Concerning dense metric subspaces of certain non-metric spaces

by

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In this paper it is shown that if Σ is a space satisfying R. L. Moore's Axioms 0 and 1, [1], then Σ contains a complete metric subspace Σ' such that the set of all points of Σ' forms a dense subset of the set of all points of Σ. A sufficient condition is given for a point set M in order that it be the set of all points of some such Σ'. The terminology used in the paper is largely that of R. L. Moore.

AXIOM 0. Every region is a point set.

AXIOM 1. There exists a sequence $G_1, G_2, G_3, \ldots$ such that

(1) for each positive integer n, G_n is a collection of regions covering the set of all points,

(2) for each positive integer n, G_{n+1} is a subcollection of G_n,

(3) if R is a region and A is a point of R and B is a point of R, there is a positive integer n such that if g is a region of G_n containing A, then $\bar{g}$ is a subset of R and, unless B is A, $\bar{g}$ does not contain B,

(4) if $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets and for each positive integer n there is a region g_n of G_n such that M_n is a subset of $\bar{g}$ and for each positive integer n, M_{n+1} is a subset of M_n, then there is a point common to all the sets of this sequence.

It has been shown that every space satisfying Axiom 0 and the following Axiom C is metric [2]:

AXIOM C. There exists a sequence $G_1, G_2, G_3, \ldots$ satisfying conditions (1), (2) and (4) of Axiom 1 together with the following condition

(3) if A is a point of a region R and B is a point of R, there is a positive integer n such that if x is a region of G_n containing A, and y is a region of G_n intersecting x, then $x+y$ is a subset of R and, unless B is A, $x+y$ does not contain B.

PROPERTY Q. A point set M is said to *have Property* Q provided it is true that if G is a collection of domains covering S, the set of all points, then there is a collection W of domains covering S with the following properties:

(1) each domain of W is a subset of some domain of G,

(2) if P is a point of M and $w_1, w_2, w_3, \ldots$ is an infinite sequence of distinct domains of W and for each positive integer n, A_n and B_n are points of w_n and the sequence $A_1, A_2, A_3, \ldots$ has P as a sequential limit point, then the sequence $B_1, B_2, B_3, \ldots$ has P as a sequential limit point.

A collection W of domains covering S and having properties (1) and (2) will be said to *have the* Q *property with respect to* G *and* M.

The statement that α is an Axiom 1 sequence means that α is a sequence $G_1, G_2, G_3, \ldots$ of collections of domains and if each domain of G_1 is called a region, then α has the properties (1) to (4) as listed in Axiom 1.

Suppose that α is an Axiom 1 sequence $G_1, G_2, G_3, \ldots$ and P is a point. The statement that α satisfies Axiom C at P means that if R is a region containing P, then there is a positive integer n such that and y are intersecting regions of G_n whose sum contains P, the r sum is a subset of R. It is to be noted here that if α satisfies A C at each point of S, then Axiom C holds in Σ, hence Σ is metric.

If α is an Axiom 1 sequence $G_1, G_2, G_3, \ldots$ and there is a point P such that α satisfies Axiom C at P, then the set of all such points will be denoted by M_α. The statement of Theorem 1 asserts that M_α exists and is an inner limiting set. This is sufficient to ensure the existence of an Axiom 1 sequence $G'_1, G'_2, G'_3, \ldots$ for a space Σ' whose points are the points of M_α such that for each n, if g is an element of G'_n, then g is a subset of an element of G_n ([1], p. 83). Since α satisfies Axiom C at each point of M_α, it follows that $G'_1, G'_2, G'_3, \ldots$ satisfies Axiom C and thus, Σ' is a complete metric space.

The statement of Theorem 3 asserts that in an Axiom 1 space Σ, if M is a point set having Property Q, then there is an Axiom 1 sequence α such that M_α contains M. From this it follows that if S, the set of all points, possesses Property Q, then there is an Axiom 1 sequence α such that M_α is S. Hence, Σ is metric if S possesses Property Q.

The statement that Σ is paracompact means that if G is a collection of domains covering S then there is a collection H of domains covering S such that each domain of H is a subset of some domain of G, and, if P is a point, there is a region containing P that does not intersect infinitely many domains of H. It is to be noted that if $h_1, h_2, h_3, \ldots$ is an infinite sequence of distinct domains of H and for each n, P_n is a point of h_n, then the set $(P_1+P_2+P_3+\ldots)$ has no limit point. This leads directly to the conclusion that if Σ is a paracompact space satisfying Axioms 0 and 1 then, since S must possess Property Q, Σ is metric.

THEOREM 1. *If space satisfies Axioms* 0 *and* 1 *and* α *is an Axiom* 1 *sequence, then* M_α *is an inner limiting set dense in* S.

Proof. Let $G_1, G_2, G_3, \ldots$ denote the elements of α. Suppose that there is a region R such that if P is a point of R and n is a positive integer, then there are two intersecting regions of G_n whose sum contains P but is not a subset of R. Suppose that P_1 is a point of R, n_1 is a positive integer such that every region of G_{n_1} that contains P_1 is a subset of R, and x_1 and y_1 are two intersecting regions of G_{n_1} such that x_1 contains P_1 and x_1+y_1 is not a subset of R. The common part of x_1 and y_1 is a subset of R and contains a point P_2. There is a positive integer n_2 greater than n_1 such that every region of G_{n_2} that contains P_2 is a subset of $(x_1\cdot y_1)$, a pair x_2 and y_2 of intersecting regions of G_{n_2} such that x_2 contains P_2, $\overline{x}_2$ is a subset of $(x_1\cdot y_1)$ and further such that x_2+y_2 is not a subset of R. This may be continued to produce a sequence $x_1, x_2, x_3, \ldots$ of regions such that there is a point A of R common to the sets of the sequence $\overline{x}_1, \overline{x}_2, \overline{x}_3, \ldots$. The point A is also a point of each region of a sequence $y_1, y_2, y_3, \ldots$ such that for no positive integer n is y_n a subset of R. This is a contradiction.

Let R_1 denote a region. There is a point P_1 of R_1 and a positive integer n_1 such that if 1 y are intersecting regions of G_{n_1} whose sum contains P_1, then $x+$ subset of R_1. Let R_2 denote a region of G_{n_1} containing P_1 such that R_2 is a subset of R_1. There are three sequences, $P_1, P_2, P_3, \ldots$, $R_1, R_2, R_3, \ldots$ and $n_1, n_2, n_3, \ldots$ such that P_1, R_1, R_2 and n_1 are as described above and for each positive integer j greater than 1, P_j is a point of R_j and n_j is a positive integer greater than n_{j-1} such that if x and y are two intersecting regions of G_{n_j} whose sum contains P_j, then $x+y$ is a subset of R_j. Further, the region R_{j+1} is a region of G_{n_j} containing P_j such that $\overline{R}_{j+1}$ is a subset of R_j. There is a point P common to all sets of the sequence $R_1, R_2, R_3, \ldots$. Let g denote a region containing P. There is a positive integer i such that R_i is a subset of g. P is a point of R_i. There is an integer k greater than i such that every region of G_{n_k} that contains P is a subset of R_{i+1}. Thus, if x and y are intersecting regions of G_{n_k} whose sum contains P, both must intersect R_{i+1}. Consequently, $x+y$ is a subset of R_i. This completes the proof that M_α is dense in S.

If M_α is S, then M_α is an inner limiting set. Suppose that there is a point of S not in M_α. Let N_α denote the point set $S-M_\alpha$. If P is a point of N_α and there is a positive integer n such that if g is a region of G_n containing P, and j is a positive integer, there are two intersecting regions of G_j whose sum contains P but is not a subset of g, and there is a region r of G_{n-1} and a positive integer i such that if x and y are two intersecting regions of G_i whose sum contains P, then $x+y$ is a subset of r; then let n_p denote the integer n. If there is no such integer, let n_p be 1.

For each integer i such that i is n_p for some point P of N_a, let N_i denote the set of all points q of N_a such that $i \geqslant n_q$. Suppose that A is a limit point of some such N_i. Let R denote a region of G_i containing A. Let j denote a positive integer greater than i such that every region of G_j containing A is a subset of R. Let g denote one such region of G_j. The region g contains a point B of N_i distinct from A. There is a positive integer k greater than j such that every region of G_k which contains B is a subset of g. Since B is a point of N_i and R is a region G_i, there are two intersecting regions of G_k such that their sum contains B but is not a subset of R. Thus, there are two intersecting regions of G_i whose sum contains A but is not a subset of R. This shows that N_i is closed and, consequently, N_a is the sum of a countable collection of closed point sets. Since N_a is $S-M_a$, M_a is an inner limiting set.

THEOREM 2. *If $\alpha_1, \alpha_2, \alpha_3, \ldots$ is a sequence of Axiom 1 sequences, then there is an Axiom 1 sequence α such that for each n, M_α contains M_{α_n}.*

Proof. For each positive integer n, let $G_{n,1}, G_{n,2}, G_{n,3}, \ldots$ denote the elements of α_n.

For each positive integer j, let $G'_{1,j}$ denote the collection such that x is an element of this collection if and only if x is an element of $G_{1,j}$ which is a subset of some region of $G_{2,j}$. Let α'_1 denote the sequence $G'_{1,1}, G'_{1,2}, G'_{1,3}, \ldots$ The sequence α'_1 is an Axiom 1 sequence. Furthermore, $M_{\alpha'_1}$ contains $M_{\alpha_1}+M_{\alpha_2}$.

For each positive integer n greater than 1 and each positive integer j, let $G'_{n,j}$ be the collection such that x is an element of this collection if and only if x is a region of $G'_{n-1,j}$ which is a subset of a region of $G_{n,j}$. Let α'_n denote the sequence $G'_{n,1}, G'_{n,2}, G'_{n,3}, \ldots$ The sequence α'_n is an Axiom 1 sequence and $M_{\alpha'_n}$ contains $M_{\alpha_1}+M_{\alpha_2}+\ldots+M_{\alpha_n}$.

For each positive integer n, $G'_{n+1,n+1}$ is a subcollection of $G'_{n,n}$. Let α denote the sequence $G'_{1,1}, G'_{2,2}, G'_{3,3}, \ldots$ If k is a positive integer and j is an integer greater than k, $G'_{j,j}$ is a subcollection of $G'_{k,j}$. Thus M_α contains M_{α_k}.

THEOREM 3. *If space satisfies Axioms 0 and 1 and M is a point set having Property Q, then there is an Axiom 1 sequence α such that M_α contains M.*

Proof. Suppose that M is a point set having Property Q and $G_1, G_2, G_3, \ldots$ is an Axiom 1 sequence. The collection G_1 covers S, so there is a collection W_1 of domains having the Q property with respect to G_1 and M. Let G'_1 denote the collection G_1. Let G'_2 denote the collection such that x is an element of G'_2 if and only if x is a region of G_2 and $\bar{x}$ is a subset of some domain of W_1. The collection G'_2 covers S, so there is a collection W_2 of domains covering S and having the Q property with respect to G'_2 and M.

There are two infinite sequences $G'_1, G'_2, G'_3, \ldots$ and $W_1, W_2, W_3, \ldots$ such that G'_1, G'_2, W_1 and W_2 are as described above and

(1) for each positive integer n greater than 1, x is an element of G'_n if and only if x is a region of G'_n such that $\bar{x}$ is a subset of some domain of W_{n-1},

(2) the collection W_n has the Q property with respect to G'_n and M. If w is a domain of W_n, then $\bar{w}$ is a subset of some domain of W_{n-1}.

For each positive integer j, let H_j denote the collection $W_j+W_{j+1}+W_{j+2}+\ldots$ For each n, H_n covers S and each domain of H_n is a subset of some region of G_n. Further, H_{n+1} is a subcollection of H_n. Therefore, $H_1, H_2, H_3, \ldots$ is an Axiom 1 sequence.

Suppose that there is a point P of M and a region R containing P such that if n is a positive integer, there are two intersecting domains of H_n whose sum contains P but is not a subset of R. There is a sequence $R_1, R_2, R_3, \ldots$ such that for each positive integer j

(1) R_j is a domain of H_j which is not a subset of R and

(2) the point set $(R_1+R_2+R_3+\ldots)$ contains P or has P as a limit point.

If there are only finitely many distinct domains represented in the sequence $R_1, R_2, R_3, \ldots$, then there is a domain d such that $\bar{d}$ contains P and d is an element of infinitely many collections of the sequence $H_1, H_2, H_3, \ldots$ This contradicts the third condition of Axiom 1. Therefore, there are infinitely many distinct domains in the sequence $R_1, R_2, R_3, \ldots$

For each positive integer j greater than 1, let d_j denote a domain of W_2 such that d_j contains R_j. There is a domain d_i of the sequence $d_2, d_3, d_4, \ldots$ such that $\bar{d}_i$ contains P since W_2 has the Q property with respect to G_2 and M. There is a region g_1 in G_1 such that g_1 contains $\bar{d}_i$. Thus, g_1 contains P and is not a subset of R.

By using the sequence $R_3, R_4, R_5, \ldots$ it can be shown that there is a region g_2 of G_2 such that g_2 contains P but is not a subset of R. Further, for each n, there is a region g_n of G_n such that g_n contains P but is not a subset of R. This is a contradiction.

Therefore, if α denotes the sequence $H_1, H_2, H_3, \ldots$, then M_α contains M.

THEOREM 4. *Suppose that S is not compact, M is an inner limiting set dense in S and there is an Axiom 1 sequence α such that M_α contains M. There is an Axiom 1 sequence α' such that $M_{\alpha'}$ is M.*

Proof. Let $G_1, G_2, G_3, \ldots$ denote the elements of α. Suppose that M is a proper subset of M_α. Let β denote the set $S-M$. The proof is divided into three cases. The first, second and third cases being those where β is degenerate, non-degenerate and closed, and not closed, respectively.

Case 1. Let P denote the point in β and let $P_1, P_2, P_3, \ldots$ denote a sequence of distinct points such that the set $(P_1+P_2+P_3+\ldots)$ has no limit point. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on P such that, for each positive integer n, R_n is a region of G_n. Let $g_1, g_2, g_3, \ldots$ denote a sequence of mutually exclusive regions such that the closure of their sum is the sum of their closures and, for each n, g_n contains P_n. For each positive integer n let d_n denote the domain (R_n+g_n-P). For each positive integer n, let G_n denote the collection consisting of all regions of G_n together with the domains $d_n, d_{n+1}, d_{n+2}, \ldots$ Let α' denote the sequence $G_1', G_2', G_3', \ldots$ If B is a point distinct from P, there is a positive integer i such that the only domains of G_1' that contain B are regions of G_i. The only domains of G_1' that contain P are regions of G_1. Thus α' satisfies the first three conditions of Axiom 1. If h and k are two intersecting domains of G_n', their common part is a subset of a region of G_n. Therefore, α' satisfies the fourth condition of Axiom 1. Hence, it is an Axiom 1 sequence. But P does not belong to $M_{\alpha'}$.

If A is a point of $S-\beta$, there is a region R containing A and a positive integer n such that every domain of G_n' that intersects R is a region of G_n. Consequently, $S-\beta$ is a subset of $M_{\alpha'}$. It follows that $M_{\alpha'}$ is M.

Case 2. There exists a sequence $H_1, H_2, H_3, \ldots$ such that for each n, H_n is a collection of regions of G_n properly covering β such that if, for each n, D_n denotes the sum of the regions of H_n then $\overline{D}_{n+1}$ is a proper subset of D_n. Suppose that A is a point of M. There is a region R containing A and a positive integer j such that R does not intersect D_j. The point set β is the boundary of M and it is the common part of the domains $D_1, D_2, D_3, \ldots$

For each positive integer n, let d_n denote the domain $D_n-\overline{D}_{n+1}$.

Suppose that P is a point of β and R is a region containing P. There is a point Z of M in R and a positive integer x such that Z belongs to D_x. There is a region R' containing Z and a positive integer j such that R' does not intersect D_j. Thus, there is a positive integer i such that Z belongs to $\overline{D}_{i-1}$ and not to $\overline{D}_i$. There is a region R'' containing Z and lying in R but not intersecting $\overline{D}_i$. Since Z is a point of $\overline{D}_{i-1}$, R'' intersects D_{i-1}. The domain $R''\cdot D_{i-1}$ is a subset of R and of $(D_{i-1}-\overline{D}_i)$. Therefore, R intersects d_{i-1}.

The sequence $d_1, d_2, d_3, \ldots$ has the following properties:

(1) if i and j are two positive integers, then d_i and d_j have no point in common,

(2) if P is a point of β and R is a region containing P, then there is a positive integer n such that d_n intersects R,

(3) if P is a point of M, there is a region R containing P and a positive integer j such that R intersects no domain of the sequence $d_1, d_2, d_3, \ldots$ with subscript greater than j.

There is an ascending sequence of positive integers $n_1, n_2, n_3, \ldots$ such that the following is true. If, for each integer j, Δ_j denotes the domain $(d_{n_j}+d_{n_j+1}+\ldots+d_{n_{j+1}-1})$ and P is a point of β, then there is a region R containing P such that, for infinitely many positive integers i, Δ_i intersects R but is not a subset of R. Also it is true that if x and y are two positive integers, then there is no point common to Δ_x and Δ_y.

For each positive integer n, let G_n' denote the collection consisting of all regions of G_n together with the domains $\Delta_n, \Delta_{n+1}, \Delta_{n+2}, \ldots$ Let α' denote the sequence $G_1', G_2', G_3', \ldots$

In the same fashion as in Case 1, it may be shown that α' is an Axiom 1 sequence. Further, α' does not have the properties stated in Axiom C at any point of β and does have these properties at each point of M. Therefore, $M_{\alpha'}$ is M.

Case 3. In this case M is not a domain but since it is an inner limiting set and $M_\alpha-M$ exists, there is a simple countable sequence $D_1, D_2, D_3, \ldots$ of domains with M as their common part, such that for each positive integer n, D_n contains D_{n+1} and $S-D_1$ intersects M_α. For each positive integer n, let β_n denote the boundary of D_n and let T_n denote the point set $\beta_n\cdot M_\alpha$. Since M is dense in S, β_n contains no domain. For each n, T_{n+1} contains T_n. The point set $(T_1+T_2+T_3+\ldots)$ is $M_\alpha-M$.

If j is a positive integer such that the common part of β_j and some region of G_j that intersects T_j is non-degenerate, let $\Delta_{j,1}, \Delta_{j,2}, \Delta_{j,3}, \ldots$ denote a sequence of domains with the properties stated in Case 2 for the sequence $\Delta_1, \Delta_2, \Delta_3, \ldots$ except that β be replaced by β_j. Let U_j denote a point set such that x is in U_j if and only if x is a point of T_j and there is a region R in G_j such that R contains x, and $R\cdot\beta_j$ is non-degenerate. Let U denote a point set such that x is in U if and only if there is a positive integer n such that x is in U_n. If there is a point P in $M_\alpha-M$ such that, for no positive integer i, P is in U_i, let the set of all such points be denoted by T.

Suppose that x is a positive integer such that U_x exists. For each point P in U_x, let $R_{x,p}$ be a region of G_x which contains P and some other point of β_x. For each positive integer n, let H_n be a collection of domains such that d is an element of H_n if and only if d is a region of G_n or for some positive integer j and point P in U_j, d is the common part of $R_{j,p}$ and $\Delta_{j,i}$ for some integer i greater than n.

Suppose that A is a point of M which is not a limit point of β. Since α satisfies Axiom C at A, there is a region R containing A and a positive integer k such that no region of G_k intersects $\bar{\beta}$ and R. Since each domain of H_k which is not a region of G_k is the common part of some $R_{j,p}$ and $\Delta_{j,i}$ where i is greater than k, this means that each point of this common part belongs to a region of G_i which intersects β_j and thus is a point of a region of G_k which intersects β. Therefore, R intersects no domain of H_k which is not a region of G_k.

Suppose that A is a point of M which is a limit point of β. Let R denote a region containing A. Since, for each i, β_i is closed, A is not a limit point of β_i. Since A belongs to M_α there is a region R_1 containing A and a positive integer n_1 such that every region of G_{n_1} that intersects R_1 is a subset of R. There is a region R_2 which is a subset of R_1 and contains A, and a positive integer x such that if $i \leqslant n_1$ and $j > x$, then R_2 does not intersect the domain $\Delta_{i,j}$. Every domain of H_{x+n_1} that intersects R_2 is a subset of R.

Suppose that A is a point of β. Let R denote a region containing A. There is a positive integer n such that every region of G_n which contains A is a subset of R. There is a positive integer j greater than n such that no region of G_j contains A and intersects $(\beta_1+\beta_2+\ldots+\beta_n)$. Each domain of H_{n+j} is a region of G_{n+j} or is of the form $(R_{x,p}\cdot\Delta_{x,i})$ where i is greater than $n+j$. Every region of G_{n+j} which contains A is a subset of R. If there is a domain $(R_{x,p}\cdot\Delta_{x,i})$ which contains A but is not a subset of R then, since $R_{x,p}$ is a region of G_x, x is less than n. Each point of $\Delta_{x,i}$ is a subset of a region of G_i which intersects β_x and since x is less than n, i must be less than j. However, i is greater than $n+j$. Consequently, every domain of H_{n+j} which contains A is a subset of R.

It follows now fairly readily that the sequence $H_1, H_2, H_3, \ldots$ satisfies the first three conditions of Axiom 1 at each point of β and the first three conditions of Axiom C at each point of M.

Suppose that $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets such that for each positive integer n, M_n contains M_{n+1} and there is a domain d_n in H_n such that M_n is a subset of $\bar{d}_n$. For each positive integer j, no two domains in the sequence $\Delta_{j,1}, \Delta_{j,2}, \Delta_{j,3} \ldots$ have a point in common. Since each domain of H_n which is not a region of G_n is of the form $(R_{x,p}\cdot\Delta_{x,i})$ where i is greater than n, there exists no positive integer m such that infinitely many domains of the sequence $d_1, d_2, d_3, \ldots$ are of the form $(R_{x,p}\cdot\Delta_{x,i})$ where $x \leqslant m$. Therefore, there is an ascending sequence $n_1, n_2, n_3, \ldots$ of positive integers and a sequence $R_1, R_2, R_3, \ldots$ of regions such that for each positive integer k, R_k is a region of G_k and contains d_{n_k}. Thus the sets of the sequence $M_1, M_2, M_3, \ldots$ have a point in common. Therefore, the sequence $H_1, H_2, H_3, \ldots$ satisfies the fourth condition in Axiom 1.

If x is a positive integer such that U_x exists and P is a point of β_x and R is a region containing P, then there is a positive integer j such that $\Delta_{x,j}$ intersects R. If A is a point of U_x, then $(R_{x,A}\cdot\beta_x)$ is non-degenerate. From these facts it follows that the sequence $H_1, H_2, H_3, \ldots$ does not satisfy the requirements of Axiom C at any point of U.

Suppose that P is a point of $M_\alpha-(M+U)$, that is to say, a point of T. For each positive integer x such that T intersects β_x, let T'_x denote $T\cdot\beta_x$. For no n does a region of G_n intersect two points of T'_n.

Suppose that T is closed. Since M is dense in S, every point of T is a limit point of M. Consequently, there is an Axiom 1 sequence $H'_1, H'_2, H'_3, \ldots$ satisfying the requirements of Axiom C at each point of M but at no point of T. This is exactly analogous to the situation in Case 2.

For each positive integer n, let G'_n denote a collection such that d is an element of G'_n if and only if d is a domain of H'_n or of H_n. Let α' denote the sequence $G'_1, G'_2, G'_3, \ldots$ It readily follows that $M_{\alpha'}$ is M.

Suppose that T is not closed but contains a limit point of itself. For each positive integer x such that T_x contains a limit point of T, let V_x be the set of all limit points of T that belong to T_x. No region of G_x covers two points of V_x. There is a well-ordered sequence ω whose terms are the points of V_x. There is a subsequence ω_1 of ω such that

(1) the first term of ω is the first term of ω_1,

(2) if Z is an initial segment of ω_1 and there is a point P of V_x such that no coherent collection of three regions of G_x covers P and some point in Z, then the first such point in ω is the first point in ω_1 to follow all the points of Z in ω_1.

If n is a positive integer and there is a point in V_x which is not in ω_i where $1 \leqslant i \leqslant n$, then let ω_{n+1} be a subsequence of ω such that

(1) the first term of ω which is not in ω_i for $1 \leqslant i \leqslant n$ is the first term of ω_{n+1},

(2) if Z is an initial segment of ω_{n+1} and there is a point P of V_x not in ω_i where $1 \leqslant i \leqslant n$, such that no coherent collection of three regions of G_{x+n} covers P and some point in Z, then the first such point in ω is the first point in ω_{n+1} to follow all the points of Z in ω_{n+1}.

Suppose that there is a point P of V_x which is in no one of the sequences $\omega_1, \omega_2, \omega_3, \ldots$ For each positive integer n, there is a coherent collection of three regions of G_n that covers P and some other point of V_x. Since P belongs to M_α, it follows that P is a limit point of V_x. This is a contradiction.

Suppose that i is a positive integer and G is a collection of regions of G_{x+i} such that each region of G contains only one point of ω_{i+1} and each point in ω_{i+1} belongs to only one region of G. Suppose that there exists a point of $\bar{G}^*$ which is not a point or limit point of any one region of G. Then there is a coherent collection of three regions of G_{x+i} that covers two points in ω_{i+1} which is impossible. Therefore, if P is a point of $\bar{G}^*$, there is only one region g in G such that P is a point of $\bar{g}$. For each positive integer j such that ω_j exists, let C_j be a collection of regions of G_{x+j} such that each region of C_j contains only one point of ω_j and each point in ω_j belongs to only one region of C_j. Let C denote the collection such that R is in C if and only if there is a positive integer n such that R is in C_n. Suppose that there is a point P of M which is a point of $\bar{C}^*$ and not a point or limit point of any region in C. Since P belongs to M_α, if R is a region containing P, there is a region R_1 containing P such that every region of C that intersects R_1 is a subset of R.

For each positive integer n such that V_n exists, there is a collection K_n of regions covering V_n with properties with respect to V_n as described in the preceeding paragraph for the collection C with respect to V_x. Suppose that P is a point of V_n. Since P is a limit point of T, if R denotes the region of K_n that contains P, then $R\cdot T$ is non-degenerate. Each point of T is a limit point of M so there is a sequence of mutually exclusive domains $d_1, d_2, d_3, \ldots$ such that

(1) for each n, d_n is a subset of R,

(2) if, for each positive integer n, G'_n denotes the collection consisting of all regions of G_n together with the domains $d_n, d_{n+1}, d_{n+2}, \ldots$, then the sequence $G'_1, G'_2, G'_3, \ldots$ is an Axiom 1 sequence which fails to satisfy Axiom C at P.

In a manner similar to the construction of the sequence $H_1, H_2, H_3, \ldots$, it may be shown that there exists an Axiom 1 sequence $H'_1, H'_2, H'_3, \ldots$ such that for each positive integer n each domain of H'_n is either a region of G_n or a subset of a region of some K_j. Further, this sequence does not have the properties as stated in Axiom C at any point of T which is a limit point of T and has these properties at each point of M.

Suppose that there is a point of T which is not a limit point of T. Let W denote the set of all such points. The point set $\bar{T}-W$ is closed. There is a sequence $E_1, E_2, E_3, \ldots$ of domains with $\bar{T}-W$ as their common part such that

(1) for each n, E_n contains E_{n+1},

(2) if P is a point of W in E_1, there is a positive integer x such that P is in E_x but not in E_{x+1}, and a region g of G_x which contains P and a point y of W distinct from P.

If there is a point of W not in E_1, the set of all such points is a closed point set. There is an Axiom 1 sequence $L_1, L_2, L_3, \ldots$ such that this sequence fails to satisfy Axiom C at each point of W not in E_1 and satisfies Axiom C at each point of M. The set of points of W that are in E_1 may be divided up into the sets of a sequence $W_1, W_2, W_3, \ldots$ which possesses the properties of the sequence $U_1, U_2, U_3, \ldots$ that made it possible to construct an Axiom 1 sequence that satisfied Axiom C at each point of M and did not satisfy Axiom C at any point of U. Therefore, there is an Axiom 1 sequence that satisfies Axiom C at each point of M but at no point of W.

If P is a point of β, then P is a point of U or of T. If P is a point of T, and T is not closed, then P is a point of W or of $T-W$. Consequently, there is an Axiom 1 sequence α' such that $M_{\alpha'}$ is M.

References

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