Extension of mappings on metric spaces

by

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Introduction. If $\mathfrak{M}$ is a subset of Hilbert space and φ is a topological map of $\mathfrak{M}$ onto $\mathfrak{M}$, it is, in general, impossible to extend φ topologically (or even continuously) over the closure $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. However, is it possible to find a suitable topological re-embedding M of $\mathfrak{M}$ in Hilbert space such that φ may be extended over $\overline{M}$? The answer to this question is in the affirmative. Actually, we shall prove far more: If Φ is a countable set of continuous mappings of the separable metrizable space M into itself, one can find a compact metric space $\overline{M}$ in which M is densely embedded, such that every continuous map of the given set may be extended continuously over $\overline{M}$.

In order to avoid unnecessary repetitions it is useful to introduce the notion of Φ-compactification. If M is a separable metrizable space and Φ a set of continuous maps of M into M, the space $\overline{M}$ is called a *Φ-compactification of M*, if $\overline{M}$ is a compactum (compact metric space) containing M densely, such that every element of Φ may be extended continuously over $\overline{M}$.

We shall investigate Φ-compactifications in § 2 and we shall e.g. find, for every set Φ closed under multiplication, necessary and sufficient conditions under which such a Φ-compactification exists (Theorem 2.12). The way in which Φ operates on M enters into these conditions. It is shown by examples and counterexamples that this is essential, and the authors believe that it is practically impossible to find necessary and sufficient conditions in terms of M alone. Applications to the case where Φ is a set of autohomeomorphisms of M are obtained as corollaries.

It is of interest to ask for conditions under which *every* autohomeomorphism of M can be extended to a suitable metric compactification $\overline{M}$ of M. We shall give in § 3 sufficient conditions — improving a little on already known results — which are believed to be rather general. Also a number of examples is given to clarify the situation.

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Another problem of a somewhat different nature is the following. The (not necessarily separable) metric space M may be embedded in its completion $\widetilde{M}$. To what extent can we extend an autohomeomorphism of M over a part of $\widetilde{M}$? In analogy to a well-known theorem of Lavrentieff we show in § 1 that there is a G_δ-set in $\widetilde{M}$, containing M, over which homeomorphic extension is possible. Our proof does not make use of Lavrentieff's result. R. Engelking — who made several useful remarks which have led to improvements of the present paper — has given a nice proof of this theorem based on Lavrentieff's theorem.

Since the G_δ-sets in complete metric spaces are precisely the topologically complete spaces, the theorem in § 1 actually says that if M is any metrizable space, and Φ a countable set of homeomorphisms from M on M, then there is a complete metric space $\widetilde{M}$ containing M densely over which all the functions in Φ can be extended to homeomorphisms.

The results in § 2 can be thought of as theorems on completely regular spaces. Thus a separable metrizable space may be thought of as a completely regular space having a base of cardinal $\aleph_0$. In 2.3 we show that if Φ is a set of $\aleph_0$ continuous maps from such a space into itself, then there is a compactification having a base of cardinal $\aleph_0$ over which all the functions in Φ can be extended continuously. Here, and in other places in § 2 (suitably modified) $\aleph_0$ can be replaced by an arbitrary infinite cardinal. The details lie outside the scope of this paper; they will appear in a later publication.

§ 1. THEOREM. *Let A be a subspace of a complete metric space (M, ϱ), and let Φ be a countable set of homeomorphisms of A on A. Then there is a G_δ-set $\widetilde{A}$ in M, containing A densely, such that every function in Φ can be extended to a homeomorphism of $\widetilde{A}$ onto $\widetilde{A}$.*

Proof. We assume, without loss of generality, that A is dense in M, and that the functions φ_k in Φ form a group under multiplication. For each k, let $V(k)$ be the set of all points of M over which φ_k can be extended *continuously*. This set is well-known to be a G_δ. We denote by V the G_δ obtained by taking $\bigcap_k V(k)$, and by $\overline{\varphi}_k$ the continuous extension of φ_k over V. Now for each $q \in M$ and each φ_k, let $\sigma(\varphi_k, q)$ be either ∞ or the infimum of the set of real numbers ε with the property that there is a neighbourhood U of q such that for x and y in $U \cap A$, $\varrho(\varphi_k x, \varphi_k y) < \varepsilon$, and define

$$V_{k,l,n} = \left\{p \in V:\ \sigma(\varphi_k, \overline{\varphi}_l p) < \frac{1}{n}\right\},$$

where k, l, n run over all the positive integers. For fixed k, l, n, we shall show that $V_{k,l,n}$ is a G_δ. Suppose $p \in V_{k,l,n}$; then there is a neighbourhood U of $\overline{\varphi}_l p$ such that

(1) $$\text{for all } x, y \text{ in } U \cap A,\ \varrho(\varphi_k x, \varphi_k y) < 1/n\,.$$

Since $p \in V$, there is a neighbourhood W of p such that $\overline{\varphi}_l(W) \subset U$. Suppose $q \in W \cap V$. Then $\overline{\varphi}_l q \in U$, hence U is a neighbourhood of $\overline{\varphi}_l q$ with the property given by (1); that is, $q \in V_{k,l,n}$. It follows that $V_{k,l,n}$ is open in V, and is therefore itself a G_δ in M.

Now set

$$\widetilde{A} = \bigcap_{k,l,n} V_{k,l,n}\,.$$

$\widetilde{A}$ is a G_δ. For each k, we set

$$\widetilde{\varphi}_k = \overline{\varphi}_k|\widetilde{A}\,.$$

Then $\widetilde{\varphi}_k$ is a one to one continuous map from $\widetilde{A}$ onto $\widetilde{A}$ for each k, and therefore (since the $\{\varphi_k\}$ form a group) a homeomorphism for each k.

§ 2. Φ-Compactification.

2.1. LEMMA. *Let (M, ϱ) be a totally bounded metric space, and let $\Phi = \{\varphi_n\}$, n running over the non-negative integers, be a set of continuous mappings of M in M, where φ_0 is the identity map. Define a real-valued function $\widetilde{\varrho}$ on $M \times M$ by*

(a) $$\widetilde{\varrho}(x, y) = \max_n 2^{-n} \varrho(\varphi_n x, \varphi_n y)\,.$$

Then:

(1) *$\widetilde{\varrho}$ is a metric on M,*

(2) *ϱ and $\widetilde{\varrho}$ induce the same topology on M,*

(3) *$(M, \widetilde{\varrho})$ is totally bounded.*

Proof. (1) $\widetilde{\varrho}$ is clearly non-negative, and $\widetilde{\varrho}(x, y) = 0$ if and only if $\varrho(x, y) = 0$; i. e., if and only if $x = y$. The symmetry of $\widetilde{\varrho}$ likewise follows from that of ϱ. It remains to show that for all $x, y, z \in M$, $\widetilde{\varrho}(x, y) \leqslant \widetilde{\varrho}(x, z) + \widetilde{\varrho}(z, y)$.

First determine k so that

$$\max_n 2^{-n} \varrho(\varphi_n x, \varphi_n y) = 2^{-k} \varrho(\varphi_k x, \varphi_k y)\,.$$

For this k,

$$\begin{aligned}\widetilde{\varrho}(x, y) &\leqslant 2^{-k}\{\varrho(\varphi_k x, \varphi_k z) + \varrho(\varphi_k z, \varphi_k y)\} \\ &\leqslant \max_n 2^{-n}\{\varrho(\varphi_n x, \varphi_n z) + \varrho(\varphi_n z, \varphi_n y)\} \leqslant \widetilde{\varrho}(x, z) + \widetilde{\varrho}(z, y)\,.\end{aligned}$$

(2) For every $r > 0$, $\widetilde{S}_r x \subset S_r x$ for each $x \in M$. Hence $\widetilde{\varrho}$ induces a finer topology than ϱ. Conversely, consider $\widetilde{S}_r x$ for any x. Choose $k > 0$ such that $D \cdot 2^{-k} < r$, where D is the ϱ-diameter of M. Since each φ_n is

continuous, there is a ϱ-neighbourhood U of x such that for every $y \in U$ and for each $n < k$, $2^{-n}\varrho(\varphi_n x, \varphi_n y) < r$. Then $\tilde{\varrho}(x, y) < r$ for all $y \in U$.

(3) Here we again make use of the following: to show that $\tilde{\varrho}(x, y) < \varepsilon$, it suffices to show that $2^{-k}\varrho(\varphi_k x, \varphi_k y) < \varepsilon$ for the finite set of all k such that $2^k\varepsilon$ is no greater than the ϱ-diameter of M.

Suppose that $(M, \tilde{\varrho})$ is not totally bounded. Then for some $\varepsilon > 0$, we can find a sequence $\{x_i\}$ such that for $j \neq m$, $\tilde{\varrho}(x_j, x_m) > \varepsilon$. (M, ϱ) *is* totally bounded, hence every sequence in M contains a ϱ-fundamental subsequence. By successive refinement, we find a subsequence $\{y_i\}$ of $\{x_i\}$ such that $\{\varphi_k y_i\}$ is a ϱ-fundamental sequence for each k for which $2^k\varepsilon$ is no greater than the ϱ-diameter of M. Then for sufficiently large j and m, $2^{-k}\varrho(\varphi_k y_j, \varphi_k y_m) < \varepsilon$ for each such k, contradicting our assumption on $\{x_i\}$.

2.2. Lemma. *If the class of mappings Φ used to define $\tilde{\varrho}$ is closed under multiplication, then each $\varphi \in \Phi$ is uniformly continuous with respect to $\tilde{\varrho}$.*

Proof. Suppose the contrary. Then there exists an $\varepsilon > 0$ such that for each n we can find x_n, y_n with

$$\tilde{\varrho}(x_n, y_n) < 1/n \quad \text{and} \quad \tilde{\varrho}(\varphi x_n, \varphi y_n) > \varepsilon .$$

Hence, for each n there is a k such that

$$\varrho(\varphi_k\varphi x_n, \varphi_k\varphi y_n) > 2^k\varepsilon .$$

Since such an inequality is only possible for a finite number of values of k, it follows that the same k is associated with infinitely many values of n. But $\varphi_k\varphi = \varphi_{k'}$ is also in Φ, and by the definition of $\tilde{\varrho}$,

$$\varrho(\varphi_k\varphi x_n, \varphi_k\varphi y_n) \leqslant 2^{k'}\tilde{\varrho}(x_n, y_n)$$

for every n; a contradiction.

2.3. Theorem. *Let M be a separable, metrizable space, and let Φ be a countable set of continuous mappings from M into itself. Then M possesses a Φ-compactification $\tilde{M}$.*

Proof. We may assume without loss of generality that the set Φ is closed under products and contains the identity. Write Φ as $\{\varphi_n\}$, where n runs over the non-negative integers and φ_0 is the identity. Since M is separable, we may introduce a totally bounded metric ϱ into M so that M and (M, ϱ) are homeomorphic. Then, if $\tilde{\varrho}$ is the totally bounded metric defined by (a), M and $(M, \tilde{\varrho})$ are homeomorphic (Lemma 2.1). Denote the completion of the metric space $(M, \tilde{\varrho})$ by $\tilde{M}$. Clearly, $\tilde{M}$ is a compact metric space containing M densely. Finally, each $\varphi \in \Phi$, being uniformly continuous with respect to $\tilde{\varrho}$ (Lemma 2.2), can be continuously extended to $\tilde{M}$.

2.4. Example. The metric (a), Lemma 2.1, depends on the class of functions Φ. In Lemma 2.2, we assumed this class to be closed under multiplication. If this is not done, the functions in Φ may fail to carry fundamental sequences to fundamental sequences (in $(M, \tilde{\varrho})$). For instance, let M be the countable discrete space, considered as the subspace of the unit interval consisting of all points with coordinate $1/n$ or $1-1/n$, where n runs through the natural numbers $\geqslant 2$. Let φ_0 be the identity map, let φ_1 interchange, for each k, the points $1/4k$ and $1-1/4k$, and let φ_2 move each point x to the point with largest coordinate less than that of x. Further, for $n \geqslant 3$, let $\varphi_n = \varphi_2^n$, and form $\tilde{\varrho}$ as in Lemma 2.1. Now ϱ and $\tilde{\varrho}$ agree on the sequence $\{1/(2n+1)\}$, hence this sequence is also fundamental in $(M, \tilde{\varrho})$. The sequence

$$\left\{\varphi_2\left(\frac{1}{2n+1}\right)\right\} = \left\{\frac{1}{2n+2}\right\},$$

however, is *not* fundamental in $(M, \tilde{\varrho})$, since for every k,

$$\tilde{\varrho}\left(\frac{1}{4k}, \frac{1}{4k+2}\right) > \frac{1}{4}.$$

2.5. Corollary. *If the space M in Theorem 2.3 is 0-dimensional, then M possesses a 0-dimensional Φ-compactification.*

Proof. A space M is separable, metrizable and 0-dimensional if and only if M admits a non-archimedian totally bounded metric ϱ, that is, a metric satisfying, for all x, y, z,

$$\varrho(x, y) \leqslant \max\{\varrho(x, z), \varrho(z, y)\} .$$

(For a discussion and references, see [4].) If the metric ϱ is non-archimedian and totally bounded, then so is the metric $\tilde{\varrho}$ introduced in Lemma 2.1. For

$$\tilde{\varrho}(x, y) = 2^{-k}\varrho(\varphi_k x, \varphi_k y) \quad \text{for some fixed } k ,$$

hence

$$\tilde{\varrho}(x, y) \leqslant \max\{2^{-k}\varrho(\varphi_k x, \varphi_k z), 2^{-k}\varrho(\varphi_k z, \varphi_k y)\} \leqslant \max\{\tilde{\varrho}(x, z), \tilde{\varrho}(z, y)\} .$$

That $\tilde{M}$, the completion of M in $\tilde{\varrho}$, is also 0-dimensional follows at once from the fact that M is dense in $\tilde{M}$; the extension of $\tilde{\varrho}$ over $\tilde{M}$ is easily seen to be non-archimedian.

Problem. The results in [5] suggest a method by which the above result might be extended to arbitrary dimension, but we have no proof of this conjecture (1).

(1) Added in proof: R. Engelking has given an affirmative solution to this problem, without using the results of [5].

Another question that suggests itself is the following. Let M be a separable metrizable 0-dimensional space, and let Φ be a (not necessarily countable) set of continuous mappings from M into M. If M possesses a Φ-compactification, does it follow that M possesses a 0-dimensional Φ-compactification? Theorem 2.12 below tells us that M possesses a Φ-compactification if and only if Φ is separable in the topology of uniform convergence with respect to some totally bounded metric ϱ on M. In view of Corollary 2.5, what we are asking is this: If M is 0-dimensional can a *non-archimedian* ϱ be found with the desired properties?

2.6. COROLLARY. *Let φ be a retraction on the separable metrizable space M. Then there is a metric compactification $\widetilde{M}$ of M such that φ can be extended to a retraction of $\widetilde{M}$ on $\widetilde{M}$.*

Proof. Apply Theorem 2.3, with $\Phi = \{\varphi^n\}$. The extension of φ to $\widetilde{M}$ is evidently a retraction.

2.7. THEOREM. *If Ψ is a finite or denumerable set of homeomorphisms of the separable, metrizable space M onto itself, then a Ψ-compactification $\widetilde{M}$ can be constructed so that each extended map φ is a homeomorphism of $\widetilde{M}$ onto $\widetilde{M}$.*

Proof. Let Φ be the *group* generated by Ψ, and apply Theorem 2.3. Then each function in Φ, and its inverse, can be extended over $\widetilde{M}$, which yields the desired result, as can easily be seen.

2.8. COROLLARY. *Every countable metrizable topological group G can be embedded in a compact metric space M in such a way that all left and right translations in G can be extended to homeomorphisms of M onto itself.*

Proof. Under the hypotheses, the space of G is separable and metrizable, so Theorem 2.7 applies. Note that in general M cannot be chosen to be a compact *group*; the additive group on the integers, in the discrete topology, for example, cannot be embedded in any compact topological group.

2.9. COROLLARY. *Every countable, non-discrete metrizable topological group G is a one-to-one continuous image of a group of homeomorphisms of the discontinuum D of Cantor, this last group furnished with the topology of uniform convergence* (see below).

Proof. The space M of G, being countable, non-discrete, homogeneous and metrizable, must be homeomorphic to the space of rationals. Hence, we may introduce a non-archimedian totally bounded metric in M ([4], [5]). G may be considered as a topological transformation group over M, the transformations being left multiplications: i. e. if f is in G, and x in M, $f(x)$ is defined to be $f \cdot x$.

We now apply Theorem 2.7, and obtain a compact $\widetilde{M}$ over which every element of G can be extended. $\widetilde{M}$ is compact, separable, dense in itself and 0-dimensional (2.5) and is therefore homeomorphic to D. So the elements of G can be considered as homeomorphisms of D onto D. Now suppose that $\{f_i\}$ is a sequence of such homeomorphisms, converging uniformly to f. Then for every $\varepsilon > 0$, there is an N such that for $n > N$, and all x, $\widetilde{\varrho}(f_n x, fx) < \varepsilon$. Hence, this inequality holds for $x = e$, the identity of G. But this means that $\widetilde{\varrho}(f_n, f) < \varepsilon$, which implies that $\{f_i\}$ converges to f in the original topology on G.

2.10. Example. Given any metric compactification $\widetilde{M}$ of the space of rational numbers M, there is a homeomorphism of M onto itself which cannot be extended over $\widetilde{M}$. For, since M is not locally compact, $\widetilde{M} \setminus M$ must contain at least two points, p_1 and p_2. Since the rationals are totally disconnected, we can find in M mutually disjoint sets A_1, A_2 and B_2, each of them both open and closed in M, such that p_1 is a limit point in $\widetilde{M}$ of A_1 but not of A_2 or B_2, and p_2 is a limit point of A_2 and B_2 but not of A_1. Now any open set in M, being countable, metrizable and dense in itself, is homeomorphic to M, hence we can define φ so that A_1 and A_2 are mapped homeomorphically on one another, and all other points of M remain fixed. Clearly, such a φ cannot be extended continuously over p_2.

As Example 2.10 shows, *Theorems* 2.3 *and* 2.7 *are in general false if the cardinality of the class of functions to be extended is uncountable.* We shall see below (Theorems 2.12 and 2.14) to what extent generalization in this direction is possible.

Let (M, ϱ) be a totally bounded metric space, and let Ψ be a set of continuous mappings from M into M. We now consider Ψ as a metric space, under two distinct well-known metrics. The first and most important is defined as follows:

$$d(f_1, f_2) = \sup_{x \in M} \varrho(f_1 x, f_2 x).$$

That d is indeed a metric is easily verified. The topology induced on Ψ by this metric is called the *topology of uniform convergence with respect to* ϱ.

We proceed to define a second metric on Ψ. First, we recall the definition of the Hausdorff metric on the class of bounded closed subsets of a metric space X. The distance between the closed sets F and G in X is defined to be the infimum of the set of positive real numbers ε such that: F lies in an ε-neighbourhood of G and G lies in an ε-neighbourhood of F; i. e. each point in either set is closer than ε to some point in the other. It is well-known that this "distance" is indeed a metric. Further, we have the following crucial property: If X is a compact metric space, then so is the set of all closed subsets of X, under the Hausdorff metric.

Now we apply the foregoing to the set Ψ. Each $f \in \Psi$ is a continuous mapping from $M \times M$ into M, hence the graph of f is a closed subset of $M \times M$. We can consider $M \times M$ as a metric space under the "product metric" ϱ_2, where

$$\varrho_2[(x_1, y_1), (x_2, y_2)] = \max\{\varrho(x_1, x_2), \varrho(y_1, y_2)\}.$$

We then define the distance d_h between f_1 and f_2 to be the Hausdorff distance between the graphs of f_1 and f_2 in $(M \times M, \varrho_2)$. Thus $d_h(f_1, f_2) < \varepsilon$ means that for every x in M, there exist points x_1 and x_2 in M such that $\varrho(x_1, x)$, $\varrho(x_2, x)$, $\varrho(f_1x_1, f_2x)$ and $\varrho(f_1x, f_2x_2)$ are all less than ε. This formulation will be needed in the following lemma.

2.11. Lemma. *Let Ψ be a set of continuous mappings from the compact metric space (M, ϱ) into itself. Then the topology of uniform convergence with respect to ϱ is equivalent to the topology given by the Hausdorff metric on Ψ.*

A proof of this lemma can by found in Kuratowski [8], § 15, VIII.

2.12. Theorem. *Let Ψ be a set of continuous mappings from the separable metrizable space M into itself; assume Ψ to be closed under multiplication. Then the following two statements are equivalent*:

(i) *For some totally bounded metric ϱ on M, there is a topology on Ψ finer than the topology of uniform convergence with respect to ϱ, under which Ψ is a separable topological semigroup.*

(ii) *M possesses a Ψ-compactification $(\widetilde{M}, \widetilde{\varrho})$.*

Proof. (i) implies (ii). Let $\{\varphi_i\}$ be a countable dense set in Ψ, closed under multiplication and containing the identity φ_0, and let $\widetilde{\varrho}$ be the metric determined by $\{\varphi_i\}$ and ϱ as in (a), Lemma 2.1. Now condition (i) says that for every $f \in \Psi$, and every $\varepsilon > 0$, there is a φ in $\{\varphi_i\}$ such that $\varrho(fx, \varphi x) < \varepsilon$ for all x; that is, $\{\varphi_i\}$ is dense in Ψ if Ψ is given the topology of uniform convergence with respect to ϱ. We show that $\{\varphi_i\}$ is also dense in Ψ in the topology of uniform convergence with respect to $\widetilde{\varrho}$.

Let f in Ψ, and $\varepsilon > 0$ be given. We must find a φ in $\{\varphi_i\}$ such that $\widetilde{d}(f, \varphi) < \varepsilon$; i. e., such that, for all x, $\widetilde{\varrho}(fx, \varphi x) < \varepsilon$. Choose k so that $2^k\varepsilon$ is greater than the ϱ-diameter of M, and for each $n \leqslant k$, define a function F_n: $\Psi \to \Psi$ by $F_n g = \varphi_n \cdot g$. Since multiplication in Ψ (in its original topology) is continuous in its second variable, each F_n is a continuous map from Ψ to Ψ. Now for each n, let

$$V_n = \{g\colon\ d(g, \varphi_n \cdot f) < 2^n\varepsilon\}.$$

By (i), each V_n is open in Ψ, hence so is

$$U = \bigcap_{n \leqslant k} F_n^{-1} V_n.$$

Therefore, U contains an element φ of $\{\varphi_i\}$. For this φ,

$$2^{-n}\varrho(\varphi_n\varphi x, \varphi_n f x) < \varepsilon$$

for all n and x; that is, $\widetilde{d}(\varphi, f) < \varepsilon$.

We now show that each f in Ψ is uniformly continuous with respect to $\widetilde{\varrho}$. Let $\varepsilon > 0$ be given, and choose n so that $\widetilde{\varrho}(fx, \varphi_n x) < \frac{1}{3}\varepsilon$ for all x. For any two points x and y in M,

$$\widetilde{\varrho}(fx, fy) \leqslant \widetilde{\varrho}(fx, \varphi_n x) + \widetilde{\varrho}(\varphi_n x, \varphi_n y) + \widetilde{\varrho}(\varphi_n y, fy)$$
$$\leqslant \tfrac{2}{3}\varepsilon + \widetilde{\varrho}(\varphi_n x, \varphi_n y).$$

Since φ_n *is* uniformly continuous with respect to $\widetilde{\varrho}$ (Lemma 2.2), we can find a $\delta > 0$ such that $\widetilde{\varrho}(x, y) < \delta$ implies $\widetilde{\varrho}(fx, fy) < \varepsilon$.

It follows (as in Theorem 2.3) that each f in Ψ can be extended continuously over the compact completion of M in the metric $\widetilde{\varrho}$.

(ii) implies (i). View the functions in Ψ as continuous maps from $\widetilde{M}$ to $\widetilde{M}$. In view of Lemma 2.11, Ψ in the topology of uniform convergence determined by $\widetilde{\varrho}$ is a subspace of a compact metric space, and is therefore separable.

We show that Ψ is a topological semigroup in this topology. Suppose that f and g are in Ψ, and let $\varepsilon > 0$ be given. Select $\delta > 0$ so that if $\widetilde{\varrho}(x_1, x_2) < \delta$, then $\widetilde{\varrho}(fx_1, fx_2) < \frac{1}{2}\varepsilon$; let U_1 be a $\frac{1}{2}\varepsilon$-neighbourhood of f, and U_2 a δ-neighbourhood of g. Then for f_1 in U_1, g_2 in U_2, we have, for each x,

$$\widetilde{\varrho}(f_1g_2x, fgx) \leqslant \widetilde{\varrho}(f_1g_2x, fg_2x) + \widetilde{\varrho}(fg_2x, fgx) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

2.13. Corollary. *Let (M, ϱ) be a totally bounded metric space, and let Ψ be a set of continuous mappings from M to M, closed under multiplication, and separable in the topology of uniform convergence with respect to ϱ. Then if the mapping taking (f, g) to fg is continuous in its second variable, it is continuous in both variables simultaneously. (Note that this mapping is automatically continuous in its first variable).*

Proof. The proof that (i) implies (ii) in Theorem 2.7 shows that M possesses a Ψ-compactification $(\widetilde{M}, \widetilde{\varrho})$. The set U there constructed is in this case an open neighbourhood of f in the topology of uniform convergence with respect to ϱ, which lies in an ε-sphere with respect to $\tilde{\varrho}$. This is the only non-trivial argument needed to show that the topologies of uniform convergence with respect to ϱ and $\tilde{\varrho}$ are equivalent. But Ψ is a topological semigroup in the latter topology, as the proof that (ii) implies (i) showed. This completes the proof.

It also follows from the proof of Theorem 2.12 that if (i) is satisfied, then Ψ is also a separable topological semigroup in the topology of uniform convergence with respect to ϱ. Notice that Theorem 2.12 contains Theorem 2.3; in the case of a countable number of mappings, the discrete topology has the necessary properties.

Just as in the countable case, the foregoing results can be applied to the problem of extending homeomorphisms.

2.14. THEOREM. *Let M be a separable metrizable space, and let Ψ be a group of homeomorphisms from M onto M. Then condition* (i), *Theorem* 2.12, *is equivalent to*

(iii) *There exists a compact metric space $(\tilde{M}, \tilde{\varrho})$ containing M densely, such that every f in Ψ can be extended to a homeomorphism from $\tilde{M}$ onto $\tilde{M}$.*

Proof. If (i) holds, then every f in Ψ and its inverse can be extended over $\tilde{M}$ by Theorem 2.12, which gives us (iii).

Conversely, (iii) implies (ii), which in turn implies (i) by the same theorem.

2.15. COROLLARY. *Let (M, ϱ) be a totally bounded metric space, and let Ψ be a group of homeomorphisms from M onto M, separable in the topology of uniform convergence with respect to ϱ. Then if the mapping taking (f, g) to fg is continuous in its second variable, Ψ is a topological group.*

Proof. We proceed just as in Corollary 2.13, and then observe that the group of all homeomorphisms of a compact metric space onto itself, in the topology under consideration, is well-known to be a topological group.

2.16. PROBLEM. Suppose that the separable metrizable space M is homotopic to a point, i. e., there is a continuous mapping F: $M \times I \to M$, where I is the closed unit interval, such that, for all x in M, $F(x, 0) = x$ and $F(x, 1) = a$ (a a fixed element of M). Does there exist a metric compactification $\tilde{M}$ of M which is also homotopic to a point? Here we must extend F to a mapping $\tilde{F}$: $\tilde{M} \times I \to \tilde{M}$, which means that we must extend continuously many functions from M into M. However, we are dealing with a special type space, and the functions are related. Nevertheless, we have not been able to prove the result, even under the additional hypothesis that as r_i converges to r in I, F_{r_i} converges to F_r *uniformly*.

§ 3. G_M-Compactification. Let M denote, again, some separable, metrizable space, and let G_M denote the group of all "autohomeomorphisms" of M; that is, homeomorphisms of M onto itself. It is natural to try to characterize those spaces M possessing a G_M-compactification. An important but restricted class of such spaces is the class of locally compact M. Indeed, for these spaces the one-point compactification is a G_M-compactification. Another, more general class has been studied by H. Freudenthal [1] and the first author [2]. They introduce an "endpoint" or "ideal" compactification which is in a certain sense "maximal" (while the one-point compactification is "minimal"). Let us briefly mention one of their main results, which we shall require in the sequel.

First, two definitions. A space M is called *semicompact* if every point in M contains arbitrarily small neighbourhoods with compact boundaries. Further, for each space M, we define the *space of quasi-components* $Q(M)$. The points of $Q(M)$ are the quasicomponents of M; the topology is given by taking as a base those sets $O \subset Q(M)$ for which $\bigcup_{q \in O} \{q\}$ is open and closed in M. In this topology, $Q(M)$ is a 0-*dimensional, regular topological space.*

The result we shall need is the following ([1], [2]) (using compact in the sense of bicompact):

(α) Every semicompact separable metrizable space M, such that $Q(M)$ is *compact*, possesses a G_M-compactification.

The proof of (α) is rather elaborate. The main purpose of this section is to extend this result to *locally* compact $Q(M)$.

3.1. THEOREM. *Every semicompact, separable metrizable space M with a locally compact space of quasicomponents $Q(M)$ possesses a G_M-compactification $\overline{M}$; i. e., there exists a compact metrizable space $\overline{M}$ containing M densely, such that every homeomorphism from M onto itself can be extended homeomorphically over $\overline{M}$.*

Proof. Consider the map

$$\tau \colon M \to Q(M)$$

which maps each m in M on its quasicomponent q_m. τ is continuous. If U is a compact open set in $Q(M)$, then $\tau^{-1}(U)$ is (open and) closed in M, and is hence semicompact (as a subspace of M). If the space of quasicomponents of a semicompact separable metrizable space is compact, it is separable and metrizable ([2], p. 63); it follows that U is separable (and metrizable). Using the Lindelöf covering theorem on M, we see that $Q(M)$ may be covered by a countable number of such U. From this it follows that $Q(M)$ is separable and metrizable.

Since $Q(M)$ is also locally compact and, we may assume, not compact, it may be compactified by a point q^* to a (0-dimensional) compactum Q^*,

$$Q^* = Q \cup \{q^*\}.$$

Let $\{W_i\} = \{W_i(q^*)\}$ be a countable base at q^* in Q^*. We now adjoin a point m^* to M, obtaining $M^* = M \cup \{m^*\}$, which we topologize as follows. Extend τ to M^* by setting $\tau^* m^* = q^*$. A set open in M will also be open in M^*, and a base for the open neighbourhoods of m^* is given by all sets of the form $\tau^{*-1}(W_i)$. It can easily be seen that M^* is semicompact (M^* is 0-dimensional at m^*), separable and metrizable, and that its space of quasicomponents is Q^*.

Since Q^* is compact, we can apply (α) to obtain a G_M-compactification $\widetilde{M}^*$ of M^*. Now every autohomeomorphism φ over M can first be extended to an autohomeomorphism φ^* over M^* by taking $\varphi^* m^* = m^*$. This is clear, since every φ^* induces a topological map $\tau^*\varphi^*\tau^{*-1}$ of Q^* onto itself, under which q^* is apparently invariant. Now by (α), φ^* can be extended over $\widetilde{M}^* = \overline{M}$.

3.2. Remarks. We note that if $Q(M)$ is locally compact but not compact, the G_M-compactification $\overline{M}$ here constructed contains a point m^* such that for every f in G_M, the extension of f over $\overline{M}$ leaves m^* fixed.

It is of interest to observe that the case of a locally compact M is not included in this theorem. Indeed, in [3] an example is given (Example 1, p. 111) of a locally compact subspace M of the plane such that $Q(M)$ is neither locally compact nor separable (though it is countable).

The condition that M be semicompact is crucial. If M denotes the interior of a circle in the plane, together with one point of its circumference, then M is not semicompact, and possesses no G_M-compactification. (In this case, $Q(M)$ consists of a single point). Semicompactness is not sufficient, however. If M is the space of the rationals, M is semicompact and has no G_M-compactification (Example 2.10). It is natural to try to arrive at a satisfactory sufficient condition by imposing conditions on $Q(M)$; we have seen that local compactness is such a condition. It seems unlikely that one can find necessary and sufficient conditions in terms of "standard" topological properties of M. We have already seen in § 2 that the way in which G_M operates on M plays an important rôle; Examples 3.3 and 3.4 below emphasize this point. It is, indeed, often advisable to consider the autohomeomorphism group when topological properties of a space are being considered. This is by no means a new idea; von Neumann, for example, showed that certain properties of Euclidean n-space (e. g. the existence or non-existence of a "measure" for subsets) are reflected in the structure of its group of isometries.

3.3. Example. If M is a rigid space, i. e. if G_M contains only the identity map, then *every* metric compactification of M is evidently a G_M-compactification. Such spaces are to be found among the 0-dimensional subsets of the line, or the connected, locally connected subsets of the plane [7]. The conditions of our theorems are satisfied in neither of these cases; in the first, M is semicompact, but $Q(M)$ is not locally compact, and in the second, M is not semicompact, though $Q(M)$ consists of a single point. Hence our sufficient conditions are by no means necessary.

3.4. Example. In the preceding example G_M was trivial. Similar examples can be constructed, however, for which G_M is large. One may consider, for example, a discrete union $M = \bigcup_i M_i$ of countably many mutually homeomorphic non-compact connected rigid subsets of the plane. Here G_M is isomorphic to the full symmetric group on the natural numbers; that is, a group of continuous order. Nevertheless, M possesses numerous G_M-compactifications.

3.5. Example. Every countable group is the group G_M for a suitable one-dimensional separable totally bounded metrizable (M, ϱ) such that G_M can be extended over the completion $\overline{M}$ of M. The metric ϱ can be even chosen in such a way that $\overline{M}$ has any chosen dimension. We shall not give details here, but refer to [6], [7], from which the results easily follow.

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