d'où nous obtenons, en vertu des lemmes 1 et 2,

\[ f^+_n(x) = \lim sup_{h \to 0} \frac{1}{h} \int_{x-h}^{x+h} g(t) dt = \lim sup_{h \to 0} \frac{1}{h} \int_{x-h}^{x+h} F(t) dt \geq \frac{1}{3}, \]

(9)

\[ f^-_n(x) = \lim_{h \to 0} \frac{1}{h} \int_{x-h}^{x+h} g(t) dt = \lim_{h \to 0} \frac{1}{h} \int_{x-h}^{x+h} F(t) dt = 0. \]

Soit \( A \) un ensemble mesurable et \( \mathcal{X}_n(x) \) la fonction caractéristique de cet ensemble.

Nous dirons que le point \( x_n \) est un point de densité 0 de l'ensemble \( A \), si

\[ \lim_{h \to 0} \frac{1}{h} \int_{x_n-h}^{x_n+h} \mathcal{X}_n(t) dt = 0. \]

Admettons encore que \( h_n(x) \) soit une fonction mesurable définie dans l'intervalle \((a, b)\).

Nous dirons que le nombre \( L_0 \) est la limite approximative supérieure à droite (à gauche) de la fonction \( h_n(x) \) au point \( x_n \in (a, b) \), si \( L_0 \) est la borne inférieure des nombres \( L \) pour lesquels l'ensemble de tous les points \( x > x_n \) pour lesquels \( h_n(x) < L \) admet en \( x_n \) un point de densité 0.

En vertu de cette définition et des relations (9), la fonction \( F(x) \) admet aux points de l'ensemble \( C \) une limite approximative supérieure à droite nulle et une limite approximative supérieure à gauche égale à 1. Étant donné que l'ensemble \( C \) a la puissance du continu, nous obtenons le théorème suivant:

**Théorème.** Il existe une fonction \( F(x) \) de la variable réelle, définie dans l'intervalle \([-1, 2]\), pour laquelle l'ensemble des nombres \( x \in [-1, 2], \) où la limite approximative supérieure à droite est inférieure à la limite approximative supérieure à gauche, a la puissance du continu.

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On the extensibility of mappings, their local properties and some of their connections with the dimension theory

by

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Introduction

The relations \( r \) and \( r_0 \) defined by Kuratowski (see [10] (1), p. 253) are in a certain sense a generalization of those stated in an important theorem of Tietze ([9], p. 117) and have been applied to the characterization of the important classes of spaces distinguished by Borsuk, such as the absolute retracts (see [2], p. 109), the absolute neighbourhood retracts (see [3], p. 222) and many others. These relations also possess a number of interesting properties (compare for instance [10] and [11]).

Special attention should be paid to the connection between the dimension of the space of arguments and the extension of the continuous mappings into an \( n \)-dimensional sphere. The above connection as well as a number of other interesting properties of the relation \( r \) have been discussed in chapter VII of book [10] by Kuratowski, and also in his paper [11] specially devoted to these problems.

In this paper (2) further properties of the relation \( r \) (see section 2) — in particular its local properties — are investigated by means of a relation \( \varphi \) (see sections 1 and 3) specially defined for this purpose; some close and natural analogies with the theory of dimension are also discussed (see section 4). These considerations show the role of the extensibility of continuous mappings not only for the dimension of sets but also for some derivative notions, for instance: the disconnection of a space, the

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

(2) This publication is a part of the dissertation presented at the University in Łódź as a Doctor's thesis in Mathematics. It has been prepared under the direction of Professor K. Kuratowski, to whom the author wishes to express his indebtedness for suggesting the problem which led to this work and the friendly care shown during its realization.

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coefficient of Urysohn, the dimension of connectivity, the manifold of Cantor, \(n\)-dimensionally connected spaces. Some possibilities of a classification of spaces, analogous to the dimensional one, are to be expected (see section 5.6).

This is in some connexion with the notions of the normal families of Hurwitz [6] and of the uni-ordered spaces of G. T. Whyburn [13].

The following considerations may be regarded as a supplement to paper [1] by P. S. Aleksandrov, in which various approaches to the notion and the theory of dimension have been discussed.

We shall use the following notation: Let \((x, y, z, \ldots)\) mean a set consisting of the elements \(x, y, z, \ldots\) in particular \((a, b) = \{a, b\}\) means an ordered pair \((a, b)\) means an open interval, whose ends are \(a\) and \(b\); \((a, b)\) means the corresponding closed interval; in particular \(I = [0, 1]\). By \(G^0\) we shall mean the \(n\)-dimensional Euclidean space, by \(G\) the set of real numbers, by \(G^+\) the set of positive integers.

By \(S_n\) we shall denote the \(n\)-dimensional sphere in the \((n + 1)\)-dimensional Euclidean space: \(S_n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}\). By \(Q_n\) we denote the \(n\)-dimensional disc in the \(n\)-dimensional Euclidean space: \(Q_n = \{x \in \mathbb{R}^{n+1} | |x| < 1\}\). \(Q_n(x)\) means a spherical neighbourhood of the point \(x\) with the radius \(\varepsilon > 0\) in the metric space \(X; Q_n(x) = B_0(x, \varepsilon^2, \varepsilon)\).

By \(R(E)\) we shall denote the complement of the relation \(E: z E y \Leftrightarrow (z, y)\), by \(R C S\) we mean the subsumption of the relation \(R\) relative to the relation \(S: R C S \Leftrightarrow (x, y) E (z, y) S (x, z)\) and by \(R \subseteq S\) the product of relations \(R\) and \(S:\) \(R \subseteq S\) if \(S (x, y) \Leftrightarrow (x, y) E (z, y) S (x, z)\).

If \(f: X \to Y\) we shall denote that \(f\) is a continuous mapping of a space \(X\) into a space \(Y\). Two mappings \(f\) and \(g\) will be equal, \(f = g\), if they assign to each value of the argument the same value \(f = g \Leftrightarrow f(x) = g(x)\). By \(f: X \to Y\) we shall denote a partial mapping (restriction of \(f\) to \(F\)), namely \(f: X \to Y\) restricted to \(F \subseteq X\). The mapping \(f^*\) is said to be an extension of the mapping \(f: X \to Y\) to the space \(X \subseteq X: f \subseteq f^*\).

Let \(f^* : Y \to Y\) and \(f^* : Y\) we shall denote the set of the mappings \(f^*\) which can be extended to the whole space \(X:\)

1. Definitions of the fundamental relations and connections between them

In [10] on p. 255 Kuratowski has introduced the relations \(\tau\) and \(\tau_0\) by means of the following definitions:

1.1. \(\mathcal{X} \tau_0 Y\) means that each continuous mapping \(f\) of an arbitrary closed subset \(F\) of the space \(X\) into the space \(Y\) can be extended to the whole space \(X:\)

\[
\mathcal{X} \tau_0 Y : \bigcap_{F \subseteq X} \bigcap_{f : F \to Y} \sum_{f \subseteq f} (f : f \subseteq f) \subseteq \mathcal{X} \tau_0 Y.
\]

or briefly:

\[
\mathcal{X} \tau_0 Y = \bigcap_{F \subseteq X} \bigcap_{f : F \to Y} (f : f \subseteq f).
\]

1.2. \(\mathcal{X} \tau_0 Y\) means that each continuous mapping \(f\) of an arbitrary closed subset \(F\) of the space \(X\) into the space \(Y\) can be extended to some open neighbourhood \(U_F\) of set \(F\) in the space \(X:\)

\[
\mathcal{X} \tau_0 Y = \bigcap_{F \subseteq X} \bigcap_{f : F \to Y} \bigcap_{U_F \subseteq X} (f : f \subseteq f) \subseteq \mathcal{X} \tau_0 Y.
\]

As we know (see [10], p. 260, th. 3), \(Y \in A\) is equivalent to \(\mathcal{X} \tau_0 Y\) for each \(X\) and \(Y \in A\) to \(\mathcal{X} \tau_0 Y\) also for each \(X\).

We introduce a new relation \(\varphi\) by means of the following definition:

1.3. The relation \(\varphi\) at the point \(x \in X\), namely

\[
\mathcal{X} \varphi x Y
\]

means that there exists an arbitrarily small neighbourhood \(U_x\) of the point \(x\) in the space \(X\) whose boundary \(\partial U_x\) and \(\mathcal{X} \varphi x Y\) are:

\[
\mathcal{X} \varphi x Y = \bigcap_{U_x \subseteq X} \sum_{\partial U_x < \varepsilon} (U_x < \varepsilon) \subseteq \mathcal{X} \varphi x Y.
\]

The relation \(\varphi\), namely

\[
\mathcal{X} \varphi x Y
\]

means that at each point \(x\) of the space \(X\) the relation \(\mathcal{X} \varphi x Y\) takes place:

\[
\mathcal{X} \varphi x Y = \bigcap_{U_x \subseteq X} \sum_{\partial U_x < \varepsilon} (d U_x < \varepsilon) \subseteq \mathcal{X} \varphi x Y.
\]

The fundamental connections between the relations defined above are formulated in the following:

1.4. Theorem. The relation \(\tau\) implies the relations \(\tau_0\) and \(\varphi\), but the converse subsumption is, in general, not true:

\[
\mathcal{X} \tau_0 Y \subseteq \mathcal{X} \tau_0 Y, \quad \mathcal{X} \varphi x Y \subseteq \mathcal{X} \varphi x Y,
\]

\[
\mathcal{X} \tau_0 Y \not\subseteq \mathcal{X} \varphi x Y.
\]
moreover the relations $\tau_0$ and $\varphi$ do not imply each other:

\[(1.4.2)\]
\[(\tau_0 \subset \varphi) \sim (\tau \subset \tau_0).\]

The proof of the subsumption (1.4.1) has been reduced to the proof of the two relations: $\tau \subset \tau_0$ and $\varphi \subset \tau_0$, which follow immediately from the definition.

Inequality $\tau \neq \tau_0, \varphi$ follows from $\mathcal{F}(\tau_0, 1)$, $\mathcal{F}(\tau_0, 1)$ and $\mathcal{F}(\varphi, 1)$, $\mathcal{F}(\varphi, 1)$.

To prove (1.4.2) we quote $\mathcal{F}(\tau_0, 1)$ and $\mathcal{F}(\varphi, 1)$ which means that $\sim (\tau_0 \subset \varphi)$. Similarly from $\mathcal{F}(\tau_0, 1)$ and $\mathcal{F}(\varphi, 1)$ it follows that $\sim (\tau \subset \tau_0).

1.5. Some auxiliary relation between the spaces $\mathcal{F}$ and $\mathcal{F}$, satisfying

\[(1.5.1)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]

which will be useful in considering the next considerations.

Let us note that

\[(1.5.2)\]
\[\mathcal{F} \times 0 \subset \mathcal{F} \times \mathcal{F},\]

for $\mathcal{F} \times 0$ is a closed subset of the Cartesian product $\mathcal{F} \times 0$; therefore $\mathcal{F} \times 0 \subset \mathcal{F}$ by (2.1.1); since $\mathcal{F} \times 0 \in \mathcal{F}$, we have $\mathcal{F} \times 0 \subset \mathcal{F}$ accordingly to (2.1.1).

The converse implication is, in general, not true:

\[(1.5.3)\]
\[\sim (\mathcal{F} \times 0 \subset \mathcal{F} \times \mathcal{F}),\]

which is proved by the fact that $\sim (\mathcal{F} \times 0 \subset \mathcal{F})$ according to (4.1.1) and (4.1.2).

2. The arithmetical properties of the relation $\tau$ and the extension of

\subsection{Theorem of the arithmetical properties of the relation $\tau$}

\[(2.1.1)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]

\[(2.1.2)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]

\[(2.1.3)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]

\[(2.1.4)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]

\[(2.1.5)\]
\[\mathcal{F} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F},\]
Proof of (2.2.10). Since \( Y \in A\mathcal{N}B \), it is connected in each dimension (see [10], p. 270), in particular in dimension 0; hence it is locally arcwise connected and as connected, it is also arcwise connected. Thus (see [11], p. 187) \( x \in X \). According to (2.2.1), \( (x) \times x \in Y, \) and thus one can apply (2.1.10).

We proceed to list some lemmas which are generalization of some known properties (cf. [8], p. 84, 94) of the \( n \)-dimensional spheres.

2.3. LEMMA. If \( X \in Y \) and \( X \in X \times Y \), then each continuous mapping of the set \( F \) into the space \( Y \) is trivial, the whole space \( X \):

\[
[(F \subseteq X \in Y) \cdot (X \times Y)] \Rightarrow Y_{\infty} = Y_{\infty} F.
\]

The proof is analogous to that in [8]. For an arbitrary \( f \in Y \), there exist by hypothesis a \( U_{\infty} \) and \( f_1 \) such that \( F \subseteq f_1 \times Y \). Let \( G = E_x \{ \{ x \in F \} \times \epsilon \} \), thus \( F \subseteq G \subseteq U_{\infty} \). By hypothesis there exists \( f_2 \) such that \( f_2 \in G \times f_1 \times Y \).

We define

\[
f_2(x) = \begin{cases} f_1(x) & \text{for } x \in G, \\ f_2(x) & \text{for } x \in X \setminus F. \end{cases}
\]

Since \( f_2 \in G \times f_1 \times Y \), \( F \subseteq f_1 \times Y \), \( i \) is defined on the union of two open sets \( G \) and \( X \setminus F \), it is continuous on \( X \).

2.4. LEMMA. If a space \( X \) is compact, \( X \in Y \) and a mapping \( f \in Y_{\infty} \) is not continuous to the whole space \( Y \) is defined on \( F \subseteq X \), then there exists a subset \( F' \) of the space \( X \) such that the mapping \( f \) is not continuous to the union \( F \cup F' \) but it is continuous to the union \( F \cup F' \), where \( F \) is an arbitrary closed proper subset of the set \( F' \):

\[
[(F \subseteq X_{\text{compact}} \in Y) \cdot (F \times Y_{\infty} = Y_{\infty} F)] \Rightarrow \Rightarrow \sum_{F' \subseteq F} f_1 \in Y_{\infty} F' \times F, \prod_{F' \subseteq F} f_1 y \times Y_{\infty} F, y = 0.
\]

The proof proceeds exactly as in the case of \( Y = \mathbb{S}_n \) (see [8]).

2.5. COROLLARY. If a space \( X \) is compact and \( X \in Y \) but \( X \in Y \), then there exist two closed subsets \( F_1 \) and \( F_2 \) of the space \( X \) such that \( F_1 \subseteq F_2 \) and a mapping \( f \in Y_{\infty} \) which is not continuous to \( F_1 \) but it is continuous to every closed proper subset \( F \) of the set \( F_2 \) containing \( F_1 \):

\[
[X_{\text{compact}} \in Y] \Rightarrow \sum_{F_1 \subseteq F_2} [(F_1 \subseteq F_2) \cdot \prod_{F' \subseteq F_1} f \in Y_{\infty} F', f = 0].
\]

2.6. LEMMA. If \( X \in Y \), then the continuous mappings \( f_1 \) and \( f_2 \) of the space \( X \) into the space \( Y \) such that \( f_1 = f_2 \) and \( X \times Y \), are homotopy:

\[
[(X \in Y) \cdot (f_1, f_2 \in Y_{\infty} \cdot (f_1 x \neq f_2 x) \times Y)] \Rightarrow f_1 = f_2.
\]

The proof is analogous to that of a similar property of \( n \)-dimensional spheres: \( Y = \mathbb{S}_n \) (cf. [8]); it is based on (2.2.6-2) and 2.3.

2.7. LEMMA. If \( X \times Y \in Y \) and the space \( X \) is the union of two closed sets \( F_1 \) and \( F_2 \) such that if \( f_1 \in Y_{\infty} \) for \( k = 1, 2 \), then \( F_{1} \setminus f_1(x) \times f_2(x) \times Y \), and thus there exists an extension of the mapping \( f_1 \) to the whole space \( X \) which is homotopic with the mapping \( f_2 \) on the set \( F_2 \):

\[
[(X \times Y) \cdot (X = F_1 + F_2) \cdot (f_1 \times Y_{\infty} \cdot (f_1 x \neq f_2 x) \times Y)] \Rightarrow \sum_{i \in X} [(f, C f) \cdot (f F_1 = f_2)].
\]

The proof is analogous to that of a similar property of the \( n \)-dimensional spheres: \( Y = \mathbb{S}_n \) (cf. [8], p. 88); it is based on 2.6, on the domain-homotopy of the relation \( \tau \) relative to the closed subsets (10), p. 234) and on the theorem of Borsuk ([4], p. 218); in the following, slightly generalized, form:

\[
[(X \times Y) \cdot (F \subseteq X) \cdot (f_1, f_2 \in Y_{\infty} \cdot (f_1 x = f_2 x) \Rightarrow \sum_{i \in X} [(f_{1, C f} \times Y_{\infty} \cdot (f_{1, x} = f_2)]) =
\]

the proof thereof is an immediate generalization of a well-known proof given by Dowker for \( Y \in A\mathcal{N}B \) (cf. [8], p. 86, or [10], p. 278).

2.8. COROLLARY. If \( X \times Y \in Y \) and the space \( X \) is the union of two closed sets \( F_1 \) and \( F_2 \) such that \( F_1 \cup F_2 \times Y \), and then if the continuous mappings \( f \) and \( g \) to the space \( X \) into the space \( Y \) are homotopic on each of the sets \( F_1 \), then they are homotopic on the whole space \( X \):

\[
[(X \times Y) \cdot (X = F_1 + F_2) \cdot (|F_1 F_2 \times Y]) \Rightarrow f = g.
\]
The proof is analogous to that of a similar property of the \( n \)-dimensional spheres: \( \mathcal{Y} = \mathcal{S}_n \) (cf. [8]); it is based on (3.2.6.1) and 2.7.

2.9. Lemma. If a space \( \mathcal{X} \) is a union of an arbitrary family \( \{G_i\}_{i \in I} \) of such open sets that \( \text{Fr} G_i \times \mathcal{Y} \not\subseteq \mathcal{Y} \) and if \( \mathcal{Y} \) is an \( \alpha \mathcal{N} \mathcal{R} \), then the continuous mapping \( f \) of a closed subset \( F \) of the space \( \mathcal{X} \) which is extendible to the union of the set \( F \) and the closure \( \text{Cl}_i \) of each set of the family \( \{G_i\}_{i \in I} \) is extendible to the whole space \( \mathcal{X} \):

\[
\left[ \bigcup_{i \in I} G_i \right] \times \mathcal{Y} \ni \left( \text{Fr} G_i \times \mathcal{Y} \not\subseteq \mathcal{Y} \right) \implies f \left( \mathcal{X} \times \mathcal{Y} \right) \supseteq \sum_{i \in I} f \left( G_i \times \mathcal{Y} \right) \, f \left( \mathcal{Y} \right).
\]

The proof, exactly as for the analogous property of the \( n \)-dimensional spheres \( \mathcal{Y} = \mathcal{S}_n \) (see [8]), is based on (2.2.3), (3.2.6.1) and 2.7.

2.10. An immediate corollary, which somewhat generalizes lemma 2.9, is the following:

\[
\left[ \bigcup_{i \in I} G_i \right] \times \mathcal{Y} \ni \left( \text{Fr} G_i \times \mathcal{Y} \not\subseteq \mathcal{Y} \right) \implies f \left( \mathcal{X} \times \mathcal{Y} \right) \supseteq \sum_{i \in I} f \left( G_i \times \mathcal{Y} \right) \, f \left( \mathcal{Y} \right) = \mathcal{Y} \left| \left( \mathcal{X} \times \mathcal{Y} \right) \right).
\]

3. The arithmetical properties of the relation \( \varphi \)

3.1. Theorem. The relation \( \mathcal{X} \varphi \mathcal{Y} \) is a topological invariant:

\[
\left( \mathcal{X} \varphi \mathcal{Y} \right) \supseteq \left( \mathcal{X} \varphi \mathcal{Z} \right) \supseteq \left( \mathcal{Y} \varphi \mathcal{Z} \right) \implies \mathcal{X} \varphi \mathcal{Z} \varphi \mathcal{Y}.
\]

Proof. First we prove that

\[
\left( \mathcal{X} \varphi \mathcal{Y} \right) \supseteq \left( \mathcal{X} \varphi \mathcal{Z} \right) \implies \mathcal{X} \varphi \mathcal{Z} \varphi \mathcal{Y}.
\]

Let \( h \) be a homeomorphism \( h(\mathcal{X}) = \mathcal{Z} \). We choose an arbitrary point \( x^* \in \mathcal{X}^* \) and an arbitrary \( \varepsilon > 0 \). Let \( \delta = h^{-1}(\varepsilon) \) and \( \delta = \varepsilon \) such that \( \mathcal{X} - \mathcal{Y} \subseteq \mathcal{Y}(\mathcal{Z}) \). By the hypothesis \( \mathcal{X} \varphi \mathcal{Y} \) there exists in \( \mathcal{X} \) such an \( U \) that \( d U < \varepsilon \) and \( \text{Fr} U \times \mathcal{Y} \). Let \( \mathcal{U} = \mathcal{U}(\mathcal{X}) \) be an open neighbourhood of the point \( x \in \mathcal{X} \) and \( h(U) \subseteq \mathcal{Y}(\mathcal{Z}) \); thus \( d h(U) < \varepsilon \). Since \( h \) is a homeomorphism of \( \mathcal{X} \) onto \( \mathcal{Z} \), we have \( h(\mathcal{U}) = \mathcal{U}(\mathcal{Z}) \). Hence by \( \text{Fr} U \times \mathcal{Y} \) and (2.1.1) it follows that \( \text{Fr}_h(U) \times \mathcal{Y} \) or \( \mathcal{X} \varphi \mathcal{Z} \varphi \mathcal{Y} \).

It remains to prove the implication

\[
\left( \mathcal{X} \varphi \mathcal{Y} \right) \supseteq \left( \mathcal{X} \varphi \mathcal{Z} \right) \implies \mathcal{X} \varphi \mathcal{Z} \varphi \mathcal{Y},
\]

which results immediately from (1.3.1) and (2.1.1).

3.2. Theorem. The relation \( \varphi \) is domain-hereditary for the closed subsets

\[
F \subset \mathcal{X} \varphi \mathcal{Y} = F \varphi \mathcal{Y}
\]

and also for the open subsets

\[
G \subset \mathcal{X} \varphi \mathcal{Y} = G \varphi \mathcal{Y}.
\]

Proof of (3.2.1). By hypothesis for an arbitrary \( x \in F \) and an arbitrary \( \varepsilon > 0 \) there exists in \( \mathcal{X} \) such an \( U_x \) that \( d U_x < \varepsilon \) and \( \text{Fr} U \times \mathcal{Y} \). The set \( U = F U_x \) is an open neighbourhood of the point \( x \) relative to the set \( F \), where \( d U < \varepsilon \). Its boundary relative to \( F \) is \( \text{Fr} U \times \mathcal{Y} \); thus, by (2.1.2), \( \text{Fr} U \times \mathcal{Y} \), which proves that \( F \varphi \mathcal{Y} \).

Proof of (3.2.2). Let \( x \in G \). Inequality \( 0 < \varepsilon < \varphi(x, \mathcal{X} - G) \) implies by hypothesis that there exists in \( X \) an \( U_x \) such that \( d U_x < \varepsilon \) and \( \text{Fr} U \times \mathcal{Y} \). Hence \( U_x \subset G \), and thus \( U_x \) is an open neighbourhood of the point \( x \) relative to the set \( G \), where \( \text{Fr} U_x = \text{Fr} U \); therefore \( \text{Fr} U_x \times \mathcal{Y} \) and \( G \varphi \mathcal{Y} \).

3.3. Theorem. The relation \( \varphi \) is finite domain-additive for the closed sets

\[
(F \varphi \mathcal{Y})^k = \bigcap_{k=1}^n F \varphi \mathcal{Y}.
\]

We shall present the proof, which in general is an inductive one, for the case \( n = 2 \), using the following decomposition in disjoint terms:

\[
F_1 \cup F_2 = F_1 \cap (F_2 - F_1) = (F_2 - F_1).
\]

If \( x \in F_1 \) and \( \varepsilon > 0 \), then there exists by hypothesis such an open neighbourhood \( U_x \) of the point \( x \) in \( F_1 \) that \( d U_x < \varepsilon \) and \( \text{Fr} U \times \mathcal{Y} \). The set \( U = U_x \) is an open neighbourhood of the point \( x \) in \( F_1 \), where \( d U < \varepsilon \) and \( \text{Fr} U_x \subset \text{Fr} U \); thus, by (2.1.1) and (2.1.2), \( \text{Fr} U \times \mathcal{Y} \), which proves that \( F_1 \varphi \mathcal{Y} \).

If \( x \in F_2 - F_1 \), then by hypothesis there exists a neighbourhood \( U \) of the point \( x \) relative to \( F_1 \) such that \( d U < \varphi(x, F_1) \) and \( \text{Fr} U \times \mathcal{Y} \). Since \( U \varphi \mathcal{Y} = \text{Fr} U \varphi \mathcal{Y} \), which is an open neighbourhood of the point \( x \) relative to \( F_1 \), means that \( F_1 \varphi \mathcal{Y} \).

For \( x \in F_2 - F_1 \), the proof is exactly as above.

3.4. Theorem. The relation \( \varphi \) is domain-additive for open sets:

\[
\left( \bigcap_{i \in I} G_i \varphi \mathcal{Y} \right) = \bigcap_{i \in I} \bigcap_{x_i \in G_i} G_i \varphi \mathcal{Y}.
\]

Proof. For an arbitrary \( x \in \bigcap_{i \in I} G_i \), there exists a certain \( G_a \) such that \( x_a \in G_a \). By hypothesis there exists in \( G_a \) an \( U_a \) such that \( d U_a < \varphi(x_a, \bigcap_{i \in I} G_i) \); thus \( U_a \subset G_a \), and \( \text{Fr} U_a \times \mathcal{Y} \). Since \( G_a \) is an open
3.5. Lemma. If $\mathcal{A} \not\subseteq \mathcal{Y}$, then in the space $\mathcal{X}$ there exists a countable sequence $(U_n)_{n \in \mathbb{N}}$ of open sets such that $\text{Fr} U_n \not\subseteq \mathcal{Y}$, which is a base for open sets of the space $\mathcal{X}$:

$$\mathcal{A} \not\subseteq \mathcal{Y} \Rightarrow \bigcup_{n \in \mathbb{N}} \left( \bigcap_{G \in \mathcal{Y}, \text{Fr} U_n \not\subseteq G} G \right) \cap \bigcap_{n \in \mathbb{N}} U_n.$$

Proof. By hypothesis there exists for each $x \in \mathcal{A}$ and for each $n \in \mathcal{N}$ an open neighbourhood $U^n_n$ in the space $\mathcal{X}$ such that $
abla U^n_n < 1/n, \text{Fr} U^n_n \not\subseteq \mathcal{Y}$ and $\mathcal{A} = \bigcap_{n \in \mathbb{N}} U^n_n$.

By virtue of Lindelöf's theorem, we may select a countable covering $(U^n_n)_{n \in \mathbb{N}}$ of the covering $(U^n_n)_{n \in \mathbb{N}}$. The double sequence $(U^n_m)_{m,n \in \mathbb{N}}$ can be ordered into the sequence $(U^n_0, U^n_1, U^n_2, \ldots) = (U^n_n)_{n \in \mathbb{N}}$ where

$$\sum_{n \in \mathbb{N}} U^n_n = \mathcal{A}, \text{Fr} U^n_n \not\subseteq \mathcal{Y} \quad \text{and} \quad \mathcal{A} = \bigcap_{n \in \mathbb{N}} U^n_n$$

with sufficiently great indexes are arbitrarily small.

Hence for any $G \in \mathcal{X}$ and each $n \in \mathcal{N}$ there exists a neighbourhood $U^n_n \in (U^n_n)$ with diameter $d U^n_n < 1/n, \text{Fr} U^n_n \not\subseteq \mathcal{Y}$. Hence $x \in \mathcal{A}$, $x \in \mathcal{X} \cap G$ or

$$G \cap \bigcap_{n \in \mathbb{N}} U^n_n \subseteq G \quad \text{and} \quad \mathcal{A} = \bigcap_{n \in \mathbb{N}} U^n_n.$$ Ordering the sequence $(U^n_n)$ according to the increasing indexes, we receive a subsequence $(U^n_n)$ satisfying the lemma.

3.6. Theorem of Decomposition. If $\mathcal{A} \not\subseteq \mathcal{Y}$ and $\mathcal{Y}$ is an ANR, then there exist sets $A$ and $B$ such that $\mathcal{X} = A + B$ and $AB = 0$, where $A \in \mathcal{A}$ and $B \in \mathcal{A}$.

$$\mathcal{A} \not\subseteq \mathcal{Y} \Rightarrow \bigcap_{A \in \mathcal{X}} \left( (A \not\subseteq \mathcal{Y}) \cdot (A \not\subseteq \mathcal{Y}) \right).$$

Remark. In view of [5], p. 392, it is enough to assume that the space $\mathcal{Y}$ be a local absolute neighbourhood retract.

Proof. By virtue of lemma 3.5 there exists in $\mathcal{X}$ a sequence $(U_n)$ of open sets which are a base in $\mathcal{X}$ and for which $\text{Fr} U_n \not\subseteq \mathcal{Y}$. Let

$$A = \bigcap_{n \in \mathbb{N}} \text{Fr} U_n; \quad B = \bigcap_{n \in \mathbb{N}} U_n,$$

hence $A$ is a $\mathcal{A}$ and from 2.14 it follows that $A \not\subseteq \mathcal{Y}$.

The set $B = \mathcal{X} - A$ is $\mathcal{A}$ and for any $U_n$ we have $B \mathcal{X} U_n \not\subseteq 0$, whence each point $x \in B$ possesses an arbitrarily small neighbourhood $U_{n(x)}$, whose boundary relative to the set $B: \text{Fr} U_{n(x)} \not\subseteq 0$, which means that $\dim B \leq 0$.

The converse implication can be proved under a weaker hypothesis:

3.7. $(A \not\subseteq \mathcal{Y} \cdot \dim B = 0) \Rightarrow A \not\subseteq \mathcal{Y}$.

Proof. For any $x \in A + B$ we have from [9], p. 173, $\dim (B - \{x\}) = 0$. Thus (see [5], p. 27) for each $\varepsilon > 0$ there exists a $U_\varepsilon$ in $A + B$ such that $d U_\varepsilon < \varepsilon$ and $\text{Fr} U_\varepsilon \subset A$. By hypothesis and (2.1.2) we have $\text{Fr} U_{n(x)} \not\subseteq \mathcal{Y}$, whence $A \not\subseteq \mathcal{Y}$.

3.8. Theorem. If a space $\mathcal{Y}$ is an ANR, then the relation $\varphi$ relative to the space $\mathcal{Y}$ is $\mathcal{A}$-domain additive:

$$F_k \varphi \mathcal{Y} \in \mathcal{A} \not\subseteq \mathcal{Y} \Rightarrow \bigcap_{k=1}^{\infty} F_k \varphi \mathcal{Y}.$$

Proof. Let $\mathcal{X} = \bigcup_{k=1}^{\infty} F_k$. We define inductively a sequence of sets $(C_m)_{m=1}^\infty$ as follows:

$$C_1 = F_1, \quad C_m = F_m - \bigcap_{k=1}^{m-1} F_k \quad \text{for} \quad m = 2, 3, \ldots$$

The sets $C_m$ are mutually disjoint and $\mathcal{X} = \bigcup_{m=1}^{\infty} C_m$. Since $C_m = F_m \{ \mathcal{X} - \bigcup_{k=1}^{m-1} F_k \}$ it is itself an $F_k$ as an intersection of two sets $F_k$. Since $C_m$ is an open set in $F_k$, in view of (3.2.2), $C_m \not\subseteq \mathcal{Y}$. By virtue of 3.5 there exist such sets $A_m$ and $B_m$ that $C_m = A_m + B_m, A_m B_m = 0, A_m$ is an $F_k$ and $A \not\subseteq \mathcal{Y}$, $B_m$ is an $G_\delta$ and $\dim B_m \leq 0$. Let us write $A = \bigcup_{m=1}^{\infty} A_m$ and $B = \bigcup_{m=1}^{\infty} B_m$. We have $\mathcal{X} = A + B$, where $\mathcal{X} \not\subseteq \mathcal{Y}$ from (3.1.5). Since $B_m \subset C_m$ and since the sets $C_m$ are mutually disjoint, we have $B_m = B_m C_m = \bigcup_{k=m}^{\infty} B_k C_m = B C_m$.

In virtue of $C_m \subset F_k$ the set $B_m$ is an $F_k$ relative to $B$, and therefore (see [9], p. 173) $\dim B \leq 0$. Hence, by 3.7, $\mathcal{X} \not\subseteq \mathcal{Y}$.

3.9. Corollary. $(A_k \subset F_k, A_k \varphi \mathcal{Y} \in \mathcal{A} \not\subseteq \mathcal{Y}) = \bigcup_{k=1}^{\infty} A_k \varphi \mathcal{Y}.$

Proof. $A_k = \bigcup_{m=1}^{\infty} F_k$, where, from (3.2.1), $F_k \varphi \mathcal{Y}$. Since $\bigcup_{k=1}^{\infty} A_k = F_k + F_k + F_k + \ldots$, we have $\bigcup_{k=1}^{\infty} A_k \varphi \mathcal{Y}$ by 3.8.
3.10. Corollary. $(\{A \in F_s, G_1\} \cdot (A, B \in \mathcal{F} \wedge \mathcal{F} \in A.N.R)) \Rightarrow A + B \in \mathcal{F}
$. Proof. Since $(A + B) - A \in F_s$, it is enough to apply 3.9.

3.11. Corollary. $F, B \in \mathcal{F} \wedge \mathcal{F} \in A.N.R \Rightarrow F + B \in \mathcal{F}$.
Proof. Since $F \in F_s, G_{1,2}$ it is enough to apply 3.10.

3.12. Corollary. $(\mathcal{F}, \mathcal{F} \in A.N.R, \mathcal{F} \neq 0) \Rightarrow \mathcal{F} + \mathcal{(x) \in \mathcal{F}}$. It results immediately from 3.11 and 4.3.

3.13. Corollary. $(\{A_1, A_2, \ldots, A_n \in \mathcal{F}, \sigma A \in \mathcal{F} \}) \Rightarrow A_1 + A_2 \mathcal{F}$. Proof. The sets $A_i = (A_1 + A_2) - A_1$ and $A_i = (A_1 + A_2) - A_2$ are open relative to $A_1 + A_2$, and therefore it is enough to apply 3.4.

3.14. Corollary. If $\mathcal{F} \in \mathcal{F}$, then $\mathcal{F} + (x) = \mathcal{F}$. If $x \in \mathcal{F}$, the point $x$ is isolated in $\mathcal{F} + \mathcal{(x) \in \mathcal{F}}$, then $\mathcal{F}$ and $(x) \in \mathcal{F}$ are separated. Hence, in view of 4.3, 3.13 can be applied.

4. The connections with dimension theory

4.1. Theorem. A space $\mathcal{F}$ is at most $n$-dimensional if and only if $\mathcal{F} \in \mathcal{S}_n$.

$$\dim \mathcal{F} \leq n \Rightarrow \mathcal{F} \in \mathcal{S}_n.$$ (4.1.1)

A compact space $\mathcal{F}$ is at most $(n-1)$-dimensional if and only if $\mathcal{F} \cap \mathcal{F} \in \mathcal{S}_{n-1}$.

$$\dim \mathcal{F} \leq n-1 \Rightarrow \mathcal{F} \cap \mathcal{F} \in \mathcal{S}_{n-1}.$$ (4.1.2)

A space $\mathcal{F}$ is at the point $x$ at most $(n-1)$-dimensional if and only if $\mathcal{F} \cap \mathcal{F} \in \mathcal{S}_n$.

$$\dim \mathcal{F} \leq n-1 \Rightarrow \mathcal{F} \cap \mathcal{F} \in \mathcal{S}_n.$$ (4.1.3-1)

A space $\mathcal{F}$ is at most $(n+1)$-dimensional if and only if $\mathcal{F} \in \mathcal{S}_n$.

$$\dim \mathcal{F} \leq n+1 \Rightarrow \mathcal{F} \in \mathcal{S}_n.$$ (4.1.3-2)

For the proof of the well-known fundamental condition (4.1.1) see [8], p. 83, or [10], p. 271.

In order to prove (4.1.2) let us note that if $\dim \mathcal{F} \leq n-1$, then $\dim(\mathcal{F} \cap \mathcal{F}) \leq n$ and (4.1.1) can be applied. Whereas if $\mathcal{F} \in \mathcal{F} \in \mathcal{S}_n$, then $\dim(\mathcal{F} \cap \mathcal{F}) \leq n$ by (4.1.1) and from the compactness of the space $\mathcal{F}$ it follows that $\dim(\mathcal{F} \cap \mathcal{F}) = \dim \mathcal{F} + 1$, whence $\dim \mathcal{F} \leq n-1$.

The hypothesis of compactness of the space $\mathcal{F}$ is indispensable only for the proof of the sufficiency of condition (4.1.2).
For the proof it is enough to assume, in 4.4, $F_1 = (v)$ for $x \in X$. In view of (16), p. 233, $(x) \neq Y$ and all the hypotheses of the lemma are satisfied.

In particular, if the space $X$ is compact and $Y = S_n$, where $n > 1$, the hypothesis of 4.5 is a sufficient condition for $\dim X \leq n$, exactly as with the inductive definition of dimension.

4.6. The theorem can be a basis for some extension of the analogy of connection between the auxiliary relation defined in 1.5 and relation $\tau$ on the one hand and the inductive definition of dimension on the other.

These relations give rise to a number of concepts, analogous to the concepts in the dimension theory, for which analogous properties can be proved (e.g., heredity or additivity).

For instance, analogously to the set $A_{n0}$ (see [9], p. 164), one can define for the set $A \subset X$ the sets:

$$A_{n0} = \prod_{x \in X} \sum_{\tau \leq 0} \sum_{U \subset X} [(d U < \sigma) \cdot (A \setminus U \tau Y)]$$

$$A_{n0} = \prod_{x \in X} \sum_{\tau \leq 0} \sum_{U \subset X} [(d U < \sigma) \cdot (A \setminus U \tau \mathcal{X} \tau Y)].$$

The analogue for the dimension-kernel (see [9], p. 180) is the $\sigma$-kernel of the space $X$ relative to the space $Y$, which is defined as $A_{n0} - A_{n0}$. Theorems analogous to those of the dimension theory can be demonstrated, e.g., that the sets $A_{n0}$ and $A_{n0}$ are $G_0$ or the implication $X \neq Y \in \text{ANR} \Rightarrow A_{n0} - A_{n0} \tau Y$.

Other examples: the decomposition-theorem (the analogue: 3.6 and 3.7), Hurewicz's extension theorem for closed sets (see [7], p. 119), the properties of the cyclic elements in the locally connected continuum (see [10], p. 335) and finally the properties or Urysohn's coefficient (see [10], p. 60).

Now we shall deal with the connections between the properties of the relations under discussion and the problem of the disconnection of a space.

4.7. Theorem. If $X \neq Y$, then each pair of different points $x_1$ and $x_2$ of the space $X$ can be separated by means of a set $F \subset X$ such that $X \neq Y$.

$$x_1, x_2 \in X \neq Y, x_1 \neq x_2 \Rightarrow \sum_{F, M_1, M_2, \mathcal{X} \in X} [((A \setminus F \cap M_1 + M_2) \cdot (M_1 \cap M_2 = 0) \cdot (F \tau Y)].$$

Proof. If $X \neq Y$, the space is uni-ordered relative to the family of the boundaries of the open neighbourhoods $U \subset X$ such that $\text{Fr } U \subset Y$ (see [13], p. 189). This implies the theorem.

4.8. Theorem. If $X \neq Y$, $Y$ is ANR and $Y \neq 0$, then each pair of the closed and disjoint sets in the space $X$ can be separated by a set $F \subset X$ such that $F \tau Y$:

$$[(F_1, F_2 \subset X \neq Y) \neq 0 \Rightarrow (F_1 \tau 0) \Rightarrow \sum_{F, M_1, M_2, \mathcal{X} \in X} [(A \setminus F \cap M_1 + M_2) \cdot (M_1 \cap M_2 = 0) \cdot (F \tau Y)].$$

Proof. There exists a continuous mapping $f$ of the space $X$ into a metric separable space $\mathbb{R}$ (e.g., a subset of Hilbert's cube) transforming the sets $F_1$ and $F_2$ into two different points of the space $\mathbb{R}$:

$$f(F_1) = x_1 \neq x_2 = f(F_2),$$

where $x_1, x_2 \epsilon f(A \setminus (F_1 + F_2))$; the partial mapping $f(A \setminus (F_1 + F_2))$ is homeomorphism (see [9], p. 139, theorem 2).

Since $X \neq (F_1 + F_2) \neq Y$ by (3.2.2), we have $f(A \setminus (F_1 + F_2)) \neq Y$ by 3.1. Since $f(A) = f(A \setminus (F_1 + F_2)) + [x_1] + [x_2]$, applying twice 3.12 we receive $f(A) \neq Y$.

In view of 4.7, there is in $f(A)$ a neighbourhood $U_1$ such that $x_2 \epsilon U_1$, $F_1 \subset U_1 \tau Y$ and that for $F_1 \subset U_1$ separates $x_1$ and $x_2$ in $X$. Since $f(U_1) = f(U_1)$, $U_1 \tau Y$ and $x_2 \neq f(U_1)$, we have $x_2 \tau U_1$ and $F_1 \tau U_1$ and on the other hand $F_1 \tau U_1$ separates the sets $F_1$ and $F_2$ in the space $X$.

In the above theorem the hypothesis $Y \neq ANR$ can be replaced by that of the compactness of the space $X$ (compare 3.12 and 3.14).

4.9. Corollary. $0 \neq Y \in \text{ANR} \Rightarrow \left[ f(A) \neq Y \Rightarrow \sum_{F, M_1, M_2, \mathcal{X} \in X} [(A \setminus F \cap G) \cdot (F \tau Y).$$

Proof. The necessity of the condition follows from the proof of 4.8. In order to prove the sufficiency let $x \in X$, $x > 0$ and let $U_0$ be an open neighbourhood in $X$ such that $\text{Fr } U_0 < e$. We put $F_1 = (x)$ and $F_2 = X - U_0$. The sets $F_1$ and $F_2$ are closed and disjoint. Hence, by hypothesis, there exists an open set $G \cap F_1 = (x)$ as well as $G \cap F_2 = 0$. The set $G$ is an open neighbourhood of the point $x$ in the space $X$ and $G \cap (X - U_0) = 0$ implies $G \subset U_0$, whence $x < e$, which together with $F_1 \tau Y$ proves that $X \neq Y$.

The hypothesis $Y \neq ANR$ can be replaced by the hypothesis of the compactness of the space $X$.

The corollary corresponds, in some sense, to a theorem of Tumarkin on the equivalence of the definition of inductive dimension, ind $X$, to that of an inductive macro-dimension of Urysohn, Ind $X$ (see [1], p. 46).
The properties considered above give rise to the following possibility of some generalization of the notion of \( n \)-dimensional Cantor manifold.

4.10. A compact space \( \mathcal{X} \) is said to be a \( \varphi \)-manifold relative to a space \( \mathcal{Y} \) if \( \mathcal{X} \not\subseteq \mathcal{Y} \) and there is no set \( A \subseteq \mathcal{X} \) such that \( A\times\mathcal{Y} \) disconnects the space \( \mathcal{X} \).

The spaces which are \( \varphi \)-manifolds relative to \( S_n \) are \((n+1)\)-dimensional Cantor manifolds.

To the well-known property of \( n \)-dimensional Cantor manifolds, namely that they are \( n \)-dimensional at every point, one can find an anagouls property, namely that if a space \( \mathcal{X} \) is a \( \varphi \)-manifold to a space \( \mathcal{Y} \), then \( \mathcal{X} \not\subseteq \mathcal{Y} \).

Let us note that a space which is a \( \varphi \)-manifold relative to a non-empty space is connected, which is an analogue to the connectivity of an \( n \)-dimensional Cantor manifold.

4.11. Lemma. The following three conditions are equivalent:

\[
\sum_{A, M, \mathcal{M} \subseteq \mathcal{X}} [(M \neq 0) \cdot (\mathcal{X} - A = M_1 + M_2) \cdot (M_1, M_2, \mathcal{M}) = 0] \cdot (A \times \mathcal{Y}),
\]

\[
\sum_{\mathcal{G} \subseteq \mathcal{X}} [(0 \neq 0) \cdot (\mathcal{G} \neq \mathcal{X}) \cdot (\mathcal{F} \times \mathcal{G} \times \mathcal{Y})],
\]

\[
\sum_{F, F' \subseteq \mathcal{X}} [(0 \neq F 
eq \mathcal{X}) \cdot (\mathcal{X} = F + F') \cdot (F, F' \times \mathcal{Y})].
\]

The proof of the three implications: (4.11.1) \( \Rightarrow \) (4.11.2) \( \Rightarrow \) (4.11.3) = \( \Rightarrow \) (4.11.1) is exactly as in [8], p. 47. It is based on (2.2.2).

12. Theorem. A compact space \( \mathcal{X} \) such that \( \mathcal{X} \not\subseteq \mathcal{Y} \) is a \( \varphi \)-manifold relative to the space \( \mathcal{Y} \) if and only if the space \( \mathcal{X} \) cannot be decomposed into the union of the closed sets \( F_1 \) and \( F_2 \) such that \( \mathcal{X} = F_1 + F_2 \), \( F_1 \neq 0 \neq F_2 \), \( \mathcal{X} - F_1 = 0 \neq \mathcal{X} - F_2 \) and \( F_1, F_2 \times \mathcal{Y} \).

The proof is contained in that of 4.11, namely in the equivalence of the conditions (4.11.1) and (4.11.3).

4.13. Theorem. If the space \( \mathcal{X} \) is compact and \( \mathcal{X} \not\subseteq \mathcal{Y} \) then the space \( \mathcal{X} \) contains a \( \varphi \)-manifold relative to the space \( \mathcal{Y} \).

Proof. It follows from the hypothesis \( \mathcal{X} \not\subseteq \mathcal{Y} \) that there exist an \( F_0 \subseteq \mathcal{X} \) and an \( f_0 \in \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \). According to 2.4 there exists an \( F \subseteq \mathcal{X} \) such that \( f_0 \in \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \), whereas \( f_0 \in \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \). The set \( F \) is a \( \varphi \)-manifold relative to the space \( \mathcal{Y} \). For, in the contrary case, there exist by (4.11.3) two closed sets \( F_1 \) and \( F_2 \) such that \( F = F_1 + F_2 \), \( 0 \neq F_1, F_2 \not\subseteq \mathcal{X} \) and \( (F_1, F_2) \times \mathcal{Y} \).

It follows from the above formulated irreducibility of the set \( \mathcal{F} \) that there exist two extensions \( F_1 \) and \( F_2 \) of the mapping \( f_0 \) on the unions \( F_1 + F_2 \) and \( F_1 + F_2 \) such that \( F_1 \subset f_0 \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \). Since the set \( \mathcal{F} \cdot \mathcal{Y} \) is open in \( F_1, F_2 \), we have \( \mathcal{F} \cdot \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \cdot \mathcal{X} \cdot \mathcal{Y} \). Hence, by 2.7, the mapping \( f_0 \) may be extended to the union \( F_1 + F_2 \), which contradicts the definition of the set \( \mathcal{F} \).

5. Remarks and problems

5.1. The properties of the relations \( \tau \) and \( \varphi \) which have been proved above suggest some connexion between these and the set-families dimensionizing the space ("les familles dimensionnantes", see [9], p. 187). Since we do not know whether those relations are for any subset denyhereditary or not, we have to assume a weaker hypothesis than those in [6] when transferring the results of the theory of these families. Another paper will be devoted to these problems.

5.2. Next to the problem \( \mathcal{F} \) of [11] the problems

\[
(5.2.1) \quad A \subseteq \mathcal{X} \not\subseteq \mathcal{Y} \Rightarrow A \varphi \mathcal{Y}
\]

and

\[
(5.2.2) \quad [(A \subseteq \mathcal{X}) \cdot (A \times \mathcal{X}) \not\subseteq \mathcal{Y}] \Rightarrow A \times \mathcal{X} \not\subseteq \mathcal{Y}
\]

can be discussed. If \( \mathcal{Y} = S_n \), both implications are true.

5.3. The problem of the existence of some analogue to the theorem on the dimension of a Cartesian product of two sets (see [9], p. 228) among the properties of the relations under consideration suggests the question whether

\[
(5.3.1) \quad \mathcal{X} \times \mathcal{Y} \in \mathcal{A} \Rightarrow \mathcal{X} \times \mathcal{Y} \not\subseteq \mathcal{Y}.
\]

If \( \mathcal{Y} = S_n \), the implication is true, namely the Cartesian product of a set at most \( n \)-dimensional by \( \mathcal{Y} \) is at most \((n+1)\)-dimensional.

If an implication converse to 4.1 is true:

\[
(5.3.2) \quad \mathcal{X} \times \mathcal{Y} \in \mathcal{A} \Rightarrow \prod_{\lambda > 0} \sum_{\mathcal{A} \subseteq \mathcal{X}} [(\text{U}_\lambda < e) \cdot (\mathcal{F} \times \mathcal{X} \times \mathcal{Y} \in \mathcal{A} \Rightarrow \mathcal{X} \times \mathcal{Y} \not\subseteq \mathcal{Y})],
\]

then the evident implication

\[
(5.3.3) \quad \prod_{\lambda > 0} \sum_{\mathcal{A} \subseteq \mathcal{X}} [(\text{U}_\lambda < e) \cdot (\mathcal{F} \times \mathcal{X} \times \mathcal{Y} \in \mathcal{A} \Rightarrow \mathcal{X} \times \mathcal{Y} \not\subseteq \mathcal{Y})]
\]

immediately implies (5.3.1).
Moreover, exactly as 3.6 and 3.7, it can be proved that (5.3.2) is equivalent to the implication

\[ (5.3.4) \quad \mathcal{I} \tau \mathcal{Y} \in A \mathcal{N} \mathcal{Y} \Rightarrow \sum_{AB} [(\mathcal{I} = A + B) \cdot (A \times \mathcal{I} \tau \mathcal{Y}) \cdot (\dim B \leq 0)], \]

which is true for \( \mathcal{Y} = \mathcal{S}_n \).

5.4. It is not known whether if the space \( \mathcal{I} \) is a \( \varphi \)-manifold relative to the space \( \mathcal{Y} \in A \mathcal{N} \mathcal{Y} \), then \( \mathcal{I} \tau \mathcal{Y} \). If \( \mathcal{Y} = \mathcal{S}_n \), the answer is positive.

5.5. It has been proved in [11] that the relation \( \tau \) is domain invariant relative to the \( \varepsilon \)-transformations. It is not known whether the same is true for the relation \( \varphi \). It is true if \( \mathcal{Y} = \mathcal{S}_n \).

5.6. The considerations of chapter 4 suggest the idea of a classification of spaces by means of the following congruence according to the relation \( \tau \).

The set \( A \) is congruent to the set \( B \) according to the relation \( \tau \);

\[ A \equiv B, \]

if and only if for each space \( \mathcal{Y} \) which is \( A \mathcal{N} \mathcal{Y} \) the following equivalence is true: \( A \tau \mathcal{Y} \equiv B \tau \mathcal{Y} \).

We shall notice that:

1. the relation \( \equiv \) belongs to the equivalence, type;

2. \( A = B \equiv \lambda = B \), but the converse implication is not true, which

is indicated by the example of spaces with different finite powers;

3. \( A = B \equiv \dim A = \dim B \), if the \( \dim X = \infty \) we consider as defined

(cf. [8], p. 24).

The following problem arises in connection with the above relationships:

What are the characteristics of spaces which are \( \tau \)-related to some defined space, for instance to the torus?

It is easy to see (4.1 and [10], p. 256) that a sufficient condition is for instance that the space be at most one-dimensional. A condition both sufficient and necessary is not known.

Bibliography


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