ergibt (siehe 4.10)

$$V'' = (B \setminus V') \cup B' \in \mathfrak{Z}''.$$ 

Wir setzen $V = V' \cup V''$. Dann ist (4.10) $V \in \mathfrak{Z}$. Weiter ist

$$V = V' \cup [(B \setminus V') \cap B']$$

$$\subseteq B \cap B' \cap E[f < \gamma] = E[f < \alpha]$$
und

$$V \subseteq V' \cup B' \cap E[f < \beta].$$

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Hyperarithmetical quantifiers *

by

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Let $HA$ be the class of hyperarithmetical functions, predicates and
sets of natural numbers [4, 5]; let $(\exists a)(a \in HA \& P(a));$
and let "$HA$" also be an abbreviation for "hyperarithmetical".

From the proof of XXVI [5], it follows that if $R$ is a recursive
predicate then there is a primitive recursive predicate $P$ (obtained
uniformly from a Gödel number of $R$) such that

$$\quad (\exists a)(a)R(a, x, a) = (a)(\exists a)P(a, x, a).$$

This result is used in [1] and is proved explicitly in [7]. In the latter
paper, Kleene asks whether the converse is true; i.e., given a recursive $P$,
can a primitive recursive $R$ be found which satisfies (0.1)? We answer
this question in the affirmative (*).

The method of proof involves an analysis of the inductive definitions
of the set $O$ of Church-Kleene ordinal notations and of the two-place
predicates $[a] = [b]$ and $|a| < |b|$, where $|a|$ is the ordinal corresponding
to $a$ via $O$ and both predicates are taken to be false if either $a$ or $b$ is
not in $O$. The techniques developed by Kleene in [2] and amended in [6]
play an important rôle. In particular we shall employ the predicate
$(a)(Ey)R(a, x, y)$ defined in § 14 of [2], which Kleene abandons in the
amended version [6] (*).

1. $O(b)$ and $Q_a$. For each natural number $b$ let $O(b)$ be the set
defined in [2] § 13 and $V(a, b, x)$ the primitive recursive predicate such
that $a \in O(b) = (\exists b)\forall^* b_x \in V(a, b, x)$. If $a = 3 \cdot 5^m b,$ let $a_0 = \Phi \langle a_0 \rangle$, i.e., if $a \in O,$ then in a manner of speaking, $a_0, a_1, a_2, \ldots$ is the fundamental
sequence whose limit is $a$. Let Def$(a, x) = [a_n]$ is defined], more precisely
Def$(a, n) = (E y) \exists^* z \phi \langle a_0 \rangle, n_0, y, y$. 

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(*) This answer appears as Corollary 2. Kreisel and Mostowski asked similar
questions, which are answered by Corollary 1 and [6] Theorem 1.

(*) It is recommended that the reader be familiar with [2] through § 14 and [6]
through § 20, and have both papers available for reference.
Lemma 1 (Kleene). If \( b \in O \) then \( O(b) = \langle a < b \rangle \) and \( O(b) \) is well-ordered by the "less-than" relation \( x < y \) with ordinal \( \beta \). (See [2, 6], § 20, for proof.)

When \( \cong \) is an equivalence relation on a set \( S \), we say that \( \preceq \) is a linear ordering of \( S \) relative to \( \cong \) if:

- (L1) \( x, y \in S \Rightarrow x \cong y \vee y \cong x \),
- (L2) \( x, y, z \in S \Rightarrow x \cong y \vee y \cong z \vee z \cong x \),
- (L3) \( x \in S \Rightarrow x \cong x \),
- (L4) \( x, y, z \in S \Rightarrow x = y \leftrightarrow x \cong y \).

If \( \cong = = \), then (L4) is automatically satisfied. Note that \( a \in O(y) \) as \( x \cong y \), \( a = x = y \), and \( O(b) \) as \( S \) satisfies (L1)-(L3) if \( b \in O \), and if \( b \notin O \) then only (L2) need be satisfied (L1) and (L3) are false for suitably chosen \( O(b) \).

We shall define a set \( Q_0 \) by amending Kleene's definition of \( Q \) to require that \( O(a) \) be linearly ordered when \( a \in Q \). In place of Kleene's (2)(35) we write

\[
a \in Q \iff a = 1 \cdot v \land a = 3 \cdot 5^{\alpha_0} \land (n)[Df(a, v) \land a \in Q \land a \in O(a_{n+1})] \land O(a) \text{ is linearly ordered by } x \cong y \text{ relative to } =.\]

One notes that the clause \( O(a) \) is linearly ordered by \( x \cong y \) relative to \( = \) does not contain the variable \( Q \) and can be written in the form \( \langle \alpha \rangle \{x, y, z\} \) where \( P \) is primitive recursive, since the closures of (L1)-(L3) assume the respective forms \( \forall \alpha \exists \exists x \exists \exists y \), \( \exists \exists x \exists \exists y \), \( \exists \exists x \exists \exists y \). Following the method of [2], § 14, we obtain a primitive recursive predicate \( R_0 \) such that the set \( Q_0 = \{ x \in \{ x, y, z \} \} \) is a solution to (1.1). Kleene [9] shows that \( O \) is the smallest set \( S \) satisfying (1.1); \( Q_0 \) can be characterized as follows:

**Lemma 2.** \( Q_0 \) is the largest set \( Q \) satisfying (1.1).

**Proof.** By the remarks above \( Q_0 \) is a solution to (1.1). Suppose the lemma is false. Then there is a set \( Q \) satisfying (1.1) and an \( a \in Q \) such that \( a \notin Q_0 \). But if \( a \in Q_0 \) then there is an \( x \) such that \( y \notin R_0(\alpha, x, y) \). Assume \( a \) and \( x \) have been chosen such that \( x \) is minimal. By examination of Kleene's (39) taking into account the modifications above, we obtain an \( a' \in O(a) \) and an \( a' < x \) such that \( a' \notin R_0(\alpha', x', y) \), which contradicts the choice of \( x \).

In obtaining \( a' \) and \( x' \) the only non-trivial case is \( a = 3 \cdot 5^{\alpha_0} \), which we now consider. Since \( a \in Q \) and \( Q \) satisfies (1.1), it follows that \( O(a) \) is linearly ordered and \( (n)[Df(a, v) \land a \in O(a_{n+1})] \). But \( a \in Q_0 \) and \( Q_0 \) also satisfies (1.1). Hence \( \forall \alpha(x) = x \in Q_0 \), i.e. the clause

\[
\forall \alpha(x) \in Q_0 \Rightarrow (a_0, x_0) \in (a_0, (E_0)(x_0, y_0)) \Rightarrow \exists \exists x_0, y_0 \in (u_0, x_0, y_0)
\]

(1.2)

obtained from Kleene's (35) is false. Choose \( x_0, x_1, y_0 \) such that

\[
\forall \alpha(x) \in Q_0 \Rightarrow (a_0, x_0) \in (a_0, (E_0)(x_0, y_0)) \Rightarrow \exists \exists x_0, y_0 \in (u_0, x_0, y_0)
\]

(1.3)

and let \( n = x_0 \). Then \( a' = a_0 = u_0 \) and \( x' = x_0 \) are the desired numbers.

Remark. This lemma and its proof suggest that it is possible to define \( Q_0 \) using the method of [2], § 8. This is accomplished by defining \( Q_0 \) as follows.

\[
(a) \; a \neq 1 \land a \neq 2^{\alpha_0} \land a \neq 3 \cdot 5^{\alpha_0} \Rightarrow a \in Q_0,
\]

\[
(b) \; a \in Q_0 \land a \neq 2^{\alpha_0} \Rightarrow \exists \exists x \in Q_0 \exists \exists y \in Q_0 \exists \exists z \in Q_0 \exists \exists w \in Q_0 \exists \exists v \in Q_0 \exists \exists u \in Q_0 \exists \exists t \in Q_0 \exists \exists s \in Q_0
\]

(1.4) \( a \in Q_0 \) only as required by (a) \( q \).

Then \( Q_0 \) is recursively enumerable in \( Df(a, y) \) \( x \in Q_0 \). Lemma 2 and the classification of \( Q_0 \) are easily obtained from this definition.

2. \( Q_0 \) in terms of \( Q_0 \).

**Lemma 3.** \( a \in O \Rightarrow = a \in Q_0 \land (E_0)(x) \in \{ x \in \{ x, y, z \} \} \). \( I \). c. \( \alpha \in O \) if and only if \( \alpha \in Q_0 \) and \( O(a) \) is well ordered.

**Proof.** The implication to the right follows from Lemmas 1 and 2. On the other hand assume \( \alpha \notin O \). If \( a \notin Q_0 \), the lemma is proved. If \( a \in Q_0 \) there is an infinite descending sequence \( \{ a_{n+1} \in O(a_n) \} \) in \( Q_0 \) \( \Rightarrow \) beginning with \( x_0 = a_0 \), using the inductive definitions of \( O \) and \( Q_0 \).

3. \( L \) and \( <_{a} \). Let \( Q_0 \) be a variable which ranges over all subsets of the natural numbers and let \( <_{a} =_{a} \) be two-place predicate variables \( \alpha \neq 0 \). We define \( L(a, Q_0, <_{a} =_{a}) \) to hold if and only if \( a = 3 \cdot 5^{\alpha_0} \), \( a \in Q_0 \), and for every \( x \) and \( y \)

\[
x \in Q_0 \Rightarrow x = x \land \forall x \epsilon = x,
\]

\[
x = x \Rightarrow x = x \land \forall x \epsilon = x < x \land y,
\]

\[
\forall a <_{a} \text{ linearly orders } Q_0 \text{ relative to } =_{a},
\]

\[
\forall a \in Q_0 \land \forall \epsilon O(y) \Rightarrow x <_{a} y,
\]

\[
\forall a \in \{ x \in \{ x, y, z \} \} \Rightarrow x <_{a} y \Rightarrow (E_0)(x \epsilon O(y) \land \forall a =_{a} x),
\]

\[
\forall a \in \{ x \in \{ x, y, z \} \} \Rightarrow x <_{a} a \Rightarrow 3^{\alpha_0} <_{a} a,
\]

\[
\forall a = 3 \cdot 5^{\alpha_0} \epsilon Q_0 \land \forall x <_{a} a \land \forall \{ x \epsilon a =_{a} x \} \Rightarrow x <_{a} y \land \forall a =_{a} y.
\]
We will not make use of the fact that each of (3.1)-(3.8) happens to be independent of the others. Let \( L(a, \leq a) = L(a, Q_a, \leq a, =a) \) where \( Q_a \) and \( =a \) are defined by (3.1) and (3.3); \( x \leq a y = x \leq a y \lor x = =a y \); and for \( \alpha \in O \)

\[
\begin{align*}
0_\alpha &= \hat{a}(0) < |a| \forall a \neq a, \\
x \leq^a y &= |x| < |y| & x, y \in O_a, \\
x \leq y &= |x| = |y| \land x \in O_a.
\end{align*}
\]

**Lemma 4.** If \( a \in O \) and \( a = 3 \cdot 5^{|a|} \), then \( L(a, O_a, \leq a, =a) \).

4. **Main theorem.** Let \( \beta \) be a variable which ranges over all two-place number-theoretic predicates, and let \( a = 3 \cdot 5^{|a|} \). Then

\[
\alpha \in O = (\beta \in L(a, \beta)) = (\beta \in L(a, \beta)).
\]

The proof of this theorem will be postponed until § 6. The solution to Kleene's question is a corollary of this theorem.

5. **Properties of \( \leq a \).** Throughout this section we assume

\[
a = 3 \cdot 5^{|a|} \land a \in O \land L(a, Q_a, \leq a, =a).
\]

Two sets \( S \) and \( T \) of natural numbers are said to be isomorphic, written \( S \cong T \), if there is a 1-1-correspondence \( xRy \) whose domain is \( S \) and whose range is \( T \) such that

\[
\text{uR} \text{u} = \text{vR} \text{v} = u \in \text{O}(v) \Rightarrow x \in \text{O}(y).
\]

An **segment** of \( S \) is a set of the form \( C(b) \cap S \) where \( b \in S \). From the classical theory of ordinals, if \( S \) and \( T \) are well-ordered by \( x \in \text{O}(y) \), then either \( S \cong T \) or one of \( S \) and \( T \) is isomorphic to a uniquely determined segment of the other. (The crucial reasons for a step in a proof are indicated at the end of the step between parentheses.)

**Lemma 5.** If \( x \leq y \), then \( \text{O}(x) \cong \text{O}(y) \).

**Proof.** Assume \( x = a y \). Then \( x, y \in Q_a \subseteq Q_b \subseteq Q_c \subseteq Q_d \), and therefore \( C(c) \) and \( O(y) \) are linearly ordered \( x \in \text{O}(y) \) satisfies (1.1)). The corresponding \( x\in \text{O}(y) \) and \( x \in \text{O}(y) \) have domain \( C(x) \) and range \( O(y) \) (3.3), (3.6)). Assume also \( v \in \text{O}(t) \). (To show \( s \in \text{O}(t) \).) Then \( s \leq t \in \text{O}(y) \); hence \( s \leq t \) (3.3). Now \( s \) and \( t \) are elements of the linearly ordered set \( O(y) \). The only way \( s \) and \( t \) can be related in that ordering consistent with \( s \leq t \), (3.5), and (3.3) is \( s \leq y \in \text{O}(y) \). Etc.

6. **Lemma 6.** If \( x \leq y \) then \( O(x) \cong \text{O}(y) \).

**Proof.** Let \( u \in \text{O}(y) \) such that \( u = a x \in \text{O}(y) \). Then \( C(x) \cong O(u) \) by the previous lemma.

**Lemma 7.** If \( a \in O \) then \( x \leq a y \Rightarrow x \equiv^a y \) and \( x \equiv^a y \Rightarrow x \equiv^a y \) (see (3.9)-(3.11)).

**Proof.** If \( x \) and \( y \) are elements of \( O_x \) and \( O_y \), then they are related the same way in one ordering as they are in the other by virtue of Lemmas 1, 6, 8. Hence it is sufficient to show that \( O_x \subseteq O_y \).

Assume \( y \in Q_a \), and by hypothesis of induction that \( C(y) \subseteq Q_x \). (To show that \( y \in Q_x \).) The case that \( y = 1 \), \( y = a \), or \( y = a^{|a|} \) is taken care of by (3.4) and (3.7). Assume also \( y = 3 \cdot 5^{|a|} \neq a \). Let \( x \in C(a) \) such that \( |x| = |y| < |a| \). Then \( x \leq a \). (see (3.5)), \( x \equiv^a y \) (hypothesis of induction), and \( x \equiv^a y \). By the remark at the beginning of the proof \( (x) \equiv (y) \); therefore \( y \equiv x \) (see (3.3)) and \( y \in Q_a \) (see (3.1), (3.3)).

**Lemma 8.** If \( a \in O \) then \( Q_a \subseteq Q_x \).

**Proof.** Assume \( a \in O \), \( x \in Q_a \), \( x \neq a \). Choose \( y \in C(a) \) such that \( x \leq a y \) (see (3.1)), (3.7)). Then \( C(y) \subseteq O_y \) (Lemma 3). But \( C(y) \) is well-ordered and \( y \in Q_x \). Hence \( x \equiv^a O \) (Lemma 3) and \( x \equiv^a O \), \( x \equiv a \). The next lemma follows from Lemmas 4, 7, 8.

**Lemma 9.** If \( a \in O \) then \( \leq^a \) is the unique relation \( \leq \) such that \( L(a, \leq a) \).

**Lemma 10.** If \( a \in O \) then \( C(x) \subseteq C(y) \) (Lemma 5). But \( C(x) \) is well-ordered and \( y \in Q_x \). Hence \( x \equiv O \) (Lemma 3) and \( x \equiv O \), \( x \equiv a \). The next lemma follows from Lemmas 4, 7, 8.

**Lemma 11.** If \( a \in O \) and \( x \equiv a y \). To show \( y \in O \), assume also \( y \in O \) (hypothesis of induction). If \( y = 1 \) or \( y = 2^{|a|} \), then \( y \in O \) (3.4), (3.7)). Assume \( y = 3 \cdot 5^{|a|} \). For each \( n \), let \( \eta(n) \in C(a) \) such that \( \eta(n) = a \eta(n) - n \cdot 3 \cdot 5^{|a|} \) (substituting \( a \) for \( y \) and \( x \) for \( z \)). Then \( \eta(n) \in C(y) \) (Lemma 5) and is therefore well-ordered. Now \( \eta(n) = y \equiv \eta(n+1) \equiv \eta(n+1) \), \( i.e. \) \( \eta(n) = \eta(n+1) \). Hence \( \eta(n) \in C(y) \) (Lemma 5) since \( C(y) \) is linearly ordered by \( u \in C(y) \), (3.5). Let \( S \) be the union of the \( C(y) \). Then \( S \subseteq C(y) \) and therefore \( y = a \eta(n) \equiv a^{|a|} \). Hence \( y = a^{|a|} \) (see (3.8)) and \( y \in O \). Hence \( Q_a \subseteq O \).

Now assume \( y \not\equiv x \), and then \( C(x) \equiv \text{O}(x) \). Hence \( x \not\equiv a y \). Then either \( y = a x \) or \( y \leq a x \). Hence \( C(y) \) is isomorphic to \( C(x) \) or to a segment of \( C(a) \). In either case \( x \equiv y \) is impossible. The last part of the lemma follows similarly.

6. **Proof of the theorem.** Assume \( a \in O \) and \( a = 3 \cdot 5^{|a|} \) (see § 4).

Then \( (E \beta) L(a, \beta) \) (Lemma 9).

Hence \( x \equiv^a y = (E \beta) L(a, \beta) \).
function \( a \). See [5], § 7, for the relativized concepts needed for this proof. Note that if \( \beta \) exists, it is HA relative to \( a \).

**Corollary 2.** There is a primitive recursive predicate \( P \) such that \( a \in 0 \iff (Ea)(x)[P(a, x, a)] \), and \( (Ea)(x)[P(a, x, a)] \), and likewise for any predicate \( (a)B(a, a) \) (in arithmetic) in place of \( a \).

Proof. Let \( A \) be the predicate obtained in Corollary 1. Then applying Lemma 12, \( A(a, a) = (E\beta)(x)[P(a, a, \beta, x)] = (E\beta)(x)[P(a, a, \beta, x)] \), where \( P \) is primitive recursive. For a given value of \( a \) the pair \( (a, \beta) \) such that \( (x)P(a, a, \beta, x) \). Among these pairs there is at most one \( \beta \) corresponding to each \( a \). Hence \( (E\beta)(x)[P(a, a, \beta, x)] \) is the existence of a unique \( \beta \) (a, \beta) such that \( (a, \beta) \). Thus if \( a \in 0 \), there is a unique pair \( (a, \beta) \) satisfying \( (x)P(a, a, \beta, x) \), and both \( \alpha \) and \( \beta \) are HA. If \( a \in 0 \) and \( (x)P(a, a, \beta, x) \), then \( a \) is not HA, and therefore the contraction of \( (a, \beta) \) to a single function is not HA. In this way the desired expression for \( (a)B(a, a) \) is obtained. The corresponding expression for \( (a)B(a, a) \) is obtained by applying [6] Theorem 1.

**Remarks.** It is known that every predicate \( H(a) \) expressed as in Lemma 11 must be HA. In fact the condition that \( a \) be unique can be omitted (see the second paragraph of this paper). Thus Lemma 11 can be used to characterize the class of HA predicates similarly to [1] for functions. Kleene [7] has obtained some results on the least segment of HA that will suffice for the range of \( a \) in \( H(a) \) (in HA) \( a \) is a fixed HA predicate, thereby strengthening our \( (Ea)(x)[P(a, x, a)] \). Kleene does not obtain \( (Ea)(x)[P(a, x, a)] \) but there does not appear to be any difficulty involved in adding uniqueness to his treatment.

**References**


Sur la compactification des espaces métriques

par

R. Engelking (Warsawa)

Dans leur travail [1], J. de Groot et R. H. McDowell ont introduit la notion de $(\Phi_n)_n$-compactification pour un espace métrique $X$, $(\Phi_n)_n$ étant une famille de transformations de $X$ dans $X$. On appelle ainsi un espace métrique compactifié $\tilde{X}$, dont $X$ est un sous-ensemble dense, s'il existe une famille $(\tilde{\Phi}_n)_n$ de prolongements des $\Phi_n$ sur $\tilde{X}$ à valeurs dans $\tilde{X}$. Ils ont aussi prouvé que pour chaque espace $X$ métrique séparable et chaque famille dénombrable $(\Phi_n)$ il existe une $(\Phi_n)$-compactification; de plus, s'il $\dim X < 0$, on peut supposer $\dim \tilde{X} < 0$. On a posé dans [1] le problème de trouver une $(\Phi_n)$-compactification $n$-dimensionnelle pour un espace $X$ de dimension $n$ et une famille $(\Phi_n)$ dénombrable. Le présent travail donne une solution de ce problème.

Nous nous proposons de prouver le théorème suivant:

**Théorème.** $X$ étant un espace métrique séparable de dimension $\leq n$ et $(\Phi_n)$ une famille de transformations de $X$ dans $X$, il existe une $(\Phi_n)$-compactification $\tilde{X}$ de $X$ telle que $\dim \tilde{X} \leq n$.

**Démonstration.** On peut supposer que les fonctions superposées $\Phi_n\tilde{\Phi}$ et l'identité $I$ de $X$ appartiennent aussi à $(\Phi_n)$. Admettons dans $X$ une métrique $\rho$ totalement bornée telle que les fonctions $\Phi_n$ soient uniformément continues dans $\rho$ (cf. [1]).

Pour chaque $m = 1, 2, \ldots$ nous définissons par induction un recouvrement $(\cdot)$ $S_m = (U_1^m, \ldots, U_n^m)$ de $X$ tel que:

(a) $\delta(U_i^m) \leq 1/m$ $(i = 1, \ldots, n)$,
(b) rang $S_m \leq m$,
(c) pour chaque $1 \leq m$ et $s \leq k_m$ il existe un $r \leq k_{m-1}$ tel que $\Phi_r(U_r^m) \subseteq U_r^{m-1}$ et une famille de fonctions $f_1^m, \ldots, f_n^m$ satisfaisant aux conditions
(d) $f_i^m : X \rightarrow [0, 1]$.

(1) Le mot "recouvrement" signifie toujours "recouvrement fini et ouvert". Un recouvrement $S$ est contenu dans $S_1$ (ou $S_0$ contient $S$) si pour chaque $U \in S$ il existe un $U_1 \in S_1$ tel que $U \subseteq U_1$. 

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