Hyperarithmetical quantifiers *

by

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Let HA be the class of hyperarithmetical functions, predicates and sets of natural numbers [4, 5]; let $(Ea)_{HA}P(a) \equiv (Ea)[a \epsilon HA \,\&\, P(a)]$; and let "$HA$" also be an abbreviation for "hyperarithmetical".

From the proof of XXVI [5], it follows that if R is a recursive predicate then there is a primitive recursive predicate P (obtained uniformly from a Gödel number of R) such that

$$(0.1) \qquad (Ea)_{HA}(x)R(a, x, a) \equiv (a)(Ex)P(a, x, a).$$

This result is used in [1] and is proved explicitly in [7]. In the latter paper, Kleene asks whether the converse is true; i. e. given a recursive P, can a primitive recursive R be found which satisfies (0.1)? We answer this question in the affirmative (1).

The method of proof involves an analysis of the inductive definitions of the set O of Church-Kleene ordinal notations and of the two-place predicates $|a| = |b|$ and $|a| < |b|$, where $|a|$ is the ordinal corresponding to a via O and both predicates are taken to be false if either a or b is not in O. The techniques developed by Kleene in [2] and amended in [6] play an important rôle. In particular we shall employ the predicate $(x)(Ey)R(a, x, y)$ defined in § 14 of [2], which Kleene abandons in the amended version [6] (2).

1. $C(b)$ **and** Q_0**.** For each natural number b let $C(b)$ be the set defined in [2] § 13 and $V(a, b, x)$ the primitive recursive predicate such that $a \epsilon C(b) \equiv (Ex)V(a, b, x)$. If $a = 3\cdot 5^{(a)_2}$, let $a_n = \Phi\big((a)_2, n_O\big)$, i. e. if $a \epsilon O$, then in a manner of speaking, $a_0, a_1, a_2, \ldots$ is the fundamental sequence whose limit is a. Let $\mathrm{Def}(a, n) \equiv [a_n$ is defined], more precisely $\mathrm{Def}(a, n) \equiv (Ey)\,T_1\big((a)_2, n_O, y\big)$.

* Research for this paper was conducted under a grant from the National Science Foundation, U. S. A.

(1) This answer appears as Corollary 2. Kreisel and Mostowski asked similar questions, which are answered by Corollary 1 and [6] Theorem 1.

(2) It is recommended that the reader be familiar with [2] through § 14 and [6] through § 20, and have both papers available for reference.

LEMMA 1 (Kleene). *If $b \in O$ then $C(b) = \hat{a}[a <_O b]$ and $C(b)$ is well-ordered by the "less-than" relation $x \in C(y)$ with ordinal $|b|$.* (See [2, 6], § 20, for proof.)

When $\doteq$ is an equivalence relation on a set S, we say that $\dot{<}$ is a *linear ordering* of S relative to $\doteq$ if [3]:

(L1) $x, y \in S \to x \doteq y \vee x \dot{<} y \vee y \dot{<} x$,
(L2) $x, y, z \in S$ & $x \dot{<} y$ & $y \dot{<} z \to x \dot{<} z$,
(L3) $x \in S \to$ not $x \dot{<} x$,
(L4) $u, v, x, y \in S$ & $x \doteq u$ & $y \doteq v \to [x \dot{<} y \equiv u \dot{<} v]$.

If $\doteq$ is $=$, then (L4) is automatically satisfied. Note that $x \in C(y)$ as $x \dot{<} y$, $=$ as $\doteq$, and $C(b)$ as S satisfy (L1)-(L3) if $b \in O$, and if $b \notin O$ then only (L2) need be satisfied ((L1) and (L3) are false for suitably chosen $C(b)$).

We shall define a set Q_0 by amending Kleene's definition of Q to require that $C(a)$ be linearly ordered when $a \in Q$. In place of Kleene's [2] (33) we write

$$a \in Q \equiv: a = 1 . \vee . a = 2^{(a)_0} \;\&\; (a)_0 \in Q . \vee .$$

$$a = 3 \cdot 5^{(a)_2} \;\&\; (n)[\mathrm{Def}(a, n) \;\&\; a_n \in Q \;\&\; a_n \in C(a_{n+1})] \;\& \tag{1.1}$$

$$[C(a) \text{ is linearly ordered by } x \in C(y) \text{ relative to } =].$$

We note that the clause $[C(a)$ is linearly ordered by $x \in C(y)$ relative to $=]$ does not contain the variable Q and can be written in the form $(x)(Ey)P(a, x, y)$ where P is primitive recursive, since the closures of (L1)-(L3) assume the respective forms $\forall[\exists \to \exists]$, $\forall[\exists \to \exists]$, $\forall[\exists \to \forall]$. Following the method of [2], § 14, we obtain a primitive recursive predicate R_0 such that the set $Q_0 = \hat{a}(x)(Ey)R_0(a, x, y)$ is a solution to (1.1). Kleene [6] shows that O is the smallest set Q satisfying (1.1); Q_0 can be characterized as follows:

LEMMA 2. *Q_0 is the largest set Q satisfying* (1.1).

Proof. By the remarks above Q_0 is a solution to (1.1). Suppose the lemma is false. Then there is a set Q satisfying (1.1) and an $a \in Q$ such that $a \notin Q_0$. But if $a \notin Q_0$ then there is an x such that $(y)\bar{R}_0(a, x, y)$. Assume a and x have been chosen such that x is minimal. By examination of Kleene's (35) taking into account the modifications above, we obtain an $a' \in C(a)$ and an $x' < x$ such that $(y)\bar{R}_0(a', x', y)$, which contradicts the choice of x.

In obtaining a' and x' the only non-trivial case is $a = 3 \cdot 5^{(a)_2}$, which we now consider. Since $a \in Q$ and Q satisfies (1.1), it follows that $C(a)$

[3] $x, y \in S$ is short for $x \in S$ & $y \in S$.

is linearly ordered and $(n)[\mathrm{Def}(a_n)$ & $a_n \in C(a_{n+1})]$. But $a \notin Q_0$ and Q_0 also satisfies (1.1). Hence $(En)[a_n \notin Q_0]$, i. e. the clause

$$(x_2)(x_3)\big[T\big((a)_2, (x_2)_O, x_3\big) \to (x_4)(Ey_2)R_0\big(U(x_3), x_4, y_2\big)\big] \tag{1.2}$$

obtained from Kleene's (35) is false. Choose x_2, x_3, x_4 such that

$$T\big((a)_2, (x_2)_O, x_3\big) \;\&\; (y_2)\bar{R}_0\big(U(x_3), x_4, y_2\big) \tag{1.3}$$

and let $n = x_2$. Then $a' = a_n = U(x_3)$ and $x' = x_4$ are the desired numbers.

Remark. This lemma and its proof suggest that it is possible to define Q_0 using the method of [2], § 8. This is accomplished by defining $\bar{Q}_0$ as follows.

($\bar{Q}$1) $a \neq 1$ & $a \neq 2^{(a)_0}$ & $a \neq 3 \cdot 5^{(a)_2} \to a \in \bar{Q}_0$,
($\bar{Q}$2) $a \in \bar{Q}_0$ & $a \neq 0 \to 2^a \in \bar{Q}_0$,
($\bar{Q}$3) $a = 3 \cdot 5^{(a)_2}$ & $[\overline{\mathrm{Def}}(a, n) \vee a_n \in \bar{Q}_0 \vee a_n \notin C(a_{n+1})] \to a \in \bar{Q}_0$,
($\bar{Q}$4) $a \in \bar{Q}_0$ only as required by ($\bar{Q}$1)-($\bar{Q}$3).

Then $\bar{Q}_0$ is recursively enumerable in $\mathrm{Def}(x, y)$, $x \in C(y)$. Lemma 2 and the classification of Q_0 are easily obtained from this definition.

2. O in terms of Q_0.

LEMMA 3. $a \in O \equiv a \in Q_0 \,\&\, (\overline{E\alpha})(x)\big[\alpha(x+1) \in C\big(\alpha(x)\big) \,\&\, \alpha(0) = a\big]$. *I. e. $a \in O$ if and only if $a \in Q_0$ and $C(a)$ is well ordered.*

Proof. The implication to the right follows from Lemmas 1 and 2. On the other hand assume $a \notin O$. If $a \notin Q_0$ the lemma is proved. If $a \in Q_0$ then it is possible to define an infinite descending sequence $\big(x_{n+1} \in C(x_n)\big)$ in $Q_0 - O$ beginning with $x_0 = a$, using the inductive definitions of O and of Q_0.

3. L and $<_a$.

Let Q_a be a variable which ranges over all subsets of the natural numbers and let $<_a$, $=_a$ be two-place predicate variables ($a \neq 0$). We define $L(a, Q_a, <_a, =_a)$ to hold if and only if $a = 3 \cdot 5^{(a)_2}$, $a \in Q_0$, and for every x and y

$$x \in Q_a \equiv x <_a a \vee x = a, \tag{3.1}$$

$$x =_a y \equiv x, y \in Q_a \;\&\; (z)[z <_a x \equiv z <_a y], \tag{3.2}$$

$$<_a \text{ linearly orders } Q_a \text{ relative to } =_a, \tag{3.3}$$

$$\{1, a\} \subseteq Q_a \subseteq Q_0, \tag{3.4}$$

$$y \in Q_a \;\&\; x \in C(y) \to x <_a y, \tag{3.5}$$

$$x <_a y \to (Ez)[z \in C(y) \;\&\; z =_a x], \tag{3.6}$$

$$x <_a a \to 2^x <_a a, \tag{3.7}$$

$$x = 3 \cdot 5^{(x)_2} \in Q_0 \;\&\; y <_a a \;\&\; (n)[x_n <_a y] \to x <_a y \vee x =_a y. \tag{3.8}$$

We will not make use of the fact that each of (3.1)-(3.8) happens to be independent of the others. Let $L(a, <_a) \equiv L(a, Q_a, <_a, =_a)$ where Q_a and $=_a$ are defined by (3.1) and (3.2); $x \leqslant_a y \equiv x <_a y \vee x =_a y$; and for $a \in O$

$$O_a \equiv \hat{x}(|x| < |a| \vee x = a)\,, \tag{3.9}$$

$$x <_a^\circ y \equiv |x| < |y| \;\&\; x, y \in O_a\,, \tag{3.10}$$

$$x =_a^\circ y \equiv |x| = |y| \;\&\; x, y \in O_a\,. \tag{3.11}$$

LEMMA 4. *If* $a \in O$ *and* $a = 3 \cdot 5^{(a)_2}$, *then* $L(a, O_a, <_a^\circ, =_a^\circ)$.

4. Main theorem. *Let* β *be a variable which ranges over all two-place number-theoretic predicates, and let* $a = 3 \cdot 5^{(a)_2}$. *Then*

$$a \in O = (E\beta)_{HA} L(a, \beta) = (E\beta) L(a, \beta)\,. \tag{4.1}$$

The proof of this theorem will be postponed until § 6. The solution to Kleene's question is a corollary of this theorem.

5. Properties of $<_a$. Throughout this section we assume

$$a = 3 \cdot 5^{(a)_2} \;\&\; a \in Q_0 \;\&\; L(a, Q_a, <_a, =_a)\,. \tag{5.1}$$

Two sets S and T of natural numbers are said to be *isomorphic*, written $S \cong T$, if there is a 1-1-correspondence xRy whose domain is S and whose range is T such that

$$uRx \;\&\; vRy \to [u \in C(v) \equiv x \in C(y)]\,. \tag{5.2}$$

A *segment* of S is a set of the form $C(b) \cap S$ where $b \in S$. From the classical theory of ordinals, if S and T are well-ordered by $x \in C(y)$, then either $S \cong T$ or one of S and T is isomorphic to a uniquely determined segment of the other. (The crucial reasons for a step in a proof are indicated at the end of the step between parentheses.)

LEMMA 5. *If* $x =_a y$, *then* $C(x) \cong C(y)$.

Proof. Assume $x =_a y$. Then $x, y \in Q_a \subset Q_0$ ((3.2), (3.4)), and therefore $C(x)$ and $C(y)$ are linearly ordered (Q_0 satisfies (1.1)). The correspondence $uRv \equiv u \in C(x) \;\&\; v \in C(y) \;\&\; u =_a v$ has domain $C(x)$ and range $C(y)$ ((3.5), (3.6)). Assume also uRs, vRt, and $u \in C(v)$. (To show $s \in C(t)$.) Then $s =_a u <_a v =_a t$ by (3.5); hence $s <_a t$ by (3.3). Now s and t are elements of the linearly ordered set $C(y)$. The only way s and t can be related in that ordering consistent with $s <_a t$, (3.5), and (3.3) is $s \in C(t)$. Etc.

LEMMA 6. *If* $x <_a y$ *then* $C(x)$ *is isomorphic to a segment of* $C(y)$.

Proof. Let $u \in C(y)$ such that $x =_a u$ (see (3.6)). Then $C(x) \cong C(u)$ by the previous lemma.

LEMMA 7. *If* $a \in O$ *then* $x <_a y \to x <_a^\circ y$ *and* $x =_a y \to x =_a^\circ y$ (see (3.9)-(3.11)).

Proof. If x and y are elements of O_a and of Q_a, then they are related the same way in one ordering as they are in the other by virtue of Lemmas 1, 5, 6. Hence it is sufficient to show that $O_a \subseteq Q_a$.

Assume $y \in O_a$ and by hypothesis of induction that $C(y) \subseteq Q_a$. (To show that $y \in Q_a$.) The case that $y = 1$, $y = a$, or $y = 2^{(y)_0}$ is taken care of by (3.4) and (3.7). Assume also $y = 3 \cdot 5^{(y)_2} \neq a$. Let $z \in C(a)$ such that $|z| = |y| < |a|$. Then $z <_a a$ (see (3.5)), $(n)[y_n \in Q_a]$ (hypothesis of induction), and $(n)[y_n <_a^\circ z]$. By the remark at the beginning of the proof $(n)[y_n <_a z]$; therefore $y \leqslant_a z$ (see (3.8)) and $y \in Q_a$ (see (3.1), (3.3)).

LEMMA 8. *If* $a \in O$ *then* $Q_a \subset O_a$.

Proof. Assume $a \in O$, $x \in Q_a$, $x \neq a$. Choose $y \in C(a)$ such that $x =_a y$ (see (3.1), (3.6)). Then $C(x) \cong C(y)$ (Lemma 5). But $C(y)$ is well-ordered and $x \in Q_0$. Hence $x \in O$ (Lemma 3) and $|x| = |y| < |a|$, i. e. $x \in O_a$.

The next lemma follows from Lemmas 4, 7, 8:

LEMMA 9. *If* $a \in O$ *then* $<_a^\circ$ *is the unique relation* $<_a$ *such that* $L(a, <_a)$.

LEMMA 10. *If* $a \notin O$ *then* $O \subset Q_a$, *and for* $y \in O$, $x <_a y \equiv |x| < |y|$ *and* $x =_a y \equiv |x| = |y|$.

Proof. Assume $a \notin O$ and $y \in O$. To show $y \in Q_a$ assuming also $C(y) \subseteq Q_a$ (hypothesis of induction). If $y = 1$ or $y = 2^{(y)_0}$, then $y \in Q_a$ ((3.4), (3.7)). Assume $y = 3 \cdot 5^{(y)_2} \in O$. For each n, let $\eta(n) \in C(a)$ such that $\eta(n) =_a y_n$ ((3.6) substituting a for y and y_n for x). Then $C(\eta(n)) \cong C(y_n)$ (Lemma 5) and is therefore well-ordered. Now $\eta(n) =_a y_n <_a y_{n+1} =_a \eta(n+1)$, i. e. $\eta(n) <_a \eta(n+1)$. Hence $\eta(n) \in C(\eta(n+1))$ since $C(a)$ is linearly ordered by $u \in C(v)$, and (3.5). Let S be the union of the $C(\eta(n))$. Then $S \subset C(a)$ and S is well-ordered by $u \in C(v)$. Thus S cannot exhaust all of $C(a)$ since the latter is not well-ordered. Let $w \in C(a) - S$. Then $S \subseteq C(w)$ and therefore $y_n =_a \eta(n) <_a w <_a a$. Hence $y \leqslant_a w$ (see (3.8)) and $y \in Q_a$. Thus $O \subset Q_a$.

Now assume $y \in O$ and $x <_a y$. Then $C(x)$ is isomorphic to a segment of $C(y)$ (Lemma 6). Hence $|x| < |y|$ (Lemma 3).

On the other hand, assume $y \in O$ and not $x <_a y$. Then either $y =_a x$ or $y <_a x$. Hence $C(y)$ is isomorphic to $C(x)$ or to a segment of $C(x)$. In either case $|x| < |y|$ is impossible. The last part of the lemma follows similarly.

6. Proof of the theorem. Assume $a \in O$ and $a = 3 \cdot 5^{(a)_2}$ (see § 4). Then $(E!\beta) L(a, \beta)$ (Lemma 9). Hence $x <_a^\circ y \equiv (E\beta)[L(a, \beta) \;\&\; x\beta y]$

$\equiv (\beta)[L(a,\beta) \rightarrow x\beta y]$. But L is certainly arithmetical, and therefore $<_a^\circ \in HA$.

On the other hand, suppose $a \notin O$ and $a = 3\cdot 5^{(a)_2}$. If $a \notin Q_0$ then $L(a,\beta)$ is false for all β. If $a \in Q_0 - O$ and $L(a,\beta)$ then O_b is recursive in β for each $b \in O$ (Lemma 10). Using also [10] or [9], Theorem 2, together with the function defined in the proof of [8], Satz 9, it follows that every HA predicate is recursive in β. Hence β is not HA. Furthermore β cannot be unique by the argument at the beginning of the proof.

Remark. The restriction that a be a limit notation is easily eliminated. One way would be to redefine L by modifying (3.7) to read $x <_a (a)_0 \rightarrow 2^x <_a a$ when a is a successor notation. Or one could reduce the successor case to the limit case using a primitive recursive function π such that $\pi(3\cdot 5^y) = 3\cdot 5^y$ and $\pi(2^y) = \pi(y)$. By either method we obtain the following (α is a function variable):

COROLLARY 1. *There is an arithmetical predicate A such that $a \in O \equiv (E\alpha)_{HA} A(\alpha, a) \equiv (E!\alpha) A(\alpha, a)$.*

LEMMA 11. *If $H(a)$ is HA then there is a primitive recursive predicate P such that $H(a) \equiv (E\alpha)(x) P(\bar{\alpha}(x), a) \equiv (E!\alpha)(x) P(\bar{\alpha}(x), a) \equiv (E\alpha)_{HA} P(\bar{\alpha}(x), a)$.*

Proof. Assume $H(a)$ is HA. Then by a theorem proved independently by Addison, Grzegorczyk, Kuznecov, and Myhill (see [1], § 3.3), the representing function of $H(a)$ can be obtained as a solution α_1 to a system E of equations which contains n function symbols and has a unique solution $\alpha_1, \alpha_2, \ldots, \alpha_n$. That $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy E can be written in the form

$$(x_1) \ldots (x_k)[S(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_n)] \tag{6.1}$$

where S is primitive recursive and $x_1, \ldots, x_k$ are the individual variables of E. Employing suitable 1-1 primitive recursive functions mapping $(x_1, \ldots, x_k)$ to x and $(\alpha_1, \ldots, \alpha_n)$ to α together with the inverse mappings we can write

$$H(a) \equiv (E\alpha)(x)[S(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_n) \;\&\; \alpha_1(a) = 0], \tag{6.2}$$

and also with $(E!\alpha)$ or $(E\alpha)_{HA}$ in place of $(E\alpha)$. The predicate P is now easily obtained using Kleene's normal form (see [6], § 24).

LEMMA 12. *If $H(\alpha, a)$ is hyperarithmetical, then there is a primitive recursive predicate P such that*

$$H(\alpha, a) \equiv (E\beta)(x) P(\alpha, a, \beta, x) \equiv (E!\beta)(x) P(\alpha, a, \beta, x).$$

Proof. The proof is similar to that of the previous lemma except that we now introduce a function symbol in E corresponding to the free variable α, and relativize the arguments above with respect to the function α. See [5], § 7, for the relativized concepts needed for this proof. Note that if β exists, it is HA relative to α.

COROLLARY 2. *There is a primitive recursive predicate P such that $a \in O \equiv (E\alpha)_{HA}(x) P(\bar{\alpha}(x), a) \equiv (E!\alpha)(x) P(\bar{\alpha}(x), a)$, and likewise for any predicate $(\alpha)B(\alpha, a)$ (B arithmetical) in place of $a \in O$.*

Proof. Let A be the predicate obtained in Corollary 1. Then applying Lemma 12, $A(\alpha, a) \equiv (E\beta)(x) P(\alpha, a, \beta, x) \equiv (E\beta)(x) P(\alpha, a, \beta, x)$, where P is primitive recursive. For a given value of a consider the pairs (α, β) such that $(x) P(\alpha, a, \beta, x)$. Among these pairs there is at most one β corresponding to each α. Hence $(E\alpha) A(\alpha, a)$ is equivalent to the existence of a unique pair (α, β) such that $(x) P(\alpha, a, \beta, x)$. Thus if $a \in O$, there is a unique pair (α, β) satisfying $(x) P(\alpha, a, \beta, x)$, and both α and β are HA. If $a \notin O$ and $(x) P(\alpha, a, \beta, x)$, then α is not HA, and therefore the contraction of (α, β) to a single function is not HA. In this way the desired expression for $a \in O$ is obtained. The corresponding expression for $(\alpha) B(\alpha, a)$ is obtained by applying [6] Theorem 1.

Remarks. It is known that every predicate $H(a)$ expressed as in Lemma 11 must be HA. In fact the condition that α be unique can be omitted (see the second paragraph of this paper). Thus Lemma 11 can be used to characterize the class of HA predicates similarly to [1] for functions. Kleene [7] has obtained some results on the least segment of HA that will suffice for the range of α in $H(a) \equiv (E\alpha)(x) P(\bar{\alpha}(x), a)$, where $H(a)$ is an arbitrary fixed HA predicate, thereby strengthening our $(E\alpha)_{HA}$. Kleene does not obtain $(E!\alpha)$, but there does not appear to be any difficulty involved in adding uniqueness to his treatment.

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Reçu par la Rédaction le 11. 5. 1959

Sur la compactification des espaces métriques

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Dans leur travail [1], J. de Groot et R. H. McDowell ont introduit la notion de $\{\Phi_\tau\}_{\tau\in T}$*-compactification* pour un espace métrique X, $\{\Phi_\tau\}_{\tau\in T}$ étant une famille de transformations de X dans X. On appelle ainsi un espace métrique compacte $\widetilde{X}$, dont X est un sous-ensemble dense, s'il existe une famille $\{\widetilde{\Phi}_\tau\}_{\tau\in T}$ de prolongements des Φ_τ sur $\widetilde{X}$ à valeurs dans $\widetilde{X}$. Ils ont aussi prouvé que pour chaque espace X métrique séparable et chaque famille dénombrable $\{\Phi_i\}$ il existe une $\{\Phi_i\}$-compactification; de plus, si $\dim X \leqslant 0$, on peut supposer $\dim \widetilde{X} \leqslant 0$. On a posé dans [1] le problème de trouver une $\{\Phi_i\}$-compactification n-dimensionnelle pour un espace X de dimension n et une famille $\{\Phi_i\}$ dénombrable. Le présent travail donne une solution de ce problème. Nous nous proposons de prouver le théorème suivant:

THÉORÈME. *X étant un espace métrique séparable de dimension $\leqslant n$ et $\{\Phi_i\}$ une famille de transformations de X dans X, il existe une $\{\Phi_i\}$-compactification $\widetilde{X}$ de X telle que $\dim \widetilde{X} \leqslant n$.*

Démonstration. On peut supposer que les fonctions superposées $\Phi_i\Phi_j$ et l'identité I de X appartiennent aussi à $\{\Phi_i\}$. Admettons dans X une métrique ϱ totalement bornée telle que les fonctions Φ_i soient uniformément continues dans ϱ (cf. [1]).

Pour chaque $m = 1, 2, \ldots$ nous définissons par induction un recouvrement [1] $\mathfrak{A}_m = \{U_1^m, \ldots, U_{k_m}^m\}$ de X tel que:

(a) $\delta(U_i^m) \leqslant 1/m$ $(i = 1, \ldots, k_m)$,

(b) rang $\mathfrak{A}_m \leqslant n$,

(c) pour chaque $l < m$ et $s \leqslant k_m$ il existe un $r \leqslant k_{m-1}$ tel que $\Phi_l(U_s^m) \subset U_r^{m-1}$

et une famille de fonctions $f_1^m, \ldots, f_{k_m}^m$ satisfaisant aux conditions

(d) $f_s^m: X \to [0, 1]$,

[1] Le mot „recouvrement" signifie toujours „recouvrement fini et ouvert". Un recouvrement $\mathfrak{A}$ est contenu dans $\mathfrak{A}_1$ (ou $\mathfrak{A}_1$ contient $\mathfrak{A}$) si pour chaque $U \in \mathfrak{A}$ il existe un $U_1 \in \mathfrak{A}_1$ tel que $U \subset U_1$.