Rates of change and functional relations *

by

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1. Introduction. In [1], Menger defines what he calls the rate of change of one fluent with respect to another, relative to a subset of a cartesian product of the domains of these fluents. If A denotes a set, by a *fluent with domain A*, Menger means a transformation from A into the real numbers. In the present note the case of two functions f and g each having a set R of real numbers as its domain is considered. The purpose is to determine a relation between the existence of a rate of change of f with respect to g and the existence of a functional relation between f and g. Throughout this note the word *interval* is used to mean closed interval and the word *segment* to mean open interval.

2. Rates of change. The rate of change of f with respect to g will be considered relative to the subset of $R\times R$ consisting of all pairs (x, x) for all numbers x in R. In this case Menger's definition is equivalent to the following:

The statement that the number c is the derivative of f with respect to g at the number x_0 in R means (1) for each number $\varepsilon > 0$ there is a number x in R such that $|x-x_0| < \varepsilon$ and $g(x) \neq g(x_0)$, and (2) if $\varepsilon > 0$ there is a number $\delta > 0$ such that if x is a number in R for which $|x-x_0| < \delta$ and $g(x) \neq g(x_0)$, then $|[f(x)-f(x_0)]/[g(x)-g(x_0)]-c| < \varepsilon$.

Such a number c will be denoted by $D_g f(x_0)$. In order that this definition be equivalent to the statement that

$$D_g f(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{g(x_0+h)-g(x_0)} \quad \text{for} \quad x_0+h \text{ in } R,$$

it is necessary and sufficient that the number x_0 have the property that $g(x) \neq g(x_0)$ for each number x in some open subset of R containing x_0. A number with this property would be called *g-discriminating* by Menger.

An example is given in [1] of functions f and g each having the Cantor ternary set C as its domain and such that (1) each number in

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the domain of g is g-discriminating, (2) $D_g f(x)$ exists for each x in the domain of g, and (3) f is not a function of g on any open subset of the domain of g. It follows from the theorem of section 4 below that there do not exist continuous functions f and g on the interval $[0,1]$ satisfying conditions (2) and (3). Indeed, if g is continuous on $[0,1]$ then there is no function f with domain $[0,1]$ such that f and g satisfy conditions (1), (2) and (3).

3. Functional relations. Suppose each of f and g is a function whose domain includes the number set R. The statement that f is a function of g on R means there is a function h with domain $g(R)$ such that for each x in R, $f(x) = h[g(x)]$. The statement that f is a function of g near the number x_0 in R means there is an open subset S of R containing x_0 such that f is a function of g on S. Clearly if f is a function of g near some number in R then the set of all such numbers is an open subset of R. The following statement is an immediate consequence of these definitions. In order that f should not be a function of g on R it is necessary and sufficient that there exist two numbers x and y in R such that $g(x) = g(y)$ and $f(x) \neq f(y)$.

The following is a simple example of differentiable functions each with domain the interval $[-2,2]$ and such that each is a function of the other near every number in $[-2,2]$ but neither is a function of the other on $[-2,2]$:

$$f(x) = x^2 \quad \text{for} \quad -2 \leqslant x \leqslant 2, \qquad g(x) = \begin{cases} x^2 & \text{for} \quad -2 \leqslant x \leqslant 1, \\ 2x-1 & \text{for} \quad 1 \leqslant x \leqslant 2. \end{cases}$$

In this example f and g are functionally related on each of two overlapping segments but not on the union of these segments.

4. The main theorem. *If each of f and g is a continuous function on the interval $[0,1]$ and $D_g f$ exists on $[0,1]$ then there is a dense open subset K of $[0,1]$ such that f is a function of g near each number in K.*

The proof of this theorem will be based on the following two lemmas.

LEMMA 1. *If each of f and g is a continuous function whose domain includes the number interval I, M denotes the subset of the cartesian plane consisting of all points $[g(x), f(x)]$ for all x in I, and f is not a function of g on any open subset of I, then some vertical line contains infinitely many points of M.*

Let T denote the continuous transformation of I onto M such that for each number x in I, $T(x)$ is $[g(x), f(x)]$, and suppose that no vertical line contains infinitely many points of M. Let a_1 and b_1 denote numbers in I such that $g(a_1) = g(b_1)$ and $f(a_1) \neq f(b_1)$. Let L_1 denote the vertical line containing $T(a_1)$ and $T(b_1)$. The image of $[a_1, b_1]$ under T is a compact continuous curve and thus contains an arc α from $T(a_1)$ to $T(b_1)$ which contains at most finitely many points of L_1. Let $T(c_1)$ denote a point of α not on L_1 and let L_1' denote a vertical line between $T(a_1)$ and $T(c_1)$. There is a last point A_1 of L_1 on α in the order from $T(a_1)$ to $T(c_1)$, and a first point A_1' of L_1' on α in the order from A_1 to $T(c_1)$. Similarly there is a last point B_1' of L_1' on α in the order from $T(c_1)$ to $T(b_1)$ and a first point B_1 of L_1 on α in the order from B_1' to $T(b_1)$. The subarcs A_1A_1' and B_1B_1' of α have no point in common and they lie, except for their endpoints, between L_1 and L_1'. Let R_1 denote a circular interior intersecting B_1B_1', lying wholly between L_1 and L_1' and containing no point of A_1A_1'. Since T is continuous, there is a subsegment S_1 of the segment (a_1, b_1) such that $T(S_1)$ is a subset of R_1 and since f is not a function of g on S_1, there exist numbers a_2 and b_2 in S_1 such that $T(a_2)$ and $T(b_2)$ are different points of the same vertical line, L_2. By a process similar to that described above it may be established that there exist a line L_2' between L_1 and L_1' and different from L_2, points A_2 and B_2 on L_2, points A_2' and B_2' on L_2', and two mutually exclusive arcs A_2A_2' and B_2B_2' such that each is a subset of M and lies except for its endpoints between L_2 and L_2'. This process may be continued to establish the existence of a sequence of pairs of vertical lines $(L_1, L_1'), (L_2, L_2'), \ldots$ and a sequence of mutually exclusive arcs $A_1A_1', A_2A_2', \ldots$ such that for each positive integer n, (1) L_{n+1} and L_{n+1}' lie between L_n and L_n' and (2) A_nA_n' has one endpoint on L_n, the other on L_n' and lies except for its endpoints between L_n and L_n'. There is a vertical line L which for each positive integer n lies between L_n and L_n'. L must intersect each arc of the sequence $A_1A_1', A_2A_2', \ldots$ and thus must contain infinitely many points of M. This completes the proof.

LEMMA 2. *If each of f and g is a continuous function whose domain includes the number interval I and $D_g f$ exists on I, then for each number x_0 in I there is an open subset S of I containing x_0 such that if x is in S and $g(x) = g(x_0)$ then $f(x) = f(x_0)$.*

Let M denote a subset of the cartesian plane and T a continuous transformation of I onto M as described in the proof of Lemma 1. Assume there is a number x_0 in I such that each open subset of I which contains x_0 also contains a number x such that $g(x) = g(x_0)$ but $f(x) \neq f(x_0)$. Let L denote the vertical line containing $T(x_0)$. Every segment containing x_0 contains a number x such that $T(x)$ is on L and is different from $T(x_0)$. It follows that there is a sequence $x_1, x_2, \ldots$ of distinct numbers in I convergent to x_0 such that (1) for each positive integer n, $T(x_n)$ is on L, (2) if m and n denote positive integers, $T(x_n) \neq T(x_m)$, and (3) the sequence $T(x_1), T(x_2), \ldots$ converges to $T(x_0)$. Let $c = D_g f(x_0)$, let $\varepsilon > 0$, and let R denote the set of all points (x, y) such that $|[y-f(x_0)]/[x-g(x_0)]-c| < \varepsilon$. No point of L except $T(x_0)$ is a limit point of R and thus for each positive integer n there is a circular interior C_n containing $T(x_n)$ but no point of R. Furthermore, for each n there is a subsegment S_n of I such that (1) S_n contains x_n, (2) S_n is of length less than $1/n$, and (3) $T(S_n)$ is a subset of C_n. But $T(S_n)$ is not a subset of L since $D_g f$ exists on I and thus S_n contains a number t_n such that $T(t_n)$ is not on L. Note that $T(t_n)$ is in C_n and not in R. Now there is a number $\delta > 0$ such that if t is in I, $|t-x_0| < \delta$, and $T(t)$ is not on L, then $T(t)$ is in R. There is, however, an integer n such that S_n is a subset of the segment $(x_0-\delta, x_0+\delta)$ and this implies that $|t_n-x_0| < \delta$ although $T(t_n)$ is not on L and not in R. This is a contradiction and the lemma follows.

Beginning now with the proof of the main theorem, let f and g denote continuous functions each having $[0,1]$ as its domain and such that $D_g f$ exists on $[0,1]$. Assume that there is a subinterval I of $[0,1]$ such that f is not a function of g near any number in I. It follows from Lemma 1 that if T denotes a transformation as defined above then there is a vertical line L which contains infinitely many points of $T(I)$. This implies that there is a number x_0 in I and an infinite sequence $x_1, x_2, \ldots$ of distinct numbers in I convergent to x_0 such that (1) for each $n \geqslant 0$, $T(x_n)$ is on L and (2) if m and n are integers, then $T(x_m) \neq T(x_n)$. But from Lemma 2 it follows that there is an open subset S of I containing x_0 such that if x is in S and $g(x) = g(x_0)$, then $f(x) = f(x_0)$. This is a contradiction and thus each subinterval of $[0,1]$ contains a number near which f is a function of g. The set of all such numbers is an open subset of $[0,1]$. This completes the proof.

5. Remarks. It can be shown that if each of f and g is a function with domain the set R of real numbers, $D_g f(x)$ exists for each x in R, and g is continuous on R, then f is continuous at each g-discriminating number in R. From this and the theorem of section 4 it follows easily that:

If each of f and g is a function with domain $[0,1]$, $D_g f$ exists on $[0,1]$, g is continuous on $[0,1]$ and each number x in $[0,1]$ is g-discriminating, then there is a dense subset K of $[0,1]$ such that f is a function of g near each number in K.

The theorem of section 4 does not hold if g is not continuous, even if f is increasing. This may be seen with the aid of the following example, suggested independently by a student, Mr. T. Engelhart.

Let $f(x) = x$ for each x in $[0,1]$,

$$g(x) = \begin{cases} 0 & \text{for each rational number } x \text{ in } [0,1], \\ 1 & \text{for each irrational number } x \text{ in } [0,1]. \end{cases}$$

In this example $D_g f(x) = 0$ for each x in $[0,1]$ but f is not a function of g on any open subset of $[0,1]$.

Lemma 2 cannot be strengthened by replacing the condition that $D_g f$ exist on I with the condition that $D_g f(x_0)$ exist. This may be seen with the aid of the following example.

Let

$$f(x) = \begin{cases} x\cdot\sin(1/x) & \text{for} \quad 0 < x \leqslant 1, \\ 0 & \text{for} \quad x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} f(x) & \text{for each } x \text{ such that } f(x) \geqslant 0, \\ 0 & \text{for each } x \text{ such that } f(x) < 0. \end{cases}$$

In this example $D_g f(0) = 1$ but every segment containing 0 contains a number x such that $g(x) = g(0) = 0$ and $f(x) \neq 0$.

Finally, it will be noted that the theorem of section 4 cannot be strengthened (even if $f = j$, the identity function on $[0,1]$) by requiring that the complement of K, if it exists, be countable. That is, there is a function g such that (1) g is continuous on $[0,1]$, (2) $D_g j$ exists on $[0,1]$, (3) each number x in $[0,1]$ is g-discriminating, and (4) the set of numbers near which j is not a function of g is uncountable. How such a function may be defined is indicated below but no proof is given to show that it has the required properties.

Let C denote the Cantor ternary set on $[0,1]$ and let H denote the collection of segments of the complement of C. For each segment s in H, let R_s denote a rhombic disc having s as one axis and whose other axis is perpendicular to s and of length the square of the length of s. Furthermore, let G_s denote a continuous function with domain $\bar{s}$ (the closure of s) such that (1) the graph of G_s is a subset of R_s, (2) G_s has derivative 0 at the endpoints of s, and (3) there exist numbers $x < y$ in s such that G_s has vertical cusps at x and y, and the derivative of G_s exists on $s-(x+y)$, being less than -1 on the segment (x, y) and positive elsewhere. Let g denote a function such that for each x in C, $g(x) = x$, and for each number x in a segment s of H, $g(x) = x + G_s(x)$. It can be shown that g has the properties stated above.

References

[1] K. Menger, *Rates of change and derivatives*, Fundamenta Mathematicae 46 (1958), p. 89-102.

[2] — *Is w a function of u?*, Colloquium Mathematicum 6 (1958), p. 41-47.

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