Concerning dense metric subspaces of certain non-metric spaces

by

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In this paper it is shown that if $\Sigma$ is a space satisfying R. L. Moore's Axioms 0 and 1, [1], then $\Sigma$ contains a complete metric subspace $\Sigma'$ such that the set of all points of $\Sigma'$ forms a dense subset of the set of all points of $\Sigma$. A sufficient condition is given for a point set $M$ in order that it be the set of all points of some such $\Sigma'$. The terminology used in the paper is largely that of R. L. Moore.

**Axiom 0.** Every region is a point set.

**Axiom 1.** There exists a sequence $G_1, G_2, G_3, \ldots$ such that

1. for each positive integer $n$, $G_n$ is a collection of regions covering the set of all points,

2. for each positive integer $n$, $G_{n+1}$ is a subcollection of $G_n$,

3. if $R$ is a region and $\mathcal{A}$ is a point of $R$ and $\mathcal{B}$ is a point of $R$, there is a positive integer $n$ such that if $g$ is a region of $G_n$ containing $A$, then $g$ is a subset of $R$ and, unless $B$ is $A$, $g$ does not contain $B$,

4. if $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets and for each positive integer $n$ there is a region $g_n$ of $G_n$ such that $g_n$ is a subset of $\omega$ and for each positive integer $n$, $M_n$ is a subset of $M_{n+1}$, then there is a point common to all the sets of this sequence.

It has been shown that every space satisfying Axiom 0 and the following Axiom C is metric [2]:

**Axiom C.** There exists a sequence $G_1, G_2, G_3, \ldots$ satisfying conditions (1), (2) and (4) of Axiom 1 together with the following condition

3. if $A$ is a point of a region $R$ and $B$ is a point of $R$, there is a positive integer $n$ such that if $s$ is a region of $G_n$ containing $A$, and $y$ is a region of $G_n$ intersecting $s$, then $s + y$ is a subset of $R$ and, unless $B$ is $A$, $s + y$ does not contain $B$.

**Property Q.** A point set $M$ is said to have Property Q provided it is true that if $G$ is a collection of domains covering $S$, the set of all
to the conclusion that if $\Sigma$ is a paracompact space satisfying Axioms 0 and 1 then, since $S$ must possess Property Q, $\Sigma$ is metric.

**Theorem 1.** If space satisfies Axioms 0 and 1 and $a$ is an Axiom 1 sequence, then $M_\xi$ is an inner limiting set dense in $S$.

**Proof.** Let $G_1, G_2, G_3, \ldots$ denote the elements of $a$. Suppose that there is a region $R$ such that if $P$ is a point of $R$ and $n$ is a positive integer, then there are two intersecting regions of $G_n$ whose sum contains $P$ but is not a subset of $R$. Suppose that $P_1$ is a point of $R$, $x_1$ is a positive integer such that every region of $G_n$ that contains $P_1$ is a subset of $R$, and $x_1 + y_1$ is a pair of intersecting regions of $G_n$ such that $x_1$ contains $P_1$ and $x_1 + y_1$ is not a subset of $R$. The common part of $x_1$ and $y_1$ is a subset of $R$ and contains a point $P_2$. There is a positive integer $n_2$ greater than $n_1$ such that $x_1$ contains $P_1$ and $x_2 + y_2$ is a subset of $R$. This may be continued to produce a sequence $x_1, x_2, x_3, \ldots$ of regions such that there is a point $A$ of $R$ common to the sets of the sequence $x_1, x_2, x_3, \ldots$. The point $A$ is also a point of each region of a sequence $y_1, y_2, y_3, \ldots$ such that for no positive integer $n$ is $y_n$ a subset of $R$. This is a contradiction.

Let $R_1$ denote a region. There is a point $P_1$ of $R_1$ and a positive integer $n_1$ such that if $x$ and $y$ are intersecting regions of $G_n$, whose sum contains $P_1$, then $x + y$ is a subset of $R_1$. Let $R_2$ denote a region of $G_n$ containing $P_1$ such that if $R_1$ is a subset of $R_2$, then $P_1$ is a point of $R_1$. There are three sequences $P_1, P_2, P_3, \ldots$, $R_1, R_2, R_3, \ldots$, and $x_1, x_2, x_3, \ldots$ such that $P_1, P_2, P_3, \ldots$, and $x_1, x_2, x_3, \ldots$ are as described above and for each positive integer $j$ greater than 1, $P_j$ is a point of $R_j$ and $x_j$ is a positive integer greater than $x_{j-1}$ such that if $x$ and $y$ are two intersecting regions of $G_n$, whose sum contains $P_j$, then $x + y$ is a subset of $R_j$. Further, the region $R_{j+1}$ is a region of $G_n$ containing $P_j$ such that $R_{j+1}$ is a subset of $R_j$. There is a point $P$ common to all sets of the sequence $R_1, R_2, R_3, \ldots$. Let $y$ denote a region containing $P$. There is a positive integer $i$ such that $R_i$ is a subset of $y$. If $P$ is a point of $R_i$, there is an integer $k$ greater than $i$ such that every region of $G_n$ containing $P$ is a subset of $R_{i+k}$. Thus, if $x$ and $y$ are intersecting regions of $G_n$, whose sum contains $P$, both must intersect $R_{i+k}$. Consequently, $x + y$ is a subset of $R_i$. This completes the proof that $M_\xi$ is dense in $S$.

If $M_\xi$ is $S$, then $M_\xi$ is an inner limiting set. Suppose that there is a point of $S$ not in $M_\xi$. Let $N_\xi$ denote the point set $S - M_\xi$. If $P$ is a point of $N_\xi$, and there is a positive integer $a$ such that if $G_n$ contains $P$, and if $j$ is a positive integer, there are two intersecting regions
of $G_i$ whose sum contains $P$ but is not a subset of $G_i$ and there is a region $r$ of $G_{x-i}$ and a positive integer $i$ such that if $x$ and $y$ are two intersecting regions of $G_i$ whose sum contains $P$, then $x + y$ is a subset of $r_i$ then let $n_i$ denote the integer $n_i$. If there is no such integer, let $n_i$ be 1.

For each integer $i$ such that $i < n_i$ for some point $P$ of $X_i$, let $N_i$ denote the set of all points $q$ of $X_i$ such that $i < n_i$. Suppose that $A$ is a limit point of some such $N_i$. Let $B$ denote a region of $G_i$ containing $A$. Let $j$ denote a positive integer greater than $i$ such that every region of $G_j$ containing $A$ is a subset of $B$. Let $g$ denote one such region of $G_i$. The region $g$ contains a point $B$ of $X_i$ distinct from $A$. There is a positive integer $k$ greater than $j$ such that every region of $G_k$ which contains $B$ is a subset of $g$. Since $B$ is a point of $X_i$ and $B$ is a region of $G_i$, there are two intersecting regions of $G_k$ such that their sum contains $B$ but is not a subset of $B$. Thus, there are two intersecting regions of $G_i$ whose sum contains $B$ but is not a subset of $B$. This shows that $N_i$ is closed and, consequently, $X_i$ is the sum of a countable collection of closed point sets. Since $X_i$ is a subset of $M_i$, $M_i$ is an inner limiting set.

**Theorem 3.** If $a_1, a_2, a_3, \ldots$ is a sequence of Axiom 1 sequences, then there is an Axiom 1 sequence $a$ such that for each $m$, $a_m$ contains $a_m$.

Proof. For each positive integer $n$, let $G_{n_1}, G_{n_2}, G_{n_3}, \ldots$ denote the elements of $a_n$.

For each positive integer $j$, let $G_j$ denote the collection such that $x$ is an element of this collection if and only if $x$ is an element of $G_j$ which is a subset of some region of $G_j$. Let $\alpha_j$ denote the sequence $G_j, G_{j+1}, G_{j+2}, \ldots$. The sequence $\alpha_j$ is an Axiom 1 sequence. Furthermore, $G_j$ contains $G_{j+1} + M_j$. For each positive integer $i > j$ such that $i$ and each positive integer $j$, let $G_i$ be the collection such that $x$ is an element of this collection if and only if $x$ is a region of $G_{i-1}$ which is a subset of a region of $G_{i-1}$. Let $\alpha_i$ denote the sequence $G_{i-1}, G_{i-2}, G_{i-3}, \ldots$. The sequence $\alpha_i$ is an Axiom 1 sequence and $\alpha_j$ contains $\alpha_j + M_j$. For each positive integer $n$, $G_{n+1} + n$ is a subcollection of $G_n$. Let $a$ denote the sequence $G_{n+1}, G_{n+2}, G_{n+3}, \ldots$. If $k$ is a positive integer and $j$ is an integer greater than $k$, $G_{j-1}$ is a subcollection of $G_{j-1}$. Thus $M_k$ contains $M_j$.

**Theorem 3.** If a point set satisfying Axioms 0 and 1 and $M$ is a point set having Property Q, then there is an Axiom 1 sequence $a$ such that $M_a$ contains $M$.

Proof. Suppose that $M$ is a point set having Property Q and $G_1, G_2, G_3, \ldots$ is an Axiom 1 sequence. The collection $G_i$ covers $S$, so there is a collection $W_i$ of domains having the Q property with respect to $G_i$, $M$. Let $G_i$ denote the collection $G_i$, and $G_i$ denote the collection such that $x$ is an element of $G_i$ if and only if $x$ is a region of $G_i$, $G_i$ is a subset of some domain of $W_i$. The collection $G_i$ covers $S_0$, so there is a collection $W_i$ of domains covering $S_0$ and having the Q property with respect to $G_i$ and $M$. There are two infinite sequences $G_1, G_2, G_3, \ldots$, $W_1, W_2, W_3, \ldots$, such that $G_i, G_2, W_1, W_2$ and $W_3$ are as described above and

1. for each positive integer $n$ greater than 1, $x$ is an element of $G_n$ if and only if $x$ is a region of $G_n$ such that $x$ is a subset of some domain of $W_{n-1}$.

2. the collection $W_n$ has the Q property with respect to $G_n$ and $M$.

If $W_n$ is a domain of $W_n$, then $w$ is a subset of some domain of $W_{n-1}$. For each positive integer $j$, let $H_j$ denote the collection $W_1 + W_{j+1} + W_{j+2} + \ldots$. For each $n$, $H_n$ covers $S$ and each domain of $H_n$ is a subset of some region of $G_n$. Further, $H_{n+1}$ is a subcollection of $H_n$. Therefore, $H_1, H_2, H_3, \ldots$ is an Axiom 1 sequence.

Suppose that there is a point $P$ of $M$ and a region $R$ containing $P$ such that if $n$ is a positive integer, there are two intersecting domains of $H_n$ whose sum contains $P$ but is not a subset of $R$. There is a sequence $R_1, R_2, R_3, \ldots$ such that for each positive integer $j$,

1. $R_j$ is a domain of $H_j$ which is not a subset of $R$ and
2. the point set $(R_1 + R_2 + R_3 + \ldots)$ contains $P$ or has $P$ as a limit point.

If there are only finitely many distinct domains represented in the sequence $R_1, R_2, R_3, \ldots$, then there is a domain $d$ such that $d$ contains $P$ and $d$ is an element of infinitely many collections of the sequence $H_1, H_2, H_3, \ldots$. This contradicts the third condition of Axiom 1. Therefore, there are infinitely many distinct domains in the sequence $R_1, R_2, R_3, \ldots$.

For each positive integer $j$ greater than 1, let $d_j$ denote a domain of $W_j$ such that $d_j$ contains $R_j$. There is a domain $d_j$ of the sequence $d_1, d_2, d_3, \ldots$ such that $d_j$ contains $P$ since $W_j$ has the Q property with respect to $G_j$ and $M$. There is a region $g_j$ in $G_j$ such that $g_j$ contains $d_j$. Thus, $g_j$ contains $P$ and is not a subset of $R$. By using the sequence $R_1, R_2, R_3, \ldots$ it can be shown that there is a region $d_j$ of $G_j$ such that $d_j$ contains $P$ but is not a subset of $R$. Further, for each $n$, there is a region $g_n$ of $G_n$ such that $g_n$ contains $P$ but is not a subset of $R$. This is a contradiction.

Therefore, if $\alpha$ denotes the sequence $H_1, H_2, H_3, \ldots$, then $M_\alpha$ contains $M$.

**Theorem 4.** Suppose that $S$ is not compact, $M$ is an inner limiting set dense in $S$ and there is an Axiom 1 sequence $\alpha$ such that $M_\alpha$ contains $M$. There is an Axiom 1 sequence $\alpha'$ such that $M_\alpha'$ is $M$. 
Proof. Let $g_1, g_2, g_3, \ldots$ denote the elements of $a$. Suppose that $M$ is a proper subset of $M_a$. Let $\beta$ denote the set $S - M$. The proof is divided into three cases. The first, second, and third cases being those where $\beta$ is degenerate, non-degenerate and closed, and not closed, respectively.

Case 1. Let $P$ denote the point in $\beta$ and let $P_1, P_2, P_3, \ldots$ denote a sequence of distinct points such that the set $(P_1 + P_2 + P_3 + \cdots)$ has no limit point. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions closing down on $P$ such that, for each positive integer $n$, $R_n$ is a region of $G_n$. Let $g_1, g_2, g_3, \ldots$ denote a sequence of mutually exclusive regions such that the closure of their sum is the sum of their closures and, for each $n$, $g_n^* \subseteq P_n$. For each positive integer $n$ let $d_n$ denote the domain $(R_n + g_n^* - P)$. For each positive integer $n$, let $G_n$ denote the collection consisting of all regions of $G_n$ together with the domains $d_n, d_{n+1}, d_{n+2}, \ldots$.

Let $\alpha'$ denote the sequence $G_1, G_2, G_3, \ldots$. If $P$ is a point distinct from $P$, there is a positive integer $i$ such that only the domains of $G_i$ that contain $R_i$ are regions of $G_i$. Thus $\alpha'$ satisfies the first three conditions of Axiom 1. If $\alpha$ and $\beta$ are two intersecting domains of $G_n^*, \alpha$'s common part is a subset of a region of $G_n^*$. Therefore, $\alpha'$ satisfies the fourth condition of Axiom 1. Hence, it is an Axiom 1 sequence. But $P$ does not belong to $M_n$.

If $\beta$ is a point of $S - \beta$, there is a region $R$ containing $\beta$ and a positive integer $n$ such that every domain of $G_n$ that intersects $R$ is a region of $G_n$. Consequently, $S - \beta$ is a subset of $M$. It follows that $M_n = M$.

Case 2. There exists a sequence $H_1, H_2, H_3, \ldots$ such that for each $n$, $H_n$ is a collection of regions of $G_n$ properly covering $\beta$ such that if $H$, for each $n$, $D_n$ denotes the sum of the regions of $H_n$ then $D_n$ is a proper subset of $D_n$. Suppose that $a$ is a point of $M$. There is a region $R$ containing $\beta$ and a positive integer $j$ such that $R$ does not intersect $D_j$. The point set $\beta$ is the boundary of $M$ and it is the common part of the domains $D_1, D_2, D_3, \ldots$.

For each positive integer $n$, let $d_n$ denote the domain $D_n - D_{n+1}$.

Suppose that $P$ is a point of $\beta$ and $R$ is a region containing $P$. There is a point $Z$ of $M$ in $R$ and a positive integer $x$ such that $Z$ belongs to $D_n$. There is a region $X'$ containing $Z$ and a positive integer $j$ such that $X'$ does not intersect $D_j$. Thus, there is a positive integer $i$ such that $Z$ belongs to $D_{i-1}$ and not to $D_i$. There is a region $X''$ containing $Z$ and lying in $R$ but not intersecting $D_1$. Since $Z$ is a point of $D_{i-1}$, $X''$ intersects $D_{i-1}$. The domain $X'' - D_{i-1}$ is a subset of $R$ and of $(D_{i-1} - D_i)$. Therefore, $R$ intersects $d_{i-1}$.

The sequence $a_1, a_2, a_3, \ldots$ has the following properties:

1. If $x$ and $y$ are two positive integers, then $a_x$ and $a_y$ have no point in common,
Suppose that $A$ is a point of $M$ which is not a limit point of $\beta$. Since $\alpha$ satisfies Axiom C at $A$, there is a region $R$ containing $A$ and a positive integer $n$ such that no region of $G_\alpha$ intersects $\beta$ and $R$. Since each domain of $G_\alpha$ which is not a region of $G_\alpha$ is the common part of some $R_{k,\alpha}$ and $\beta$, where $i$ is greater than $n$, this means that each point of this common part belongs to a region of $G_\alpha$ which intersects $\beta$ and thus is a point of a region of $G_\alpha$ which intersects $\beta$. Therefore, $R$ intersects no domain of $H_\alpha$ which is not a region of $G_\alpha$.

Suppose that $A$ is a point of $M$ which is a limit point of $\beta$. Let $R$ denote a region containing $A$. Since, for each $\tau$, $\beta_{\tau}$ is closed, $A$ is not a limit point of $\beta_{\tau}$. Since $A$ belongs to $M_\alpha$, there is a region $R_\alpha$ containing $A$ and a positive integer $n$, such that every region of $G_\alpha$ intersects $R_\alpha$ is a subset of $R$. There is a region $R_\beta$ which is a subset of $R_\alpha$ and contains $A$, and a positive integer $x$ such that if $i \leq n$ and $j > x$, then $R_\beta$ does not intersect the domain $\beta_{\tau}$, Every domain of $H_{\alpha+1}$ which intersects $R_\alpha$ is a subset of $R$.

Suppose that $A$ is a point of $\beta$. Let $R$ denote a region containing $A$. There is a positive integer $n$ such that every region of $G_\alpha$ which contains $A$ is a subset of $R$. There is a positive integer $\iota$ greater than $n$ such that no region of $G_\alpha$ contains $A$ and intersects $(\beta_{\iota} - \beta_{\iota-1} + \ldots + \beta_{\iota})$. Each domain of $H_{\alpha+1}$ is a region of $G_{\alpha+1}$ or is of the form $(R_{k,\alpha},R_{k,\alpha})$ where $i$ is greater than $n+\iota$. Every region of $H_{\alpha+1}$ which contains $A$ is a subset of $R$. If there is a domain $(R_{k,\alpha},R_{k,\alpha})$ which contains $A$ but is not a subset of $R$ then, since $R_{k,\alpha}$ is a region of $G_\alpha$, $x$ is less than $n$. Each point of $R_{k,\alpha}$ is a subset of a region of $G_\alpha$ which intersects $R_{k,\alpha}$ and since $x$ is less than $n$, $\iota$ must be less than $i$. However, $\iota$ is greater than $n+\iota$. Consequently, every domain of $H_{\alpha+1}$ which contains $A$ is a subset of $R$.

It follows now fairly readily that the sequence $H_1,H_2,H_3,\ldots$ satisfies the first three conditions of Axiom 1 at each point of $\alpha$ and the first three conditions of Axiom C at each point of $M$.

Suppose that $M_1,M_2,M_3,\ldots$ is a sequence of closed point sets such that for each positive integer $n$, $M_n$ contains $M_{n+1}$, and there is a domain $d_{n}$ in $H_\alpha$ such that $M_n$ is a subset of $d_{n}$. For each positive integer $\iota$, no two domains in the sequence $A_{\iota},A_{\iota+1},A_{\iota+2}$ have a point in common. Since each domain of $H_\alpha$ which is not a region of $G_\alpha$ is of the form $(R_{k,\alpha},R_{k,\alpha})$ where $i$ is greater than $n$, there exists no positive integer $m$ such that infinitely many domains of the sequence $A_1,A_2,A_3,\ldots$ are of the form $(R_{k,\alpha},R_{k,\alpha})$ where $\alpha \leq m$. Therefore, there is an ascending sequence $\eta_1,\eta_2,\eta_3,\ldots$ of positive integers and a sequence $R_1,R_2,R_3,\ldots$ of regions such that for each positive integer $k$, $R_k$ is a region of $G_\alpha$ and contains $d_n$. Thus the sets of the sequence $M_1,M_2,M_3,\ldots$ have a point in common. Therefore, the sequence $H_1,H_2,H_3,\ldots$ satisfies the fourth condition in Axiom 1.

If $\alpha$ is a positive integer such that $U_\alpha$ exists and $P$ is a point of $\beta_{\alpha}$ and $R$ is a region containing $P$, then there is a positive integer $\iota$ such that $A_{\iota}$ intersects $R$. If $A$ is a point of $U_\alpha$, then $(R_{\alpha+1},\beta_{\alpha})$ is non-degenerate. From these facts it follows that the sequence $H_1,H_2,H_3,\ldots$ does not satisfy the requirements of Axiom C at any point of $U$.

Suppose that $P$ is a point of $M_\alpha - (M_\alpha + U)$, that is to say, a point of $T$. For each positive integer $\alpha$ such that $T$ intersects $\beta_{\alpha}$, let $T_{\alpha}$ denote $T \cap \beta_{\alpha}$. For no $\alpha$ does a region of $G_\alpha$ intersect two points of $T_{\alpha}$.

Suppose that $T$ is closed. Since $M$ is dense in $S$, every point of $T$ is a limit point of $M$. Consequently, there is an Axiom 1 sequence $H_1,H_2,H_3,\ldots$, satisfying the requirements of Axiom C at each point of $M$ but at no point of $T$. This is exactly analogous to the situation in Case 3.

For each positive integer $n$, let $G_n$ denote a collection such that $d$ is an element of $G_n$ if and only if $d$ is a domain of $H_n$ or of $H_{n+1}$. Let $a'$ denote the sequence $G_1,G_2,G_3,\ldots$. It readily follows that $M_\alpha$ is $M$.

Suppose that $T$ is not closed but contains a limit point of itself. For each positive integer $n$ such that $T_{\alpha}$ contains a limit point of $T$, let $V_\alpha$ be the set of all limit points of $T$ that belong to $T_{\alpha}$. No region of $G_{\alpha+1}$ covers two points of $V_\alpha$. There is a well-ordered sequence $\omega$ whose terms are the points of $V_\alpha$. There is a subsequence $\omega_\alpha$ of $\omega$ such that

1. the first term of $\omega$ is the first term of $\omega_\alpha$;
2. if $\alpha$ is an initial segment of $\omega_\alpha$ and there is a point $P$ of $V_\alpha$ such that no coherent collection of three regions of $G_{\alpha+1}$ covers $P$ and some point in $Z$, then the first such point in $\omega$ is the first point in $\omega_\alpha$ to follow all the points of $Z$ in $\omega_\alpha$.

If $\alpha$ is a positive integer and there is a point in $V_\alpha$ which is not in $\omega_\alpha$, then let $\omega_{\alpha+1}$ be a subsequence of $\omega$ such that

1. the first term of $\omega$ which is not in $\omega_\alpha$ for $1 \leq i \leq \alpha$ is the first term of $\omega_{\alpha+1}$;
2. if $\alpha$ is an initial segment of $\omega_{\alpha+1}$, and there is a point $P$ of $V_\alpha$ not in $\omega_\alpha$ where $1 \leq i \leq \alpha$, such that no coherent collection of three regions of $G_{\alpha+1}$ covers $P$ and some point in $Z$, then the first such point in $\omega$ is the first point in $\omega_{\alpha+1}$ to follow all the points of $Z$ in $\omega_{\alpha+1}$.

Suppose that there is a point $P$ of $V_\alpha$ which is in no one of the sequences $\omega_1,\omega_2,\omega_3,\ldots$. For each positive integer $n$, there is a coherent collection of three regions of $G_n$ that covers $P$ and some other point of $V_\alpha$. Since $P$ belongs to $M_\alpha$, it follows that $P$ is a limit point of $V_\alpha$. This is a contradiction.

Suppose that $i$ is a positive integer and $G$ is a collection of regions of $G_{\alpha+1}$ such that each region of $G$ contains only one point of $\omega_{\alpha+1}$ and
each point in $a_{n+1}$ belongs to only one region of $G$. Suppose that there exists a point of $G'$ which is not a point or limit point of any one region of $G$. Then there is a coherent collection of three regions of $G_{n+1}$ that covers two points in $a_{n+1}$ which is impossible. Therefore, if $P$ is a point of $G'$, there is only one region $g$ in $G$ such that $P$ is a point of $g$. For each positive integer $j$ such that $a_j$ exists, let $G_j$ be a collection of regions of $G_{n+1}$ such that each region of $G_j$ contains only one point of $a_j$ and each point in $a_j$ belongs to only one region of $G_j$. Let $C$ denote the collection such that $R$ is in $C$ if and only if there is a positive integer $s$ such that $R$ is in $C_s$. Suppose that there is a point $P$ of $M$ which is a point of $G'$ and not a point or limit point of any region in $G$. Since $P$ belongs to $M_s$, if $R$ is a region containing $P$, there is a region $R_j$ containing $P$ such that every region of $C$ that intersects $R_j$ is a subset of $R$.

For each positive integer $n$ such that $V_n$ exists, there is a collection $K_n$ in $V_n$ with properties with respect to $V_n$ as described in the preceding paragraph for the collection $C$ with respect to $V_n$. Suppose that $P$ is a point of $V_n$. Since $P$ is a limit point of $T$, if $R$ denotes the region of $K_n$ that contains $P$, then $R$ is non-degenerate. Each point of $T$ is a limit point of $M$ so there is a sequence of mutually exclusive domains $d_1, d_2, d_3, ...$ such that

(1) for each $n$, $d_n$ is a subset of $R$,

(2) if, for each positive integer $n$, $G_n$ denotes the collection consisting of all regions of $G_n$ together with the domains $d_n, d_{n+1}, d_{n+2}, ...$, then the sequence $G_1, G_2, G_3, ...$ is an Axiom 1 sequence which fails to satisfy Axiom C at $P$.

In a manner similar to the construction of the sequence $B_1, B_2, B_3, ...$, it may be shown that there exists an Axiom 1 sequence $B_1', B_2', B_3', ...$ such that for each positive integer $n$, each domain of $B_n$ is either a region of $G_n$ or a subset of a region of some $K_j$. Further, this sequence does not have the properties as stated in Axiom C at any point of $T$ which is a limit point of $T$ and has these properties at $M$.

Suppose that there is a point of $T$ which is not a limit point of $T$. Let $W$ denote the set of all such points. The point set $T - W$ is closed. There is a sequence $E_1, E_2, E_3, ...$ of domains with $T - W$ as their common part such that

(1) for each $n$, $E_n$ contains $E_{n+1}$,

(2) if $P$ is a point of $W$ in $E_1$, there is a positive integer $s$ such that $P$ is in $E_s$, but not in $E_{s+1}$, and a region $g$ of $G_s$ which contains $P$ and a point $g$ of $W$ distinct from $P$.

If there is a point of $W$ not in $E_1$, the set of all such points is a closed point set. There is an Axiom 1 sequence $L_1, L_2, L_3, ...$ such that this sequence fails to satisfy Axiom C at each point of $W$ not in $E_1$ and satisfies Axiom C at each point of $M$. The set of points of $W$ that are in $E_1$ may be divided up into the sets of a sequence $W_1, W_2, W_3, ...$ which possesses the properties of the sequences $U_1, U_2, U_3, ...$ that made it possible to construct an Axiom 1 sequence that satisfied Axiom C at each point of $M$ and did not satisfy Axiom C at any point of $U$. Therefore, there is an Axiom 1 sequence that satisfies Axiom C at each point of $M$ but at no point of $W$.

If $P$ is a point of $T$, then $P$ is a point of $U$ or of $T$. If $P$ is a point of $T$, and $T$ is closed, then $P$ is a point of $W$ or of $T - W$. Consequently, there is an Axiom 1 sequence $a'$ such that $M_{a'}$ is $M$.

References


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