrm{a}_3'') \qquad \ulcorner y = [(x \cap x) \cup (x' \cap x')] \cap y\urcorner,$$

$$(\mathrm{a}_4'') \qquad \ulcorner [(x \cap x) \cup (x' \cap x')] \cap x = [(x \cap x) \cup (x' \cap x')] \cap x\urcorner$$

and

$$(a_5'') \qquad \ulcorner[(x \cap x) \cup (x' \cap x')] \cap y = [(x \cap x) \cup (x' \cap x')] \cap y\urcorner.$$

Hence it follows that $\mathfrak{A}_0$ fulfils the condition (S) for $p=3$ and for $\tau^1(x,y) = \tau^2(x,y) = x \cap y$, $\vartheta_1^1 = (x \cap x) \cup (x' \cup x')$, $\vartheta_2^1 = x$, $\vartheta_1'^1 = \vartheta_1^1$, $\vartheta_2'^1 = \vartheta_2^1$, $\vartheta_1^2 = (x \cap y) \cup (x' \cap y')$, $\vartheta_2^2 = y$, $\vartheta_1'^2 = (x \cap x) \cup (x' \cap x')$, $\vartheta_2'^2 = y$, $\varphi^2(x,y) = \varphi^3(x,y) = x \cap y$, $\delta_1^2 = (x \cap y) \cup (x' \cap y')$, $\delta_2^2 = x$, $\delta_1'^2 = (x \cap x) \cup (x' \cap x')$, $\delta_2'^2 = x$, $\delta_1^3 = (x \cap x) \cup (x' \cap x')$, $\delta_2^3 = y$, $\delta_1'^3 = \delta_1^3$, $\delta_2'^3 = y$.

Thus by the theorem 11 we obtain (5.4). Moreover we remark that from (5.4) it follows that a set $N$ is $(x \cap y) \cup (x' \cap y')$-normal in an boolean algebra $\boldsymbol{B}$ if and only if $N$ is a filter in $\boldsymbol{B}$.

IV. Now we shall give an example of an equationally definable class of algebras with one constant element definable by equation for which the form of congruences is not determined by any term. Let $\mathfrak{A}$ be the class of all algebras of the type $\langle 2,1\rangle$ with the operation signs $\cdot$ and $^{-1}$. Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be the class of all algebras in which the following equations are valid:

$$(1) \qquad \ulcorner x\cdot(y\cdot z) = (x\cdot y)\cdot z\urcorner,$$

$$(2) \qquad \ulcorner x\cdot x^{-1} = y\cdot y^{-1}\urcorner.$$

Let $\psi(x,y)$ be any $\mathfrak{A}$-term with two different variables $x$ and $y$. We have the following theorem:

(5.5) *There exists no term $\psi(x,y)$ which determines the form of congruences in $\mathfrak{A}_0$.*

**Proof.** It is easy to see that in $E_{\mathfrak{A}}(\mathfrak{A}_0)$ is not any equation of the form $\ulcorner x = \tau\urcorner$, where $\tau \neq x$. Hence and from the theorem 11 (see condition (S)) it follows the theorem (5.5).

V. Finally we give a simple sufficient condition for to be the form of congruences in equationally definable class $\mathfrak{A}_0 \subset \mathfrak{A}$ determined by the term $\psi(x,y)$.

(5.6) *Let $\mathfrak{A}_0 \subset \mathfrak{A}$ be an arbitrary equationally definable class of algebras which fulfils the following condition* (R):

(R) *to $E_{\mathfrak{A}}(\mathfrak{A}_0)$ belong the following equations*

$$(S_1^*) \qquad \ulcorner\psi(x,x) = \psi(y,y)\urcorner,$$

$$(a_1^*) \qquad \ulcorner x = \tau\big(\psi(x,y),y\big)\urcorner$$

*where $\tau$ is some $\mathfrak{A}$-term with two different variables. Then the term $\psi(x,y)$ determines the form of congruences in $\mathfrak{A}_0$.*

**Proof.** From (R) it follows that to $E_{\mathfrak{A}}(\mathfrak{A}_0)$ belong also the following equations:



$$(a_3^*) \qquad \ulcorner y = y\urcorner,$$

$$(a_4^*) \qquad \ulcorner y = \tau\big(\psi(x,x),y\big)\urcorner.$$

From $(S_1^*)$, $(a_1^*)$, $(a_3^*)$ and $(a_4^*)$ results that the condition (S) is fulfilled for $p=2$, $\tau^1 = \tau$, $\vartheta_1^1 = \psi(x,y)$, $\vartheta_2^1 = y$, $\vartheta_1'^1 = \psi(x,x)$, $\vartheta_2'^1 = y$ and $\varphi^2 = y$, $\delta_1^1 = \delta_1'^1 = y$. Thus by the theorem 11 the term $\psi(x,y)$ determines the form of congruences in the class $\mathfrak{A}_0$ and the theorem (5.6) is proved.

## References


[1] G. Birkhoff, *On the combination of subalgebras*, Proc. Cambridge Phil. Soc. 29 (1933), p. 441-464.

[2] — *On the structure of abstract algebras*, ibidem 31 (1935), p. 433-454.

[3] — *Lattice theory*, New York 1948.

[4] J. Słomiński, *The theory of abstract algebras with infinitary operations*, Rozprawy Matem. 18 (1959).



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