On the extensibility of mappings, their local properties and some of their connections with the dimension theory

by

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Introduction

The relations τ and τ_v defined by Kuratowski (see [10] (1), p. 252) are in a certain sense a generalization of those stated in an important theorem of Tietze ([9], p. 117) and have been applied to the characterization of the important classes of spaces distinguished by Borsuk, such as the absolute retracts (see [2], p. 159), the absolute neighbourhoods retracts (see [3], p. 222) and many others. These relations also possess a number of interesting properties (compare for instance [10] and [11]).

Special attention should be paid to the connection between the dimension of the space of arguments and the extension of the continuous mappings into an n-dimensional sphere. The above connection as well as a number of other interesting properties of the relation τ have been discussed in chapter VII of book [10] by Kuratowski, and also in his paper [11] specially devoted to these problems.

In this paper (2) further properties of the relation τ (see section 2) — in particular its local properties — are investigated by means of a relation φ (see sections 1 and 3) specially defined for this purpose; some close and natural analogies with the theory of dimension are also discussed (see section 4). These considerations show the role of the extensibility of continuous mappings not only for the dimension of sets but also for some derivative notions, for instance: the disconnection of a space, the

(1) The numbers in brackets refer to the bibliography at the end of the paper.

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coefficient of Urysohn, the dimension of connectivity, the manifold of Cantor, n-dimensionally connected spaces. Some possibilities of a classification of spaces, analogous to the dimensional one, are to be expected (see section 5.6).

This is in some connexion with the notions of the normal families of Hurewicz [6] and of the uni-ordered spaces of G. T. Whyburn [13].

The following considerations may be regarded as a supplement to paper [1] by P. S. Aleksandrov, in which various approaches to the notion and the theory of dimension have been discussed.

We shall use the following notation: Let $\{x, y, z, \ldots\}$ mean a set consisting of the elements $x, y, z, \ldots$; in particular $\langle a, b\rangle = \big\{\{a\}, \{a, b\}\big\}$ means an ordered pair. (a, b) means an open interval, whose ends are a and b, $[a, b]$ means the corresponding closed interval; in particular $\mathcal{I} = [0, 1]$. By $\mathcal{E}^n$ we shall mean the n-dimensional Euclidean space, by $\mathcal{E}$ — the set of real numbers, by $\mathcal{N}$ — the set of positive integers.

By $\mathcal{S}_n$ we shall denote the n-dimensional sphere in the $(n+1)$-dimensional Euclidean space: $\mathcal{S}_n = \underset{x}{E}[x \in \mathcal{E}^{n+1}\cdot |x| = 1]$. By $\mathcal{Q}_n$ we denote an n-dimensional disk in the n-dimensional Euclidean space: $\mathcal{Q}_n = \underset{x}{E}[x \in E^{n+1}\cdot |x| \leqslant 1]$, $\mathcal{Q}_\varepsilon(x)$ means a spherical neighbourhood of the point x with the radius $\varepsilon > 0$ in the metric space $X: \mathcal{Q}_\varepsilon(x) = \underset{y}{E}[y \in X\cdot \varrho(x, y) < \varepsilon]$.

By $\bar{R}$ we shall denote the complement of the relation $R: x\bar{R}y \Longleftrightarrow \sim(xRy)$, by $R \subset S$ we mean the subsumption of the relation R relative to the relation $S: R \subset S \Longleftrightarrow (xRy \Rightarrow xSy)$ and by $R\cdot S$ the product of relations R and $S: x(R\cdot S)y \Longleftrightarrow [(xRy)\cdot(xSy)]$.

By $f \in \mathcal{Y}^{\mathcal{X}}$ we shall denote that f is a continuous mapping of a space $\mathcal{X}$ into a space $\mathcal{Y}$. Two mappings f and g will be equal, $f = g$, if they assign to each value of the argument the same value $f = g \Longleftrightarrow (f(x) = g(x))$. By $f|F$ we shall denote a partial mapping (restriction of f to F), namely $f \in \mathcal{Y}^{\mathcal{X}}$ restricted to $F \subset \mathcal{X}$. The mapping f^* is said to be an *extension* of the mapping $f \in \mathcal{Y}^A$ to the space $\mathcal{X} \supset A$:

$$f \subset f^*,$$

if $f^* \in \mathcal{Y}^{\mathcal{X}}$ and $f^*|A = f$. By $\mathcal{Y}^{\mathcal{X}}|F$ we shall denote the set of the mappings $f \in \mathcal{Y}^F$ which can be extended to the whole space $\mathcal{X}$:

$$\mathcal{Y}^{\mathcal{X}}|F = \underset{f}{E}\,[(f \in \mathcal{Y}^F)\cdot \sum_{f^*}(f \subset f^* \in \mathcal{Y}^{\mathcal{X}})]\,.$$

$\mathcal{X}_1 \underset{\text{top}}{=} \mathcal{X}_2$ will denote that the sets X_1 and X_2 are homeomorphic, and $f_0 \simeq f_1$ that the mappings $f_0, f_1 \in \mathcal{Y}^{\mathcal{X}}$ are homotopic.

Closed sets will be denoted by F and open sets by G, with an index, if necessary; the open neighbourhoods for the point x by U_x and those for the set F by U_F. The boundary of the set A relative to the set C will be denoted by $\mathrm{Fr}_C A$. The diameter of the set A will be denoted by dA.

We introduce the following abbreviations: AR will mean the absolute retract and ANR the absolute neighbourhood retract.

We restrict our considerations to the *metric separable spaces*, though the proofs of many of the properties are also valid for more general spaces. Only some additional hypotheses, for instance compactness or connectivity, will be named explicitely.

1. Definitions of the fundamental relations and connections between them

In [10] on p. 252 Kuratowski has introduced the relations τ and τ_v by means of the following definitions:

1.1. $\mathcal{X}\,\tau\,\mathcal{Y}$ means that each continuous mapping f of an arbitrary closed subset F of the space $\mathcal{X}$ into the space $\mathcal{Y}$ can be extended to the whole space $\mathcal{X}$:

$$\mathcal{X}\tau\mathcal{Y} \overset{\mathrm{df}}{=} \prod_{F\subset\mathcal{X}}\ \prod_{f\in\mathcal{Y}^F}\ \sum_{f^*} f \subset f^* \in \mathcal{Y}^{\mathcal{X}}$$

or briefly:

$$\mathcal{X}\,\tau\,\mathcal{Y} \Longleftrightarrow \prod_{F\subset\mathcal{X}} (\mathcal{Y}^F = \mathcal{Y}^{\mathcal{X}}|F)\,.$$

1.2. $\mathcal{X}\,\tau_v\,\mathcal{Y}$ means that each continuous mapping f of an arbitrary closed subset F of the space $\mathcal{X}$ into the space $\mathcal{Y}$ can be extended to some open neighbourhood U_F of the set F in the space $\mathcal{X}$:

$$\mathcal{X}\,\tau_v\,\mathcal{Y} \overset{\mathrm{df}}{=} \prod_{F\subset\mathcal{X}}\ \prod_{f\in\mathcal{Y}^{\mathcal{X}}}\ \sum_{U_F\subset\mathcal{X}}\ \sum_{f^*} f \subset f^* \in \mathcal{Y}^{U_F}.$$

As we know (see [10], p. 260, th. 5), $\mathcal{Y} \in AR$ is equivalent to $\mathcal{X}\,\tau\,\mathcal{Y}$ for each $\mathcal{X}$ and $\mathcal{Y} \in ANR$ to $\mathcal{X}\,\tau_v\,\mathcal{Y}$ also for each $\mathcal{X}$.

We introduce a new relation φ by means of the following definition:

1.3. The relation φ_x at the point $x \in \mathcal{X}$, namely

$$\mathcal{X}\,\varphi_x\,\mathcal{Y}\,,$$

means that there exists an arbitrarily small neighbourhood U_x of the point x in the space $\mathcal{X}$ whose boundary $\mathrm{Fr}\,U_x\,\tau\,\mathcal{Y}$:

(1.3.1) $$\mathcal{X}\,\varphi_x\,\mathcal{Y} \overset{\mathrm{df}}{=} \prod_{\varepsilon>0}\ \sum_{U_x\subset\mathcal{X}} (\mathrm{d}\,U_x < \varepsilon\cdot \mathrm{Fr}\,U_x\,\tau\,\mathcal{Y})\,.$$

The relation φ, namely

$$\mathcal{X}\,\varphi\,\mathcal{Y}\,,$$

means that at each point x of the space $\mathcal{X}$ the relation $\mathcal{X}\,\varphi_x\,\mathcal{Y}$ takes place:

(1.3.2) $$\mathcal{X}\,\varphi\,\mathcal{Y} \overset{\mathrm{df}}{=} \prod_{\varepsilon>0}\ \prod_{x\in\mathcal{X}}\ \sum_{U_x\subset\mathcal{X}} (\mathrm{d}\,U_x < \varepsilon\cdot \mathrm{Fr}\,U_x\,\tau\,\mathcal{Y})\,.$$

The fundamental connections between the relations defined above are formulated in the following:

1.4. THEOREM. *The relation τ implies the relations τ_v and φ, but the converse subsumption is, in general, not true:*

(1.4.1) $$\tau \underset{\neq}{\subset} (\tau_v\cdot\varphi)\,,$$

moreover the relations τ_v and φ do not imply each other:

(1.4.2) $$\sim(\tau_v \subset \varphi)\cdot \sim(\varphi \subset \tau_v)\,.$$

The proof of the subsumption (1.4.1) has been reduced to the proof of the two relations: $\tau \subset \tau_v$ and $\tau \subset \varphi$, which follow immediately from the definition.

Inequality $\tau \neq \tau_v \cdot \varphi$ follows from $\mathscr{I}\,\overline{\tau}\,\{0,1\}$, $\mathscr{I}\,\tau_v\,\{0,1\}$ and $\mathscr{I}\,\varphi\,\{0,1\}$.

To prove (1.4.2) we quote $\mathscr{I}^2\,\tau_v\,\{0,1\}$ and $\mathscr{I}^2\,\overline{\varphi}\,\{0,1\}$ which means that $\sim(\tau_v \subset \varphi)$. Similarly from $\mathscr{I}\,\overline{\tau}_v\,\{0,\ldots,\frac{1}{3},\frac{1}{2},1\}$ and $\mathscr{I}\,\varphi\,\{0,\ldots,\frac{1}{3},\frac{1}{2},1\}$ it follows that $\sim(\varphi \subset \tau_v)$.

1.5. Some auxiliary relation between the spaces $\mathscr{X}$ and $\mathscr{Y}$, satisfying the condition

(1.5.1) $$\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}$$

will be useful in the following considerations.

Let us note that

(1.5.2) $$\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y} \Rightarrow \mathscr{X}\,\tau\,\mathscr{Y}\,,$$

for $\mathscr{X}\times\{0\}$ is a closed subset of the Cartesian product $\mathscr{X}\times\mathscr{I}$; therefore $\mathscr{X}\times\{0\}\,\tau\,\mathscr{Y}$ by (2.1.2); since $\mathscr{X}\times\{0\} \underset{\text{top}}{=} \mathscr{X}$, we have $\mathscr{X}\,\tau\,\mathscr{Y}$ according to (2.1.1).

The converse implication is, in general, not true:

(1.5.3) $$\sim(\mathscr{X}\,\tau\,\mathscr{Y} \Rightarrow \mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y})\,,$$

which is proved by the fact that $\{x\}\,\tau\,\mathscr{S}_0$ according to (4.1.1) and $\{x\}\times\mathscr{I}\,\overline{\tau}\,\mathscr{S}_0$ according to (4.1.2).

2. The arithmetical properties of the relation τ and the extension of mappings

2.1. Theorem *of the arithmetical properties of the relation τ:*

(2.1.1) $$[(\mathscr{X} \underset{\text{top}}{=} \mathscr{X}^*)\cdot(\mathscr{Y} \underset{\text{top}}{=} \mathscr{Y}^*)\cdot(\mathscr{X}\,\tau\,\mathscr{Y})] \Rightarrow \mathscr{X}^*\,\tau\,\mathscr{Y}^*,$$

(2.1.2) $$F \subset \mathscr{X}\,\tau\,\mathscr{Y} \Rightarrow F\,\tau\,\mathscr{Y}\,,$$

(2.1.3) $$\Big[(F_k\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{n} F_k\Big)\Big] \Rightarrow \mathscr{X}\,\tau\,\mathscr{Y}\,,$$

(2.1.4) $$\Big[(F_k\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{\infty} F_k\,\tau_v\,\mathscr{Y}\Big)\Big] \Rightarrow \mathscr{X}\,\tau\,\mathscr{Y}\,,$$

(2.1.5) $$\Big[(A_k \in F_\sigma)\cdot(A_k\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{\infty} A_k\,\tau_v\,\mathscr{Y}\Big)\Big] \Rightarrow \mathscr{X}\,\tau\,\mathscr{Y}\,,$$

(2.1.6) $$[(G \subset \mathscr{X}\,\tau\,\mathscr{Y})\cdot(G\,\tau_v\,\mathscr{Y})] \Rightarrow G\,\tau\,\mathscr{Y}\,,$$

(2.1.7) $$\Big[(G_t\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{t\in T} G_t\,\tau_v\,\mathscr{Y}\Big)\Big] \Rightarrow \mathscr{X}\,\tau\,\mathscr{Y}\,,$$

(2.1.8) $$[(A \in F_\sigma\cdot G_\delta)\cdot(A\,,B\,\tau\,\mathscr{Y})\cdot(A+B\,\tau_v\,\mathscr{Y})] \Rightarrow A+B\,\tau\,\mathscr{Y}\,,$$

(2.1.9) $$[(F\,,B\,\tau\,\mathscr{Y})\cdot(F+B\,\tau_v\,\mathscr{Y})] \Rightarrow F+B\,\tau\,\mathscr{Y}\,,$$

(2.1.10) $$[(\mathscr{X}\,\tau\,\mathscr{Y} \neq 0)\cdot(\mathscr{X}+\{x\}\,\tau_v\,\mathscr{Y})] \Rightarrow \mathscr{X}+\{x\}\,\tau\,\mathscr{Y}\,.$$

From these properties the following are known: (2.1.1) (see [2], p. 170), (2.1.2) (see [10], p. 254), (2.1.3) (see [11], p. 187), (2.1.4) (see [11], p. 187); with a somewhat stronger hypothesis $\mathscr{Y} \in ANR$ (see [14], p. 680).

The proofs of the properties (2.1.5)-(2.1.9) are immediate.

To prove (2.1.10) let us be noted that $\{x\}\,\tau\,\mathscr{Y}$ according to [10], p. 253 and apply (2.1.9).

2.2. Theorem *of the arithmetic properties of the auxiliary relation defined in* 1.5:

(2.2.1) $$[(\mathscr{X} \underset{\text{top}}{=} \mathscr{X}^*)\cdot(\mathscr{Y} \underset{\text{top}}{=} \mathscr{Y}^*)\cdot(\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y})] \Rightarrow \mathscr{X}^*\times\mathscr{I}\,\tau\,\mathscr{Y}^*,$$

(2.2.2) $$[(F \subset \mathscr{X})\cdot(\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y})] \Rightarrow F\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.3) $$\Big[(F_k\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{n} F_k\Big)\Big] \Rightarrow \mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.4) $$\Big[(F_k\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{\infty} F_k\Big)\cdot(\mathscr{X}\times\mathscr{I}\,\tau_v\,\mathscr{Y})\Big] \Rightarrow \mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.5) $$\Big[(A_k \in F_\sigma)\cdot(A_k\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{k=1}^{\infty} A_k\Big)\cdot(\mathscr{X}\times\mathscr{I}\,\tau_v\,\mathscr{Y})\Big] \Rightarrow \mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.6-1) $$[(G \subset \mathscr{X})\cdot(G\times\mathscr{I}\,\tau_v\,\mathscr{Y})\cdot(\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y})] \Rightarrow G\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.6-2) $$[(G \subset \mathscr{X})\cdot(G\times \operatorname{Int}\mathscr{I}\,\tau_v\,\mathscr{Y})\cdot(\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y})] \Rightarrow G\times \operatorname{Int}\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.7) $$\Big[(G_t\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\Big(\mathscr{X} = \sum_{t\in T} G_t\Big)\cdot(\mathscr{X}\times\mathscr{I}\,\tau_v\,\mathscr{Y})\Big] \Rightarrow \mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.8) $$\big[(A \in F_\sigma\cdot G_\delta)\cdot(A\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot(B\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\big((A+B)\times\mathscr{I}\,\tau_v\,\mathscr{Y}\big)\big] \Rightarrow (A+B)\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.9) $$\big[(F\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot(B\times\mathscr{I}\,\tau\,\mathscr{Y})\cdot\big((F+B)\times\mathscr{I}\,\tau_v\,\mathscr{Y}\big)\big] \Rightarrow (F+B)\times\mathscr{I}\,\tau\,\mathscr{Y}\,,$$

(2.2.10) $$[(\mathscr{X}\times\mathscr{I}\,\tau\,\mathscr{Y} \neq 0)\cdot(\mathscr{Y}_{\text{connected}} \in ANR)] \Rightarrow (\mathscr{X}+\{x\})\times\mathscr{I}\,\tau\,\mathscr{Y}\,.$$

The properties (2.2.1)-(2.2.9) are consequences of the corresponding properties of the relation τ and of the topological properties of the Cartesian product.

Proof of (2.2.10). Since $\mathcal{Y} \in ANR$, it is connected in each dimension (see [10], p. 270), in particular in dimension 0; hence it is local arcwise connected and as connected, it is also arcwise connected. Thus (see [11], p. 187) $\mathcal{J}\tau\mathcal{Y}$. According to (2.2.1), $\{x\}\times\mathcal{J}\tau\mathcal{Y}$, and thus one can apply (2.1.10).

We proceed to list some lemmas which are generalization of some known properties (cf. [8], p. 84, 94) of the n-dimensional spheres.

2.3. LEMMA. *If $\mathfrak{X}\tau_v\mathcal{Y}$ and $\mathfrak{X}-F\tau\mathcal{Y}$, then each continuous mapping of the set F into the space $\mathcal{Y}$ is extensible to the whole space X:*

$$[(F\subset\mathfrak{X}\tau_v\mathcal{Y})\cdot(\mathfrak{X}-F\tau\mathcal{Y})]\Rightarrow\mathcal{Y}^F=\mathcal{Y}^{\mathfrak{X}}|F.$$

The proof is analogous to that in [8]. For an arbitrary $f\in\mathcal{Y}^F$ there exist by hypothesis a U_F and f_1 such that $f\subset f_1\in\mathcal{Y}^{U_F}$. Let $G=\underset{x}{E}[\varrho(x,F)<\varrho(x,\mathfrak{X}-U_F)]$; thus $F\subset G\subset\bar{G}\subset U_F$. By hypothesis there exists f_2 such that $f_1|\bar{G}(\mathfrak{X}-F)\subset f_2\in\mathcal{Y}^{\mathfrak{X}-F}$.

We define

$$f^*(x)=\begin{cases}f_1(x) & \text{for}\quad x\in G,\\ f_2(x) & \text{for}\quad x\in\mathfrak{X}-F.\end{cases}$$

Since $f_2|\bar{G}(\mathfrak{X}-F)=f_1|\bar{G}(\mathfrak{X}-F)$, f^* is defined on the union of two open sets G and $\mathfrak{X}-F$, it is continuous on $\mathfrak{X}$.

2.4. LEMMA. *If a space $\mathfrak{X}$ is compact, $\mathfrak{X}\tau_v\mathcal{Y}$ and a mapping $f_0\in\mathcal{Y}^{F_0}$ non-extensible to the whole space $\mathfrak{X}$ is defined on $F_0\subset\mathfrak{X}$, then there exists a subset F^* of the space $\mathfrak{X}$ such that the mapping f_0 is not extensible to the union F_0+F^* but it is extensible to the union F_0+F, where F is an arbitrary closed proper subset of the set F^*:*

$$[(F_0\subset\mathfrak{X}_{\text{compact}}\tau_v\mathcal{Y})\cdot(f_0\in\mathcal{Y}^{F_0}-\mathcal{Y}^{\mathfrak{X}}|F_0)]\Rightarrow$$
$$\Rightarrow\sum_{F^*\subset\mathfrak{X}}\Big(f_0\,\bar{\in}\,\mathcal{Y}^{F_0+F^*}|F_0\cdot\prod_{F\subsetneq F^*}f_0\in\mathcal{Y}^{F_0+F}|F_0\Big).$$

The proof proceeds exactly as in the case of $\mathcal{Y}=\mathcal{S}_n$ (see [8]).

2.5. COROLLARY. *If a space $\mathfrak{X}$ is compact and $\mathfrak{X}\bar{\tau}\mathcal{Y}$ but $\mathfrak{X}\tau_v\mathcal{Y}$, then there exist two closed subsets F_1 and F_2 of the space $\mathfrak{X}$ such that $F_1\subset F_2$ and a mapping $f\in\mathcal{Y}^{F_1}$ which is not extensible to F_2 but it is extensible to every closed proper subset F of the set F_2 containing F_1:*

$$\mathfrak{X}_{\text{compact}}(\bar{\tau}\cdot\tau_v)\mathcal{Y}\Rightarrow\sum_{F_1,F_2\subset\mathfrak{X}}\Big[(F_1\subset F_2)\cdot\sum_f\Big(f\in\mathcal{Y}^{F_1}-\mathcal{Y}^{F_2}|F_1\cdot$$
$$\cdot\prod_{F\subsetneq F_2}(F_1\subset F\Rightarrow f\in\mathcal{Y}^F|F_1)\Big)\Big].$$

For the proof it is enough to note that

$$\mathfrak{X}\bar{\tau}\mathcal{Y}\Leftrightarrow\sum_{F_1\subset\mathfrak{X}}\sum_f f\in\mathcal{Y}^{F_1}-\mathcal{Y}^{\mathfrak{X}}|F_1$$

and then to apply 2.4 putting $F_0=F_1$ and $F_0+F^*=F_2$.

2.6. LEMMA. *If $\mathfrak{X}\tau_v\mathcal{Y}$, then the continuous mappings f_0 and f_1 of the space $\mathfrak{X}$ into the space $\mathcal{Y}$ such that $\underset{x}{E}[f_0(x)\neq f_1(x)]\times\mathcal{J}\tau\mathcal{Y}$, are homotopic:*

$$[(\mathfrak{X}\tau_v\mathcal{Y})\cdot(f_0,f_1\in\mathcal{Y}^{\mathfrak{X}})\cdot(\underset{x}{E}[f_0(x)\neq f_1(x)]\times\mathcal{J}\tau\mathcal{Y})]\Rightarrow f_0\simeq f_1.$$

The proof is analogous to that of a similar property of n-dimensional spheres: $\mathcal{Y}=\mathcal{S}_n$ (cf. [8]); it is based on (2.2.6-2) and 2.3.

2.7. LEMMA. *If $\mathfrak{X}\times\mathcal{J}\tau_v\mathcal{Y}$ and the space $\mathfrak{X}$ is the union of two closed sets F_1 and F_2 such that if $f_k\in\mathcal{Y}^{F_k}$ for $k=1,2$, then $\underset{x}{E}[f_1(x)\neq f_2(x)]\times\times\mathcal{J}\tau\mathcal{Y}$, and thus there exists an extension of the mapping f_1 to the whole space $\mathfrak{X}$ which is homotopic with the mapping f_2 on the set F_2:*

$$[(\mathfrak{X}\times\mathcal{J}\tau_v\mathcal{Y})\cdot(\mathfrak{X}=F_1+F_2)\cdot(f_k\in\mathcal{Y}^{F_k})\cdot(\underset{x}{E}[f_1(x)\neq f_2(x)]\times$$
$$\times\mathcal{J}\tau\mathcal{Y})]\Rightarrow\sum_{f\in\mathcal{Y}^{\mathfrak{X}}}[(f_1\subset f)\cdot(f|F_2\simeq f_2)].$$

The proof is analogous to that of a similar property of the n-dimensional spheres: $\mathcal{Y}=\mathcal{S}_n$ (cf. [8], p. 88); it is based on 2.6, on the domain-heredity of the relation τ_v relative to the closed subsets ([10], p. 254) and on the theorem of Borsuk ([4], p. 218); in the following, slightly generalized, form:

$$\Big[(\mathfrak{X}\times\mathcal{J}\tau_v\mathcal{Y})\cdot(F\subset\mathfrak{X})\cdot(f_1,f_2\in\mathcal{Y}^F)\cdot(f_1\simeq f_2)\cdot\sum_{f_1^*}(f_1\subset f_1^*\in\mathcal{Y}^{\mathfrak{X}})\Big]\Rightarrow$$
$$\Rightarrow\sum_{f_2^*}(f_2\subset f_2^*\in\mathcal{Y}^{\mathfrak{X}}\cdot f_1^*\simeq f_2^*);$$

the proof thereof is an immediate generalization of a well-known proof given by Dowker for $\mathcal{Y}\in ANR$ (cf. [8], p. 86, or [10], p. 278).

2.8. COROLLARY. *If $\mathfrak{X}\times\mathcal{J}\tau_v\mathcal{Y}$ and the space $\mathfrak{X}$ is the union of two closed sets F_1 and F_2 such that $(F_1F_2\times\mathcal{J})\times\mathcal{J}\tau\mathcal{Y}$, then if the continuous mappings f and g of the space $\mathfrak{X}$ into the space $\mathcal{Y}$ are homotopic on each of the sets F_k, then they are homotopic on the whole space $\mathfrak{X}$:*

$$[(\mathfrak{X}\times\mathcal{J}\tau_v\mathcal{Y})\cdot(\mathfrak{X}=F_1+F_2)\cdot([(F_1F_2)\times\mathcal{J}]\times\mathcal{J}\tau\mathcal{Y})\cdot$$
$$\cdot(f,g\in\mathcal{Y}^{\mathfrak{X}})\cdot(f|F_1\simeq g|F_1)\cdot(f|F_2\simeq g|F_2)]\Rightarrow f\simeq g.$$

The proof is analogous to that of a similar property of the n-dimensional spheres: $\mathcal{Y} = \mathcal{S}_n$ (cf. [8]); it is based on (2.2.6-1) and 2.7.

2.9. Lemma. *If a space $\mathcal{X}$ is a union of an arbitrary family $\{G_t\}_{t\in T}$ of such open sets that $\mathrm{Fr}\, G_t \times \mathcal{I}\, \tau\, \mathcal{Y}$ and if $\mathcal{Y}$ is an ANR, then the continuous mapping f of a closed subset F of the space $\mathcal{X}$ which is extensible to the union of the set F and the closure $\overline{G}_t$ of each set of the family $\{G_t\}_{t\in T}$ is extensible to the whole space $\mathcal{X}$*:

$$\Big[\Big(F \subset \mathcal{X} = \sum_{t\in T} G_t\Big)\cdot(\mathrm{Fr}\, G_t \times \mathcal{I}\,\tau\,\mathcal{Y} \in ANR)\cdot \Big(f \in \mathcal{Y}^F \Rightarrow \sum_{f_t} f \subset f_t \in \mathcal{Y}^{F+\overline{G}_t}\Big)\Big] \Rightarrow \sum_{f^*} f \subset f^* \in \mathcal{Y}^{\mathcal{X}}.$$

The proof, exactly as for the analogous property of the n-dimensional spheres $\mathcal{Y} = \mathcal{S}_n$ (see [8]), is based on (2.2.3), (2.2.6-1) and 2.7.

2.10. An immediate corollary, which somewhat generalizes lemma 2.9, is the following:

$$\Big[\Big(F \subset \mathcal{X} = \sum_{t\in T} G_t\Big)\cdot(\mathrm{Fr}\, G_t \times \mathcal{I}\,\tau\,\mathcal{Y} \in ANR)\cdot(\mathcal{Y}^F = \mathcal{Y}^{F+\overline{G}_t}|F)\Big] \Rightarrow \mathcal{Y}^F = \mathcal{Y}^{\mathcal{X}}|F.$$

3. The arithmetical properties of the relation φ

3.1. Theorem. *The relation $\mathcal{X}\varphi\mathcal{Y}$ is a topological invariant*:

$$[(\mathcal{X} \underset{\mathrm{top}}{=} \mathcal{X}^*)\cdot(\mathcal{Y} \underset{\mathrm{top}}{=} \mathcal{Y}^*)\cdot(\mathcal{X}\varphi\mathcal{Y})] \Rightarrow \mathcal{X}^*\varphi\mathcal{Y}^*.$$

Proof. First we prove that

$$[(\mathcal{X} \underset{\mathrm{top}}{=} \mathcal{X}^*)\cdot(\mathcal{X}\varphi\mathcal{Y})] \Rightarrow \mathcal{X}^*\varphi\mathcal{Y}.$$

Let h be a homeomorphism $h(\mathcal{X}) = \mathcal{X}^*$. We choose an arbitrary point $x^* \in \mathcal{X}^*$ and an arbitrary $\varepsilon^* > 0$. Let $x = h^{-1}(x^*)$ and $\varepsilon = \varrho\big(x, \mathcal{X} - h^{-1}(Q_{\varepsilon^*/2}(x^*))\big)$. By the hypothesis $\mathcal{X}\varphi_x\mathcal{Y}$ there exists in $\mathcal{X}$ such an U_x that $\mathrm{d}\, U_x < \varepsilon$ and $\mathrm{Fr}\, U_x\,\tau\,\mathcal{Y}$.

$h(U_x)$ is an open neighbourhood of the point x^* in $\mathcal{X}^*$ and $h(U_x) \subset Q_{\varepsilon^*/2}(x^*)$; thus $\mathrm{d}\, h(U_x) < \varepsilon^*$. Since h is a homeomorphism of $\mathcal{X}$ onto $\mathcal{X}^*$, we have $h(\mathrm{Fr}\, U_x) = \mathrm{Fr}\, h(U_x)$. Hence by $\mathrm{Fr}\, U_x\,\tau\,\mathcal{Y}$ and (2.1.1) it follows that $\mathrm{Fr}\, h(U_x)\,\tau\,\mathcal{Y}$ or $\mathcal{X}^*\varphi_{x^*}\mathcal{Y}$.

It remains to prove the implication

$$[(\mathcal{Y} \underset{\mathrm{top}}{=} \mathcal{Y}^*)\cdot(\mathcal{X}\varphi\mathcal{Y})] \Rightarrow \mathcal{X}\varphi\mathcal{Y}^*,$$

which results immediately from (1.3.1) and (2.1.1).

3.2. Theorem. *The relation φ is domain-hereditary for the closed subsets*

(3.2.1) $$F \subset \mathcal{X}\varphi\mathcal{Y} \Rightarrow F\varphi\mathcal{Y}$$

and also for the open subsets

(3.2.2) $$G \subset \mathcal{X}\varphi\mathcal{Y} \Rightarrow G\varphi\mathcal{Y}.$$

Proof of (3.2.1). By hypothesis for an arbitrary $x \in F$ and an arbitrary $\varepsilon > 0$ there exists in $\mathcal{X}$ such an U_x that $\mathrm{d}\, U_x < \varepsilon$ and $\mathrm{Fr}\, U_x\,\tau\,\mathcal{Y}$. The set $U = FU_x$ is an open neighbourhood of the point x relative to the set F, where $\mathrm{d}\, U < \varepsilon$. Its boundary relative to F: $\mathrm{Fr}_F\, U_x$ is a closed subset of $\mathrm{Fr}\, U_x$; thus, by (2.1.2), $\mathrm{Fr}_F\, U\,\tau\,\mathcal{Y}$, which proves that $F\varphi_x\mathcal{Y}$.

Proof of (3.2.2). Let $x \in G$. Inequality $0 < \varepsilon < \varrho(x, \mathcal{X} - G)$ implies by hypothesis that there exists in X an U_x such that $\mathrm{d}\, U_x < \varepsilon$ and $\mathrm{Fr}\, U_x\,\tau\, Y$. Hence $\overline{U}_x \subset G$, and thus U_x is an open neighbourhood of the point x relative to the set G, where $\mathrm{Fr}_G\, U_x = \mathrm{Fr}\, U_x$; therefore $\mathrm{Fr}_G\, U_x\,\tau\,\mathcal{Y}$ and $G\varphi_x\mathcal{Y}$.

3.3. Theorem. *The relation φ is finite domain-additive for the closed sets*

$$(F_k\varphi\mathcal{Y}.\ k = 1, 2, \ldots, n) \Rightarrow \sum_{k=1}^{n} F_k\varphi\mathcal{Y}.$$

We shall present the proof, which in general is an inductive one, for the case $n = 2$, using the following decomposition in disjoint terms: $F_1 + F_2 = F_1F_2 + (F_1 - F_2) + (F_2 - F_1)$.

If $x \in F_1F_2$ and $\varepsilon > 0$, then there exists by hypothesis such an open neighbourhood U_k $(k = 1, 2)$ of the point x in F_k that $\mathrm{d}\, U_k < \varepsilon/2$ and $\mathrm{Fr}\, U_k\,\tau\,\mathcal{Y}$. The set $U_1 + U_2$ is an open neighbourhood of the point x in $F_1 + F_2$, where $\mathrm{d}(U_1 + U_2) < \varepsilon$ and $\mathrm{Fr}(U_1 + U_2) \subset \mathrm{Fr}\, U_1 + \mathrm{Fr}\, U_2$; thus, by (2.1.3) and (2.1.2), $\mathrm{Fr}(U_1 + U_2)\,\tau\,\mathcal{Y}$, which proves that $F_1 + F_2\varphi_x\mathcal{Y}$.

If $x \in F_1 - F_2$, then by hypothesis there exists a neighbourhood U of the point x relative to F_1 such that $\mathrm{d}\, U < \varrho(x, F_2)$ and $\mathrm{Fr}\, U\,\tau\,\mathcal{Y}$. Since $\overline{U}F_2 = 0$, U is an open neighbourhood of the point x relative to $F_1 + F_2$, which means that $F_1 + F_2\varphi_x\mathcal{Y}$.

For $x \in F_2 - F_1$ the proof is exactly as above.

3.4. Theorem. *The relation φ is domain-additive for open sets*:

$$(G_t\varphi Y\cdot t \in T) \Rightarrow \sum_{t\in T} G_t\varphi\mathcal{Y}.$$

Proof. For an arbitrary $x_0 \in \sum_{t\in T} G_t$ there exists a certain G_{t_0} such that $x_0 \in G_{t_0}$. By hypothesis there exists in G_{t_0} a U_{x_0} such that $\mathrm{d}U_{x_0} < \varrho(x_0, \sum_{t\in T} G_t - G_{t_0})$; thus $\overline{U}_{x_0} \subset G_{t_0}$ and $\mathrm{Fr}\, U_{x_0}\,\tau\,\mathcal{Y}$. Since G_{t_0} is an open set, U_{x_0} as an open set relative to G_{t_0} is an open neighbourhood of the point x_0 in $\sum_{t\in T} G_t$ and also $\mathrm{Fr}_{G_{t_0}} U_{x_0} = G_{t_0}\overline{U}_{x_0} - U_{x_0} = \mathrm{Fr}\, U_{x_0}$, and thus $\mathrm{Fr}\, U_{x_0} \tau \mathcal{Y}$, which proves that $\sum_{t\in T} G_t \varphi_{x_0} \mathcal{Y}$.

3.5. Lemma. *If $\mathfrak{X}\varphi\mathcal{Y}$, then in the space $\mathfrak{X}$ there exists a countable sequence $\{U_k\}_{k\in\mathcal{N}}$ of open sets such that $\mathrm{Fr}\, U_k \tau \mathcal{Y}$, which is a base for open sets of the space $\mathfrak{X}$*:

$$\mathfrak{X}\varphi\mathcal{Y} \Rightarrow \sum_{(U_k)_{k\in\mathcal{N}}} \Big[(U_k\subset\mathfrak{X})\cdot(\mathrm{Fr}\, U_k \tau \mathcal{Y})\cdot \prod_{G\subset X}\sum_{\{m_k\}\subset\mathcal{N}}\Big(G=\sum_{k=1}^{\infty} U_{m_k}\Big)\Big].$$

Proof. By hypothesis there exists for each point $x\in\mathfrak{X}$ and for each $n\in\mathcal{N}$ an open neighbourhood $U_x^{(n)}$ in the space $\mathfrak{X}$ such that $\mathrm{d}\, U_x^{(n)} < 1/n$, $\mathrm{Fr}\, U_x^{(n)} \tau \mathcal{Y}$ and $\mathfrak{X} = \sum_{x\in\mathfrak{X}} U_x^{(n)}$.

By virtue of Lindelöf's theorem we may select a countable covering $\{U_m^{(n)}\}_{m\in\mathcal{N}}$ from the covering $\{U_x^{(n)}\}_{x\in\mathfrak{X}}$. The double sequence $\{U_m^{(n)}\}_{m,n\in\mathcal{N}}$ can be ordered into the sequence $\{U_1^{(1)}, U_1^{(2)}, U_1^{(1)}, \ldots\} = \{U_k\}_{k\in\mathcal{N}}$ where $\sum_{k=1}^{\infty} U_k = \mathfrak{X}$, $\mathrm{Fr}\, U_k \tau \mathcal{Y}$ and the diameters of the terms of the sequence with sufficiently great indexes are arbitrarily small.

Hence for any $G\subset\mathfrak{X}$ and any $x\in G$ there exists a neighbourhood $U_{k(x)}\in\{U_k\}$ with diameter $\mathrm{d}\, U_{k(x)} < \varrho(x, \mathfrak{X}-G)$. Hence $x\in U_{k(x)}\subset G$ or $G\subset\sum_{x\in G}\{x\}\subset\sum_{k(x)} U_{k(x)}\subset G$, and thus $G=\sum_{k(x)} U_{k(x)}$. Ordering the sequence $\{U_{k(x)}\}$ according to the increasing indexes, we receive a subsequence $\{U_{m_k}\}$ satisfying the lemma.

3.6. Theorem of decomposition. *If $\mathfrak{X}\varphi\mathcal{Y}$ and $\mathcal{Y}$ is ANR, then there exist sets A and B such that $\mathfrak{X}=A+B$ and $AB=0$, where A is F_σ and $A\tau Y$, B is G_δ and $\dim B\leqslant 0$*:

$$\mathfrak{X}\varphi\mathcal{Y}\in ANR \Rightarrow \sum_{A,B}[(\mathfrak{X}=A+B)\cdot(AB=0)\cdot(A\in F_\sigma\cdot A\tau\mathcal{Y})\cdot$$
$$\cdot(B\in G_\delta\cdot\dim B\leqslant 0)].$$

Remark. In view of [5], p. 392, it is enough to assume that the space $\mathcal{Y}$ be a local absolute neighbourhood retract.

Proof. By virtue of lemma 3.5 there exists in $\mathfrak{X}$ a sequence $\{U_k\}$ of open sets which are a base in $\mathfrak{X}$ and for which $\mathrm{Fr}\, U_k\tau\mathcal{Y}$. Let $A=\sum_{k=1}^{\infty}\mathrm{Fr}\, U_k$; hence A is a F_σ and from 2.1.4 it follows that $A\tau\mathcal{Y}$.

The set $B=X-A$ is G_δ and for any U_k we have $B\,\mathrm{Fr}\, U_k=0$, whence each point $x\in B$ possesses an arbitrarily small neighbourhood

$U_{k(x)}$, whose boundary relative to the set B: $\mathrm{Fr}_B U_{k(x)}$ is empty, which means that $\dim_x B\leqslant 0$.

The converse implication can be proved under a weaker hypothesis:

3.7. $(A\tau\mathcal{Y}\cdot\dim B=0)\Rightarrow A+B\varphi\mathcal{Y}$.

Proof. For any $x\in A+B$ we have from [9], p. 173 $\dim(B+\{x\})=0$. Thus (see [8], p. 27) for each $\varepsilon>0$ there exists an U_x in $A+B$ such that $\mathrm{d}\, U_x<\varepsilon$ and $\mathrm{Fr}\, U_x\subset A$. By hypothesis and (2.1.2) we have $\mathrm{Fr}\, U_x\tau\mathcal{Y}$, whence $A+B\varphi_x\mathcal{Y}$.

3.8. Theorem. *If a space $\mathcal{Y}$ is an ANR, then the relation φ relative to the space $\mathcal{Y}$ is F_σ-domain additive*:

$$F_k\varphi\mathcal{Y}\in ANR\Rightarrow\sum_{k=1}^{\infty}F_k\varphi\mathcal{Y}.$$

Proof. Let $\mathfrak{X}=\sum_{k=1}^{\infty}F_k$. We define inductively a sequence of sets $\{C_m\}_{m\in\mathcal{N}}$ as follows:

$$C_1=F_1,\qquad C_m=F_m-\sum_{k=1}^{m-1}F_k\quad\text{for}\quad m=2,3,\ldots$$

The sets C_m are mutually disjoint and $\mathfrak{X}=\sum_{m=1}^{\infty}C_m$. Since $C_m=F_m\cdot(\mathfrak{X}-\sum_{k=1}^{m-1}F_k)$, it is itself an F_σ as an intersection of two sets F_σ. Since C_m is an open set in F_m, in view of (3.2.2), $C_m\varphi\mathcal{Y}$. By virtue of 3.6 there exist such sets A_m and B_m that $C_m=A_m+B_m$, $A_mB_m=0$, A_m is an F_σ and $A\tau\mathcal{Y}$, B_m is an G_δ and $\dim B_m\leqslant 0$. Let us write $A=\sum_{m=1}^{\infty}A_m$ and $B=\sum_{m=1}^{\infty}B_m$. We have $\mathfrak{X}=A+B$, where $A\tau\mathcal{Y}$ from (2.1.5). Since $B_m\subset C_m$ and since the sets C_m are mutually disjoint, we have $B_m=B_mC_m=\sum_{k=1}^{\infty}B_k\cdot C_m=BC_m$.

In virtue of $C_m\in F_\sigma$ the set B_m is an F_σ relative to B, and therefore (see [9], p. 172) $\dim B\leqslant 0$. Hence, by 3.7, $\mathfrak{X}\varphi\mathcal{Y}$.

3.9. Corollary. $(A_k\in F_\sigma\cdot A_k\varphi\mathcal{Y}\in ANR)\Rightarrow\sum_{k=1}^{\infty}A_k\varphi\mathcal{Y}$.

Proof. $A_k=\sum_{m=1}^{\infty}F_{km}$, where, from (3.2.1), $F_{km}\varphi\mathcal{Y}$. Since $\sum_{k=1}^{\infty}A_k=F_{11}+F_{12}+F_{21}+\ldots$, we have $\sum_{k=1}^{\infty}A_k\varphi\mathcal{Y}$ by 3.8.

3.10. Corollary. $[(A\in F_\sigma\cdot G_\delta)\cdot(A, B\,\varphi\,\mathcal{Y}\in ANR)]\Rightarrow A+B\,\varphi\,\mathcal{Y}$.

Proof. Since $(A+B)-A\in F_\sigma$, it is enough to apply 3.9.

3.11. Corollary. $F, B\,\varphi\,\mathcal{Y}\in ANR\Rightarrow F+B\,\varphi\,\mathcal{Y}$.

Proof. Since $F\in F_\sigma\cdot G_\delta$, it is enough to apply 3.10.

3.12. Corollary. $(\mathcal{X}\varphi\,\mathcal{Y}\in ANR.\ \mathcal{Y}\neq 0)\Rightarrow\mathcal{X}+\{x\}\,\varphi\,\mathcal{Y}$.

It results immediately from 3.11 and 4.3.

3.13. Corollary. $[(\bar{A}_1A_2+A_1\bar{A}_2=0)\cdot(A_k\varphi\,\mathcal{Y})]\Rightarrow A_1+A_2\,\varphi\,\mathcal{Y}$.

Proof. The sets $A_1=(A_1+A_2)-\bar{A}_2$ and $A_2=(A_1+A_2)-\bar{A}_1$ are open relative to A_1+A_2, and therefore it is enough to apply 3.4.

3.14. Corollary. *If $\mathcal{X}\varphi\,\mathcal{Y}\neq 0$ and $\mathcal{X}$ is a compact space, then the union $\mathcal{X}+\{x\}$ satisfies the relation φ as well*:

$$\mathcal{X}_{\text{compact}}\,\varphi\,\mathcal{Y}\neq 0\Rightarrow\mathcal{X}+\{x\}\,\varphi\,\mathcal{Y}.$$

Proof. If $x\in\mathcal{X}$, then $\mathcal{X}+\{x\}=\mathcal{X}$. If $x\bar{\in}\mathcal{X}$, the point x is isolated in $\mathcal{X}+\{x\}$, then $\mathcal{X}$ and $\{x\}$ are separated. Hence, in view of 4.3, 3.13 can be applied.

4. The connections with dimension theory

4.1. Theorem. *A space $\mathcal{X}$ is at most n-dimensional if and only if $\mathcal{X}\tau\mathcal{S}_n$*:

$$\dim\mathcal{X}\leqslant n\Leftrightarrow\mathcal{X}\tau\mathcal{S}_n. \tag{4.1.1}$$

A compact space $\mathcal{X}$ is at most $(n-1)$-dimensional if and only if $\mathcal{X}\times\mathcal{I}\tau\mathcal{S}_n$:

$$\dim\mathcal{X}\leqslant n-1\Leftrightarrow\mathcal{X}\times\mathcal{I}\tau\mathcal{S}_n. \tag{4.1.2}$$

A space $\mathcal{X}$ is at the point x at most $(n+1)$-dimensional if and only if $\mathcal{X}\varphi_x\mathcal{S}_n$:

$$\dim_x\mathcal{X}\leqslant n+1\Leftrightarrow\mathcal{X}\varphi_x\mathcal{S}_n. \tag{4.1.3-1}$$

A space $\mathcal{X}$ is at most $(n+1)$-dimensional if and only if $\mathcal{X}\varphi\mathcal{S}_n$.

$$\dim\mathcal{X}\leqslant n+1\Leftrightarrow\mathcal{X}\varphi\mathcal{S}_n. \tag{4.1.3-2}$$

For the proof of the well-known fundamental condition (4.1.1) see [8], p. 83, or [10], p. 271.

In order to prove (4.1.2) let us note that if $\dim\mathcal{X}\leqslant n-1$, then $\dim(\mathcal{X}\times\mathcal{I})\leqslant n$ and (4.1.1) can be applied. Whereas if $\mathcal{X}\times\mathcal{I}\tau\mathcal{S}_n$, then $\dim(\mathcal{X}\times\mathcal{I})\leqslant n$ by (4.1.1) and from the compactness of the space $\mathcal{X}$ it follows that $\dim(\mathcal{X}\times\mathcal{I})=\dim\mathcal{X}+1$, whence $\dim\mathcal{X}\leqslant n-1$.

The hypothesis of compactness of the space $\mathcal{X}$ is indispensable only for the proof of the sufficiency of condition (4.1.2).

For the proof of (4.1.3-1) let $x\in\mathcal{X}$, $\varepsilon>0$ and $\dim_x\mathcal{X}\leqslant n+1$. By the inductive definition of dimension ([9], p. 162) there exists an U_x in $\mathcal{X}$ such that $\mathrm{d}\,U_x<\varepsilon$ and $\dim\operatorname{Fr}U_x\leqslant n$; thus, in view of (4.1.1), $\operatorname{Fr}U_x\tau\mathcal{S}_n$, which means $\mathcal{X}\varphi_x\mathcal{Y}$.

Conditions (4.1.2), (4.1.1) and (4.1.3) indicate that the relation defined in 1.5 as well as the relations τ and φ form not only a subsumption sequence but also — to a certain degree — an inductive sequence for the dimension of a space. By passing from one relation to the other we may augment — with $\mathcal{Y}=\mathcal{S}_n$ — the dimension of the space at most by one.

4.2. Corollary. $\dim\mathcal{X}=n+1\Leftrightarrow\mathcal{X}(\bar{\tau}\cdot\varphi)\mathcal{S}_n$.

It results immediately from (4.1.1) and (4.1.3-2).

4.3. Corollary. *A space $\mathcal{X}$ is at most one-dimensional if and only if $\mathcal{X}\varphi\,\mathcal{Y}$ for any non-empty space $\mathcal{Y}$*

$$\dim X\leqslant 1\Leftrightarrow\prod_{\mathcal{Y}\neq O}\mathcal{X}\varphi\,\mathcal{Y}.$$

Proof. For any $x\in\mathcal{X}$ and any $\varepsilon>0$ there exists in $\mathcal{X}$ an U_x such that $\dim_x\operatorname{Fr}U_x\leqslant 0$, which is equivalent (see [10], p. 253) to $\operatorname{Fr}U_x\tau\,\mathcal{Y}$ for each $\mathcal{Y}\neq 0$.

4.4. Lemma. *Let $\mathcal{X}=\sum_{t\in T}F_t$ and for each $t\in T$ is $F_t\tau\,\mathcal{Y}$, $\mathcal{Y}$ being ANR. If in every open neighbourhood $U_t\supset F_t$ there is an open neighbourhood $V_t\supset F_t$ such that $\bar{V}_t\subset U_t$ and $\operatorname{Fr}V_t\times\mathcal{I}\tau\,\mathcal{Y}$, then $\mathcal{X}\tau\,\mathcal{Y}$*:

$$\Big[\Big(\mathcal{X}=\sum_{t\in T}F_t\Big)\cdot(F_t\tau\,\mathcal{Y}\in ANR)\cdot\prod_{U_t\supset F_t}\sum_{V_t}\big((F_t\subset V_t\subset\bar{V}_t\subset U_t)\cdot(\operatorname{Fr}V_t\times\mathcal{I}\tau\,\mathcal{Y})\big)\Big]\Rightarrow\mathcal{X}\tau\,\mathcal{Y}.$$

The proof is analogous to that in [8], p. 90. Let $F\subset\mathcal{X}$ and $f\in\mathcal{Y}^F$. In view of (2.1.6), $(F+F_t)-F\tau\,\mathcal{Y}$, and thus by 2.3 there exists an f_t such that $f\subset f_t\in\mathcal{Y}^{F+F_t}$. Since $\mathcal{Y}\in ANR$, there exist in $\mathcal{X}$ a neighbourhood U_t of $F+F_t$ and a mapping $\bar{f}_t$ such that $f_t\subset\bar{f}_t\in\mathcal{Y}^{U_t}$.

By hypothesis there is in $\mathcal{X}$ an open set V_t such that $F_t\subset V_t\subset\bar{V}_t\subset U_t$ and $\operatorname{Fr}V_t\times\mathcal{I}\tau\,\mathcal{Y}$. Therefore $F+\bar{V}_t\subset U_t$ and $f\subset\bar{f}_t|(F+\bar{V}_t)\in\mathcal{Y}^{F+\bar{V}_t}$. Since $\mathcal{X}=\sum_{t\in T}V_t$ and $\operatorname{Fr}V_t\times\mathcal{I}\tau\,\mathcal{Y}$, by 2.9 there exists an extension $f^*\in\mathcal{Y}^{\mathcal{X}}$ of the mapping f to the whole space $\mathcal{X}$, which proves that $\mathcal{X}\tau\,\mathcal{Y}$.

4.5. Theorem. *If for each point $x\in\mathcal{X}$ there exists an arbitrarily small neighbourhood U_x in the space $\mathcal{X}$ such that $\operatorname{Fr}U_x\times\mathcal{I}\tau\,\mathcal{Y}$ and if the space $\mathcal{Y}$ is non-empty and an ANR, then $\mathcal{X}\tau\,\mathcal{Y}$*:

$$\prod_{\varepsilon>0}\prod_{x\in\mathcal{X}}\sum_{U_x\subset\mathcal{X}}[(\mathrm{d}\,U_x<\varepsilon)\cdot(\operatorname{Fr}U_x\times\mathcal{I}\tau\,\mathcal{Y}\in ANR)\cdot(\mathcal{Y}\neq 0)]\Rightarrow\mathcal{X}\tau\,\mathcal{Y}.$$

For the proof it is enough to assume, in 4.4, $F_t = \{x\}$ for $x \in \mathfrak{X}$. In view of [10], p. 253, $\{x\}\tau\,\mathcal{Y}$ and all the hypotheses of the lemma are satisfied.

In particular, if the space $\mathfrak{X}$ is compact and $\mathcal{Y} = \mathcal{S}_n$, where $n > -1$, the hypothesis of 4.5 is a sufficient condition for $\dim X \leqslant n$, exactly as with the inductive definition of dimension.

4.6. The theorem can be the base for some extension of the analogy of connection between the auxiliary relation defined in 1.5 and relation τ on the one hand and the inductive definition of dimension on the other.

These relations give rise to a number of concepts, analogous to the concepts in the dimension theory, for which analogous properties can be proved (e. g. heredity or additivity).

For instance, analogously to the set $A_{(n)}$ (see [9], p. 164), one can define for the set $A \subset \mathfrak{X}$ the sets:

$$A_{(\varphi,\mathcal{Y})} = \mathop{E}_{x\in\mathfrak{X}} \prod_{\varepsilon>0} \sum_{U_x\subset\mathfrak{X}} [(\mathrm{d}\,U_x < \varepsilon)\cdot(A\,\mathrm{Fr}\,U_x\,\tau\,\mathcal{Y})],$$

$$A_{(\tau,\mathcal{Y})} = \mathop{E}_{x\in\mathfrak{X}} \prod_{\varepsilon>0} \sum_{U_x\subset\mathfrak{X}} [(\mathrm{d}\,U_x < \varepsilon)\cdot(A\,\mathrm{Fr}\,U_x \times \mathcal{I}\,\tau\,\mathcal{Y})].$$

The analogue for the dimension-kernel (see [9], p. 186) is the φ-kernel of the space $\mathfrak{X}$ relative to the space $\mathcal{Y}$, which is defined as $\mathfrak{X}_{(\varphi,\mathcal{Y})} - \mathfrak{X}_{(\tau,\mathcal{Y})}$. Theorems analogous to those of the dimension theory can be demonstrated, e. g. that the sets $A_{(\varphi,\mathcal{Y})}$ and $A_{(\tau,\mathcal{Y})}$ are G_δ or the implication

$$\mathfrak{X}\,\varphi\,\mathcal{Y} \in ANR \Rightarrow \mathfrak{X}_{(\varphi,\mathcal{Y})} - \mathfrak{X}_{(\tau,\mathcal{Y})}\,\varphi\,\mathcal{Y}.$$

Other examples: the decomposition-theorem (the analogue: 3.6 and 3.7), Hurewicz's extension theorem for closed sets (see [7], p. 146), the properties of the cyclic elements in the local connected continuum (see [10], p. 235) and finally the properties or Urysohn's coefficient (see [10], p. 60).

Now we shall deal with the connections between the properties of the relations under discussion and the problem of the disconnection of a space.

4.7. Theorem. *If $\mathfrak{X}\,\varphi\,\mathcal{Y}$, then each pair of different points x_1 and x_2 of the space $\mathfrak{X}$ can be separated by means of a set $F \subset \mathfrak{X}$ such that $\mathfrak{X}\,\tau\,\mathcal{Y}$*:

$$(x_1, x_2 \in \mathfrak{X}\,\varphi\,\mathcal{Y}.\ x_1 \neq x_2) \Rightarrow \sum_{F,M_1,M_2\subset\mathfrak{X}} [(\mathfrak{X} - F \supset M_1 + M_2)\cdot(x_k \in M_k)\cdot (\bar{M}_1 M_2 + M_1\bar{M}_2 = 0)\cdot(F\,\tau\,\mathcal{Y})].$$

Proof. If $\mathfrak{X}\,\varphi\,\mathcal{Y}$, the space is uni-ordered relative to the family of the boundaries of the open neighbourhoods $U_x \subset \mathfrak{X}$ such that $\mathrm{Fr}\,U_x\,\tau\,\mathcal{Y}$ (see [13], p. 169). This implies the theorem.

4.8. Theorem. *If $\mathfrak{X}\,\varphi\,\mathcal{Y}$, $\mathcal{Y}$ is ANR and $\mathcal{Y} \neq 0$, then each pair of the closed and disjoint sets in the space $\mathfrak{X}$ can be separated by a set $F \subset \mathfrak{X}$ such that $F\,\tau\,\mathcal{Y}$*:

$$[(F_1, F_2 \subset \mathfrak{X}\,\varphi\,\mathcal{Y} \neq 0)\cdot(\mathcal{Y} \in ANR)\cdot(F_1F_2 = 0)] \Rightarrow$$
$$\Rightarrow \sum_{F,M_1,M_2\subset\mathfrak{X}} [(\mathfrak{X} - F \supset M_1 + M_2)\cdot(F_k \subset M_k)\cdot(\bar{M}_1 M_2 + M_1\bar{M}_2 = 0)\cdot(F\,\tau\,\mathcal{Y})].$$

Proof. There exists a continuous mapping f of the space $\mathfrak{X}$ into a metric separable space $\mathfrak{X}^*$ (e. g. into a subset of Hilbert's cube) transforming the sets F_1 and F_2 into two different points of the space $\mathfrak{X}^*$:

$$f(F_1) = \{x_1\} \neq \{x_2\} = f(F_2),$$

where $x_1, x_2 \bar{\in} f(\mathfrak{X} - (F_1 + F_2))$; the partial mapping $f|(\mathfrak{X} - (F_1 + F_2))$ is homeomorphism (see [9], p. 139, theorem 2).

Since $\mathfrak{X} - (F_1 + F_2)\,\varphi\,\mathcal{Y}$ by (3.2.2), we have $f(\mathfrak{X} - (F_1 + F_2))\,\varphi\,\mathcal{Y}$ by 3.1. Since $f(\mathfrak{X}) = f(\mathfrak{X} - (F_1 + F_2)) + \{x_1\} + \{x_2\}$, applying twice 3.12 we receive $f(\mathfrak{X})\,\varphi\,\mathcal{Y}$.

In view of 4.7, there is in $f(\mathfrak{X})$ a neighbourhood U_{x_1} such that $x_2 \bar{\in} \bar{U}_{x_1}$, $\mathrm{Fr}\,U_{x_1}\,\tau\,\mathcal{Y}$ and that $\mathrm{Fr}\,U_{x_1}$ separates x_1 and x_2 in $f(\mathfrak{X})$. We have $f^{-1}(\mathrm{Fr}\,U_{x_1}) = \mathrm{Fr}\,f^{-1}(U_{x_1}) \subset \mathfrak{X} - (F_1 + F_2)$ and, by (2.1.1), $f^{-1}(\mathrm{Fr}\,U_{x_1})\,\tau\,\mathcal{Y}$. Since $f(F_1) = \{x_1\} \subset U_{x_1}$, we have $F_1 \subset f^{-1}(U_{x_1})$ and on the other hand $F_2\,\overline{f^{-1}(U_{x_1})} = f^{-1}(\{x_2\}\bar{U}_{x_1}) = 0$. Hence the set $f^{-1}(\mathrm{Fr}\,U_{x_1})$ separates the sets F_1 and F_2 in the space $\mathfrak{X}$.

In the above theorem the hypothesis $\mathcal{Y} \in ANR$ can be replaced by that of the compactness of the space $\mathfrak{X}$ (compare 3.12 and 3.14).

4.9. Corollary. $0 \neq \mathcal{Y} \in ANR \Rightarrow \Big[\mathfrak{X}\,\varphi\,\mathcal{Y} \Longleftrightarrow \prod_{F_1,F_2\subset\mathfrak{X}} \Big(F_1F_2 = 0 \Rightarrow \sum_{G\subset\mathfrak{X}} \big((F_1 \subset G)\cdot(\bar{G}F_2 = 0)\cdot(\mathrm{Fr}\,G\,\tau\,\mathcal{Y})\big)\Big)\Big]$.

Proof. The necessity of the condition follows from the proof of 4.8. In order to prove the sufficiency let $x \in \mathfrak{X}$, $\varepsilon > 0$ and let U_x be an open neighbourhood in $\mathfrak{X}$ such that $\mathrm{d}\,U_x < \varepsilon$. We put $F_1 = \{x\}$ and $F_2 = \mathfrak{X} - U_x$. The sets F_1 and F_2 are closed and disjoint. Hence, by hypothesis, there exists an open set $G \supset F_1 = \{x\}$ as well as $\bar{G}F_2 = 0$ and $\mathrm{Fr}\,G\,\tau\,\mathcal{Y}$. The set G is an open neighbourhood of the point x in the space $\mathfrak{X}$ and $\bar{G}(X - U_x) = 0$ implies $\bar{G} \subset U_x$, whence $\mathrm{d}\,G < \varepsilon$, which together with $\mathrm{Fr}\,G\,\tau\,\mathcal{Y}$ proves that $\mathfrak{X}\,\varphi_x\,\mathcal{Y}$.

The hypothesis $\mathcal{Y} \in ANR$ can be replaced by the hypothesis of the compactness of the space $\mathfrak{X}$.

The corollary corresponds, in some sense, to a theorem of Tumarkin on the equivalence of the definition of inductive dimension, $\mathrm{ind}\,\mathfrak{X}$, to that of an inductive macro-dimension of Urysohn, $\mathrm{Ind}\,\mathfrak{X}$ (see [1], p. 46).

The properties considered above give rise to the following possibility of some generalization of the notion of n-dimensional Cantor manifold.

4.10. A compact space $\mathcal{X}$ is said to be a *φ-manifold relative to a space* $\mathcal{Y}$ if $\mathcal{X}\varphi\mathcal{Y}$ and there is no set $A\subset\mathcal{X}$ such that $A\times\mathcal{I}\tau\mathcal{Y}$ disconnects the space $\mathcal{X}$.

The spaces which are φ-manifolds relative to $\mathcal{S}_n$ are $(n+1)$-dimensional Cantor manifolds.

To the well-known property of n-dimensional Cantor manifolds, namely that they are n-dimensional at every point, one can find an analogous property, namely that if a space $\mathcal{X}$ is a φ-manifold to a space $\mathcal{Y}$, then $\mathcal{X}\varphi\mathcal{Y}$.

Let us note that a space which is a φ-manifold relative to a non-empty space is connected, which is an analogue to the connectivity of an n-dimensional Cantor manifold.

4.11. Lemma. *The following three conditions are equivalent*:

$$\sum_{A,M_1,M_2\subset\mathcal{X}}[(M_k\neq 0)\cdot(\mathcal{X}-A=M_1+M_2)\cdot(\overline{M}_1M_2+M_1\overline{M}_2=0)\cdot(A\times\mathcal{I}\tau\mathcal{Y})], \tag{4.11.1}$$

$$\sum_{G\subset\mathcal{X}}[(0\neq G)\cdot(\overline{G}\neq\mathcal{X})\cdot(\operatorname{Fr}G\times\mathcal{I}\tau\mathcal{Y})], \tag{4.11.2}$$

$$\sum_{F_1,F_2\subset\mathcal{X}}[(0\neq F_k\neq\mathcal{X})\cdot(\mathcal{X}=F_1+F_2)\cdot(F_1F_2\times\mathcal{I}\tau\mathcal{Y})]. \tag{4.11.3}$$

The proof of the three implications: (4.11.1)$\Rightarrow$(4.11.2)$\Rightarrow$(4.11.3)$\Rightarrow$(4.11.1) is exactly as in [8], p. 47. It is based on (2.2.2).

4.12. Theorem. *A compact space $\mathcal{X}$ such that $\mathcal{X}\varphi\mathcal{Y}$ is a φ-manifold relative to the space $\mathcal{Y}$ if and only if the space $\mathcal{X}$ cannot be decomposed into the union of the closed sets F_1 and F_2 such that $\mathcal{X}=F_1+F_2$, $F_1\neq 0\neq F_2$, $\mathcal{X}-F_1\neq 0\neq\mathcal{X}-F_2$ and $F_1F_2\times\mathcal{I}\tau\mathcal{Y}$.*

The proof is contained in that of 4.11, namely in the equivalence of the conditions (4.11.1) and (4.11.3).

4.13. Theorem. *If the space $\mathcal{X}$ is compact and $\mathcal{X}(\overline{\tau}\cdot\tau_v\cdot\varphi)\mathcal{Y}$ then the space $\mathcal{X}$ contains a φ-manifold relative to the space $\mathcal{Y}$.*

Proof. It follows from the hypothesis $\mathcal{X}\overline{\tau}\mathcal{Y}$ that there exist an $F_0\subset\mathcal{X}$ and an $f_0\in\mathcal{Y}^{F_0}-\mathcal{Y}^{\mathcal{X}}|F_0$. According to 2.4 there exists an $F^*\subset\mathcal{X}$ such that $f_0\,\overline{\epsilon}\,\mathcal{Y}^{F_0+F^*}|F_0$, whereas $f_0\in\mathcal{Y}^{F_0+F}|F_0$ for each $F\subsetneqq F^*$.

The set F^* is a φ-manifold relative to the space $\mathcal{Y}$. For, in the contrary case, there exist by (4.11.3) two closed sets F_1 and F_2 such that $F^*=F_1+F_2$, $0\neq F_k\neq F^*$ and $(F_1F_2)\times\mathcal{I}\tau\mathcal{Y}$.

It follows from the above formulated irreducibility of the set F^* that there exist two extensions f_1 and f_2 of the mapping f_0 on the unions F_0+F_1 and F_0+F_2: $f_0\subset f_k\in\mathcal{Y}^{F_0+F_k}$. Since the set $\underset{x}{E}[f_1(x)\neq f_2(x)]$ is open in F_1F_2, we have $\underset{x}{E}[f_1(x)\neq f_2(x)]\times\mathcal{I}\tau\mathcal{Y}$, (by 2.2.6-1). Hence, by 2.7, the mapping f_1 may be extended to the union F_0+F^*, which contradicts the definition of the set F^*.

5. Remarks and problems

5.1. The properties of the relations τ and φ which have been proved above suggest some connexion between these and the set-families dimensionizing the space ("les familles dimensionnantes", see [9], p. 187). Since we do not know whether those relations are for any subset domain-hereditary or not, we have to assume a weaker hypothesis than those in [6] when transferring the results of the theory of these families. Another paper will be devoted to these problems.

5.2. Next to the problem P 86 of [11] the problems

$$A\subset\mathcal{X}\varphi\mathcal{Y}\Rightarrow A\varphi\mathcal{Y} \tag{5.2.1}$$

and

$$[(A\subset\mathcal{X})\cdot(\mathcal{X}\times\mathcal{I}\tau\mathcal{Y})]\Rightarrow A\times\mathcal{I}\tau\mathcal{Y} \tag{5.2.2}$$

can be discussed. If $\mathcal{Y}=\mathcal{S}_n$, both implications are true.

5.3. The problem of the existence of some analogue to the theorem on the dimension of a Cartesian product of two sets (see [9], p. 225) among the properties of the relations under consideration suggests the question whether

$$\mathcal{X}\tau\mathcal{Y}\in ANR\Rightarrow\mathcal{X}\times\mathcal{I}\varphi\mathcal{Y}. \tag{5.3.1}$$

If $\mathcal{Y}=\mathcal{S}_n$, the implication is true, namely the Cartesian product of a set at most n-dimensional by $\mathcal{I}$ is at most $(n+1)$-dimensional.

If an implication converse to 4.4 is true:

$$\mathcal{X}\tau\mathcal{Y}\in ANR\Rightarrow\prod_{\varepsilon>0}\prod_{x\in\mathcal{X}}\sum_{U_x\subset\mathcal{X}}[(\mathrm{d}\,U_x<\varepsilon)\cdot(\operatorname{Fr}U_x\times\mathcal{I}\tau\mathcal{Y})], \tag{5.3.2}$$

then the evident implication

$$\prod_{\varepsilon>0}\prod_{x\in\mathcal{X}}\sum_{U_x\subset\mathcal{X}}[(\mathrm{d}\,U_x<\varepsilon)\cdot(\operatorname{Fr}U_x\times\mathcal{I}\tau\mathcal{Y}\in ANR)]\Rightarrow\mathcal{X}\times\mathcal{I}\varphi\mathcal{Y} \tag{5.3.3}$$

immediately implies (5.3.1).

Moreover, exactly as 3.6 and 3.7, it can be proved that (5.3.2) is equivalent to the implication

(5.3.4) $$\mathcal{X}\,\tau\,\mathcal{Y}\in ANR \Rightarrow \sum_{A,B}[(\mathcal{X}=A+B)\cdot(A\times\mathcal{J}\,\tau\,\mathcal{Y})\cdot(\dim B\leqslant 0)],$$

which is true for $\mathcal{Y}=\mathcal{S}_n$.

5.4. It is not known whether if the space $\mathcal{X}$ is a φ-manifold relative to the space $\mathcal{Y}\in ANR$, then $\mathcal{X}\,\overline{\tau}\,\mathcal{Y}$.

If $\mathcal{Y}=\mathcal{S}_n$, the answer is positive.

5.5. It has been proved in [11] that the relation $\overline{\tau}$ is domain invariant relative to the ε-transformations. It is not known whether the same is true for the relation $\overline{\varphi}$. It is true if $\mathcal{Y}=\mathcal{S}_n$.

5.6. The considerations of chapter 4 suggest the idea of a classification of spaces by means of the following congruence according to the relation τ.

The set A is *congruent* to the set B according to the relation τ:

$$A \underset{(\tau)}{\equiv} B,$$

if and only if for each space $\mathcal{Y}$ which is ANR the following equivalence is true: $A\,\tau\,\mathcal{Y} \Leftrightarrow B\,\tau\,\mathcal{Y}$.

We shall notice that:

1. the relation $\underset{(\tau)}{\equiv}$ belongs to the equivalence type;

2. $A \underset{\text{top}}{=} B \Rightarrow A \underset{(\tau)}{\equiv} B$, but the converse implication is not true, which is indicated by the example of spaces with different finite powers;

3. $A \underset{(\tau)}{\equiv} B \Rightarrow \dim A = \dim B$, if the $\dim X=\infty$ we consider as defined (cf. [8], p. 24).

The following problem arises in connection with the above relationships:

What are the characteristics of spaces which are τ-related to some defined space, for instance to the torus?

It is easy to see (4.1 and [10], p. 256) that a sufficient condition is for instance that the space be at most one-dimensional. A condition both sufficient and necessary is not known.

Bibliography

[1] P. S. Aleksandrov, *Современное состояние теории размерности*, Усп. мат. наук VI. 5 (45) (1951), p. 43-68.

[2] K. Borsuk, *Sur les retractes*, Fund. Math. 17 (1931), p. 152-170.

[3] — *Über eine Klasse von lokal zusammenhängenden Räumen*, ibidem 19 (1932), p. 220-242.

[4] K. Borsuk, *Un théorème sur les prolongements de fonctions continues*, Ann. Soc. Pol. Math. 16 (1937), p. 218.

[5] O. Hanner, *Some theorems on ANR*, Arkiv för Math. 1.5 (1951), p. 389-408.

[6] W. Hurewicz, *Normalbereiche und Dimensionstheorie*, Math. Annalen 96 (1927), p. 760.

[7] — *Über Abbildungen topologischer Räume auf die $\mathcal{S}_n$*, Fund. Math. 24 (1935), p. 144-150.

[8] W. Hurewicz and D. Wallman, *Dimension theory*, Princeton 1948.

[9] C. Kuratowski, *Topologie I*, Warszawa 1948.

[10] — *Topologie II*, Warszawa 1950.

[11] — *Sur quelques problèmes topologiques concernant le prolongement des fonctions continues*, Coll. Math. 2 (1951), p. 186-191.

[12] A. D. Wallace, *Dimensional types*, Bull. Amer. Math. Soc. 51 (1943), p. 679-681.

[13] G. T. Whyburn, *Potentially regular point sets*, Fund. Math. 16 (1930), p. 160-172.

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