Extension of the set on which mappings into S^n are homotopic

by

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§ 1. Introduction. The following question has been raised by A. Granas [see „The New Scottish Book", Wrocław, 1946-1958, problem 179]:

"The function $f(x)$ is defined on a compact space X and its values lie on the n-dimensional sphere S^n. If $X_0 \subset X$ denotes a set on which $f(x)$ is homotopic to a constant, does there exist an open set G which contains X_0 and on which $f(x)$ is also homotopic to a constant?"

In this paper we answer the above question affirmatively for X a metric space (compactness is not used in the proof of our theorem).

If H is a homotopy connecting f and g on X_0, f and g being mappings on X into S^n, one might hope to find an open set G containing X_0 and a homotopy connecting f and g on G which is an extension of H. However, there is an example which shows that such an extension may not exist. In the proof of our theorem, we make use of H and an averaging process to construct a homotopy M which connects f and g on an open set G which contains X_0. It is seen that we may make M be as close as we please to H on X_0 (within any preassigned positive distance), although we cannot require M and H to agree on X_0.

§ 2. Main results. Let (X, d) be a metric space, and let f and g be mappings on X into the n-sphere S^n. We assume that d is a bounded metric for X, and that f is homotopic to g on a subset X_0 of X. We let I be the closed unit interval $[0, 1]$, and we embed S^n as the unit sphere in Euclidean $(n+1)$-space E^{n+1}. If $v \in E^{n+1}$, we let $|v|$ denote the norm of v.

LEMMA. *If $\mu, u_1, u_2, u_3, \ldots$ are members of S^n, $|u_i - \mu| < \theta < 1/2$ for each i, $\lambda_i \geqslant 0$ for each i, $\sum_{i=1}^{\infty} \lambda_i$ converges, $A = \sum_{i=1}^{\infty} \lambda_i u_i$, and $\lambda_i > 0$ for some i, then:*

$$(1-\theta) \sum_{i=1}^{\infty} \lambda_i \leqslant |A| \leqslant (1+\theta) \sum_{i=1}^{\infty} \lambda_i,$$

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and

$$|(A/|A|)-\mu| < 4\theta .$$

Proof.

$$|A| \leqslant \Big|\sum_{i=1}^{\infty} \lambda_i(u_i-\mu)\Big| + \Big|\sum_{i=1}^{\infty}\lambda_i\mu\Big|$$
$$\leqslant \theta \sum_{i=1}^{\infty}\lambda_i + \sum_{i=1}^{\infty}\lambda_i = (1+\theta)\sum_{i=1}^{\infty}\lambda_i .$$

Also,

$$\sum_{i=1}^{\infty}\lambda_i = \Big|\sum_{i=1}^{\infty}\lambda_i\mu\Big| \leqslant \Big|\sum_{i=1}^{\infty}\lambda_i(\mu-u_i)\Big| + \Big|\sum_{i=1}^{\infty}\lambda_i u_i\Big| .$$

Thus,

$$\sum_{i=1}^{\infty}\lambda_i \leqslant \theta\sum_{i=1}^{\infty}\lambda_i + |A|, \quad \text{and} \quad (1-\theta)\sum_{i=1}^{\infty}\lambda_i \leqslant |A| .$$

This proves our first conclusion.

Now,

$$(A/|A|)-\mu \leqslant \Big(\Big|\sum_{i=1}^{\infty}\lambda_i(u_i-\mu)\Big| + \Big|\sum_{i=1}^{\infty}\lambda_i\mu - |A|\mu\Big|\Big)\Big/|A|$$
$$\leqslant \Big(\theta\sum_{i=1}^{\infty}\lambda_i + \Big|\sum_{i=1}^{\infty}\lambda_i - |A|\Big|\Big)\Big/|A|$$
$$\leqslant \Big(\theta\sum_{i=1}^{\infty}\lambda_i + \theta\sum_{i=1}^{\infty}\lambda_i\Big)\Big/(1-\theta)\sum_{i=1}^{\infty}\lambda_i$$
$$\leqslant 2\theta/(1-\theta) < 4\theta .$$

THEOREM. *If* $H: X_0\times I\to S^n$ *is a homotopy such that* $H(x,0)=f(x)$ *and* $H(x,1)=g(x)$ *for* $x\in X_0$, *and* $\varepsilon>0$, *then there exists an open set* G *containing* X_0 *and a homotopy* $M: G\times I\to S^n$ *such that* $M(x,0)=f(x)$ *and* $M(x,1)=g(x)$ *for* $x\in G$, *and* $|M(x,t)-H(x,t)|<\varepsilon$ *for* $x\in X_0$ *and* $t\in I$.

Proof. We may assume without loss of generality that $\varepsilon<1$. We define $\eta=\varepsilon/42$.

Since I is compact, for each $p\in X_0$ there exists $\delta(p)>0$ such that if $U(p)$ is the $\delta(p)$-neighborhood of p in X_0, then the diameter of $H(U(p),t)$ is less than η for each $t\in I$. me define $V(p)$ to be the $\delta(p)/2$-neighborhood of p in X. me assume that $\delta(p)$ has been chosen small enough so that $X-V(p)\neq\emptyset$, and $f(V(p))$ and $g(V(p))$ each have diameter less than η.

If $p\in X_0$, $q\in X_0$ and $V(p)\cap V(q)\neq\emptyset$, then either $p\in U(q)$ or $q\in U(p)$, and hence $|H(p,t)-H(q,t)|<\eta$ for each $t\in I$.

We define $G=\bigcup_{p\in X_0}V(p)$. For each $p\in X_0$ we define a mapping α_p on G into the set of non negative numbers by

$$\alpha_p(x)=\inf\{d(x,y)\,|\,y\in X-V(p)\} .$$

It is easy to verify that

$$|\alpha_p(x_1)-\alpha_p(x_2)|\leqslant d(x_1,x_2)$$

for all $p\in X_0$, $x_1\in G$, $x_2\in G$.

We let F be the set of all mappings of I into S^n, and metrize F by

$$\varrho(u,v)=\sup\{|u(t)-v(t)|\,\big|\,t\in I\} .$$

The space (F,ϱ) is separable, and contains a countable dense subset D.

Now, for each $p\in X_0$, we choose a member φ_p of D such that

$$|H(p,t)-\varphi_p(t)|<\eta$$

for each $t\in I$. The set of all members of D which are chosen can be arranged in a sequence $u_1,u_2,u_3,\dots$ We define a sequence $\beta_1,\beta_2,\beta_3,\dots$ of real valued functions on G by letting

$$E_i=\{p\,|\,\varphi_p=u_i\}$$

and

$$\beta_i(x)=\sup\{\alpha_p(x)\,|\,p\in E_i\} .$$

It is a simple matter to verify that

$$|\beta_i(x_1)-\beta_i(x_2)|\leqslant\sup\{|\alpha_p(x_1)-\alpha_p(x_2)|\,\big|\,p\in E_i\}\leqslant d(x_1,x_2) .$$

Thus, each β is continuous.

Since we have assumed that d is a bounded metric, the functions α_p, $p\in X_0$, are uniformly bounded. Thus, we may define $k: G\times I\to E^{n+1}$ by

$$k(x,t)=\sum_{i=1}^{\infty}\beta_i(x)2^{-i}u_i(t) .$$

If $x\in G$, then there exists i such that $\beta_i(x)\neq 0$. Suppose $\beta_j(x)\neq 0\neq\beta_h(x)$ for some $x\in V$. Then there exist points p and q in X_0 such that $\alpha_p(x)\neq 0\neq\alpha_q(x)$ and $\varphi_p=u_j$, $\varphi_q=u_h$. It follows that $x\in V_p\cap V_q$. This implies that $|H(p,t)-H(q,t)|<\eta$ for each $t\in I$, and hence $\varrho(u_i,u_j)<3\eta$.

If we now apply our Lemma to the series $\sum_{i=1}^{\infty}\beta_i(x)2^{-i}u_i(t)$, letting μ be one of the $u_i(t)$ for which $\beta_i(x)\neq 0$, we see that $|k(x,t)|>0$.

Thus, we may define $K: G\times I\to S^n$ by

$$K(x,t)=k(x,t)/|k(x,t)| .$$

It also follows from our Lemma that if $\beta_i(x) \neq 0$, then

$$|K(x,t)-u_i(t)| < 12\eta .$$

We now obtain for $x \in G$:

$$|f(x)-f(p)| < \eta \text{ for some } p \in X_0,$$
$$f(p) = H(p,0),$$
$$|H(p,0)-\varphi_p(0)| < \eta,$$
$$\varphi_p(0) = u_i(0) \text{ for some } i \text{ such that } \beta_i(x) \neq 0,$$

and finally

$$|u_i(0)-K(x,0)| < 12\eta .$$

Thus, it follows that $|f(x)-K(x,0)| < 14\eta = \varepsilon/3$. Likewise, $|g(x)-K(x,1)| < \varepsilon/3$.

If $x \in X_0$, $|H(x,t)-\varphi_x(t)| < \eta$ for all $t \in I$. We have $\varphi_x = u_i$ for some i such that $\beta_i(x) \neq 0$, and hence $|u_i(t)-K(x,t)| < 12\eta$. It follows that

$$|H(x,t)-K(x,t)| < 13\eta < \varepsilon/3$$

for $x \in X_0$ and $t \in I$.

We now define $m\colon G \times I \to E^{n+1}$ by

$$m(x,t) = K(x,t)+(1-t)[f(x)-K(x,0)]+t[g(x)-K(x,1)] .$$

It follows that $|m(x,t)-K(x,t)| < \varepsilon/3$ for all $x \in G$, $t \in I$, and $m(x,0) = f(x)$, $m(x,1) = g(x)$. Thus, we may define $M\colon G \times I \to S^n$ by letting $M(x,t) = m(x,t)/|m(x,t)|$. We obtain $M(x,0) = f(x)$ and $M(x,1) = g(x)$.

Since $M(x,t)$ is the point on S^n nearest $m(x,t)$, we have $|M(x,t)-K(x,t)| \leqslant |M(x,t)-m(x,t)|+|m(x,t)-K(x,t)| < 2\varepsilon/3$.

Finally, for $x \in X_0$ and $t \in I$,

$$|M(x,t)-H(x,t)| \leqslant |M(x,t)-K(x,t)|+|K(x,t)-H(x,t)| < 2\varepsilon/3+\varepsilon/3 = \varepsilon .$$

This concludes the proof that M has the desired properties.

COROLLARY. *If f and g are homotopic on a subset X_0 of X, then there is an open set W on which f and g are homotopic such that $W \supset X_0$ and W is dense in X.*

Proof. By our Theorem, there exists an open set $G \supset X_0$ and a homotopy $M\colon G \times I \to S^n$ such that $M(x,0) = f(x)$ and $M(x,1) = g(x)$.

We let π be the set of all homotopies $N\colon G(N) \times I \to S^n$ for which: $G(N)$ is open and $G(N) \supset G$, N is an extension of M, and $N(x,0) = f(x)$, $N(x,1) = g(x)$ for $x \in G(N)$.

It is possible to partially order π by defining $N_1 < N_2$ if and only if $G(N_1) \subset G(N_2)$ and N_2 is an extension of N_1. Every chain in the partially ordered system $(\pi, <)$ has an upper bound, so by Zorn's lemma, there is a maximal element N^*. It is easy to see that, because N^* is maximal, $G(N^*)$ is dense in X. Hence, we obtain the desired set by letting $W = G(N^*)$.

Example. We define E to be the mapping of the real number system into S^1 (thought of as the group of complex numbers of unit modulus) which is defined by $E(x) = e^{ix}$. We let X be the closed interval $[0,1]$, and define $f = E|X$.

Next, we let A_n be the open interval $(2^{-n-1}, 2^{-n})$, and let $X_0 = \{0\} \cup \bigcup_{n=0}^{\infty} A_n$. We define r_n to be the mid point of A_n.

A homotopy $H\colon X_0 \times I \to S^1$ can be defined by

$$H(x,t) = \begin{cases} E(tr_n+t^2[x-r_n]) & \text{for } x \in A_n, \\ 1 & \text{for } x = 0 . \end{cases}$$

Clearly, $H(x,1) = f(x)$ and $H(x,0) = 1$ for all $x \in X_0$.

Now suppose that there exists an open (relative to X) set $G \supset X_0$ and an extension M of H such that $M\colon G \times I \to S^1$. It is easy to see that $2^{-n} \in G$ for all large n, since $0 \in G$, and hence $M(2^{-n},t)$ is defined for large n and all $t \in I$. Moreover, we must have $E(tr_n+t^2[2^{-n}-r_n]) = M(2^{-n},t) = E(tr_{n-1}+t^2[2^{-n}-r_{n-1}])$. This implies that

$$tr_n+t^2[2^{-n}-r_n] = tr_{n-1}+t^2[2^{-n}-r_{n-1}]$$

and hence $(t-t^2)(r_n-r_{n-1}) = 0$. This is impossible for $0 < t < 1$.

Our example shows that, in general, the homotopy M of our Theorem cannot be obtained by extending the homotopy H.

Remark. In the proof of our Theorem, the fact that S^n is the unit sphere in E^{n+1} is used in the following way: Given a set of points on S^n having sufficiently small diameter, the number of points in the set being finite or enumerable, and a positive weight for each point of the cluster, we take a weighted average of the points in E^{n+1} and project this weighted average radially from the origin onto S^n. This procedure is used in defining K, and a similar technique is employed in defining M.

Now, if S^n is replaced by any finite n-dimensional polyhedron P^n, we can employ a similar technique. First of all, P^n can be embedded in E^{2n+1}. Then, since every finite polyhedron is an absolute neighborhood retract, there exists an open set W (in E^{2n+1}) containing P^n and a retraction R of W onto P^n. Hence, for all sufficiently small (in diameter) countable sets of points on P^n, we can take weighted averages in E^{2n+1}, and then, each such weighted average being in W, retract it by R onto P^n.

Thus, it is possible to replace the hypothesis in the statement of our Theorem that f and g are mappings into the n-sphere by the more general hypothesis that they are mappings into a finite polyhedron.

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Résolution d'un problème de M. Z. Zahorski sur les limites approximatives

par

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Du théorème de Young sur la symétrie de la structure d'une fonction résulte la conséquence suivante:

Pour chaque fonction $f(x)$ de la variable réelle, définie dans un certain intervalle fermé, l'ensemble de toutes les valeurs x, pour lesquelles la limite supérieure à droite est inférieure à la limite supérieure à gauche, est tout au plus dénombrable.

M. Zahorski a demandé si ce théorème reste vrai quand on y remplace les limites supérieures par les limites supérieures approximatives.

Ce travail a pour objet de résoudre le problème de M. Zahorski. Nous y montrons, en effet, qu'il est possible de trouver une fonction de la variable réelle $f(x)$, définie pour chaque x, pour laquelle l'ensemble des points, dont la limite approximative supérieure à droite est inférieure à la limite approximative supérieure à gauche, a la puissance du continu.

La construction de cette fonction se composera de 2 parties. Dans la première, on construit dans l'intervalle $[-1, 2]$ l'image géométrique d'une fonction $f(x)$ non décroissante et bornée, qui admet, en tout point d'un ensemble non dense C ayant la puissance du continu, une dérivée à droite nulle et un nombre dérivé de Dini à gauche positif. En outre, cette fonction remplit la condition de Lipschitz dans l'intervalle de définition. Dans la seconde partie de la construction on détermine, à l'aide de la dérivée de la fonction $f(x)$, la fonction caractéristique $F(x)$ d'un certain ensemble, qui représentera la fonction cherchée.

I^{re} partie de la construction

Construisons l'image de la fonction $f(x)$ comme le produit d'une suite déscendante d'ensembles fermés, bornés et non vides A_n. Les ensembles A_n sont connexes, se composent d'un nombre fini de segments rectilignes et de certains quadrilatères concaves. Nous définissons les ensembles A_n par induction de la façon suivante