On superpositions of simple mappings

by

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1. Introduction. K. Borsuk and R. Molski considered in [4] a class of continuous mappings called *simple mappings*. A continuous mapping f of a space X onto the space Y *is of order* $\leqslant k$ if for every point $y \in Y$ the set $f^{-1}(y)$ contains at most k points (cf. [8], p. 52). The mappings of order $\leqslant 2$ are said to be *simple mappings*. In [4] the authors raise the following question (p. 92, No 4): does there exist a continuous mapping of a finite order which is not a superposition of a finite number of simple mappings?

The purpose of this paper is to prove that every continuous mapping f of a finite order defined on the compact space X of a finite dimension is a superposition of a finite number of simple mappings. On the other hand, we shall construct a compact infinite dimensional space X and a continuous mapping of a finite order f defined on X which will not be a superposition of a finite number of simple mappings.

2. Auxiliary definitions and notations.

Definition 2.1. A collection of subsets of a space X constitutes a *decomposition* $\mathfrak{W}$ of X if the sets of $\mathfrak{W}$ are disjoint and non-empty, and if they fill up X. The decomposition $\mathfrak{W}$ is said to be *upper semi-continuous* if for every closed subset A of X the union of all sets of $\mathfrak{W}$ intersecting A is closed in X (see [8], p. 42).

P. Alexandroff ([1] and [2]; cf. also [8], p. 42) has proved the following theorem: *In order that a decomposition $\mathfrak{W}$ of a compact space X be upper semicontinuous, it is sufficient and necessary that there exist a space Y and a continuous mapping f of X onto Y such that the sets belonging to $\mathfrak{W}$ are the same as the sets $f^{-1}(y)$ where $y \in Y$.*

Let $\{A_i\}$ $(i = 1, 2, \ldots)$ denote a sequence of subsets of the space X and let $\operatorname{Lim}_{i\to\infty} A_i$ be defined as in [7], p. 241-245. We shall use the following important properties of the notion of this limit:

(i) The generalized Bolzano-Weierstrass theorem. *If the space is separable, then from every sequence of its subsets we can choose a convergent subsequence (may be to the empty set)* (see [7], p. 246).

(ii) *If the space X is compact and sets $A, A_i \subset X$ $(i = 1, 2, \ldots)$ are non-empty and closed, then the condition $A = \operatorname{Lim}_{i\to\infty} A_i$ is equivalent to $\lim_{i\to\infty} \operatorname{dist}(A, A_i) = 0$, where*

$$\operatorname{dist}(A, B) = \max\left(\sup_{a\in A}\inf_{b\in B}\varrho(a, b),\ \sup_{b\in B}\inf_{a\in A}\varrho(a, b)\right)$$

([8], p. 21).

(iii) *In order that the decomposition $\mathfrak{W}$ of the compact space X be upper semicontinuous, it is necessary and sufficient that for every convergent sequence $\{W_i\}$, $W_i \in \mathfrak{W}$ $(i = 1, 2, \ldots)$, there exists a set $W_0 \in \mathfrak{W}$ satisfying $\operatorname{Lim}_{i\to\infty} W_i \subset W_0$.*

Definition 2.2. Let $\mathfrak{U} = \{U_\varkappa\}$ and $\mathfrak{V} = \{V_\lambda\}$ be two decompositions of the space X. The decomposition of X consisting of all non-empty intersections $U_\varkappa \cap V_\lambda$ will be denoted by $\mathfrak{U} \cap \mathfrak{V}$.

LEMMA 2.3. *If the decompositions $\mathfrak{U} = \{U_\varkappa\}$ and $\mathfrak{V} = \{V_\lambda\}$ of the compact space X are upper semicontinuous, then the decomposition $\mathfrak{W} = \mathfrak{U} \cap \mathfrak{V}$ is also upper semicontinuous.*

Proof. Let $\{W_{\tau_i}\}$ be a convergent sequence of sets belonging to $\mathfrak{W}$. For every $i = 1, 2, \ldots$ there exist $U_{\varkappa_i} \in \mathfrak{U}$ and $V_{\lambda_i} \in \mathfrak{V}$ such that $W_{\tau_i} = U_{\varkappa_i} \cap V_{\lambda_i}$. By property (i) we can assume that the sequence $U_{\varkappa_i}$ converges to U_0' and V_{λ_i} converges to V_0'. Using property (iii) we can find $U_0' \subset U_0 \in \mathfrak{U}$, $V_0' \subset V_0 \in \mathfrak{V}$. Since X is compact and since $U_{\varkappa_i} \cap V_{\lambda_i} \neq 0$ for $i = 1, 2, \ldots$, we obtain $U_0 \cap V_0 \neq 0$. Then there exists $W_0 = U_0 \cap V_0 \in \mathfrak{W}$ and $\operatorname{Lim}_{i\to\infty} W_{\tau_i} \subset W_0$. By property (iii) this proves the lemma.

Definition 2.4. Let $\mathfrak{U}$ and $\mathfrak{V}$ be upper semicontinuous decompositions of X and let m be a natural number. If for each set of $\mathfrak{U}$ there exists a set of $\mathfrak{V}$ containing it and for each set of $\mathfrak{V}$ there exist at most m sets of $\mathfrak{U}$ contained in it, then we shall write $\mathfrak{U} \underset{m}{\subset} \mathfrak{V}$.

3. Expressing the problem in terms of semicontinuous decompositions. We now prove

LEMMA 3.1. *Let us consider the mappings: f_1 of $X = X_1$ onto X_2, f_2 of X_2 onto $X_3, \ldots, f_r$ of X_r onto $X_{r+1} = Y$, where X is a compact space and let $\varphi_l = f_l \circ f_{l-1} \circ \ldots \circ f_2 \circ f_1$ $(l \leqslant r)$. Then in order that all the functions $f_1, f_2, \ldots, f_r$ be continuous it is necessary and sufficient that all the functions $\varphi_1, \varphi_2, \ldots, \varphi_r$ be continuous.*

Proof. It is obvious that the condition is necessary. To prove its sufficiency suppose that $\varphi_1, \varphi_2, \ldots, \varphi_r$ are continuous and f_{l_0} is the first non-continuous function in the sequence $f_1, f_2, \ldots, f_r$. Of course $l_0 > 1$.

Then there exists a sequence $\{x_i\}$, $x_i \in X_{l_0}$, $\lim_{i\to\infty} x_i = x_0 \in X_{l_0}$, such that $\lim_{i\to\infty} f_{l_0}(x_i)$, if it exists, differs from $f_{l_0}(x_0)$. Since X_{l_0+1}, as the continuous image of a compact space, is also compact, we can assume that

$$x_i \in X_{l_0}, \quad \lim_{i\to\infty} x_i = x_0, \tag{1}$$

and there exists $\lim_{i\to\infty} f_{l_0}(x_i) \neq f_{l_0}(x_0)$.

Let ξ_i denote an arbitrary point of the (non-empty) set $\varphi_{l_0-1}^{-1}(x_i) \subset X$. Then

$$\varphi_{l_0-1}(\xi_i) = x_i. \tag{2}$$

Let us choose a convergent subsequence

$$\lim_{i\to\infty} \xi_{j_i} = \xi_0. \tag{3}$$

With regard to the continuity of φ_{l_0-1} and from (2) we obtain $\varphi_{l_0-1}(\xi_0) = \lim_{i\to\infty} \varphi_{l_0-1}(\xi_{j_i}) = \lim_{i\to\infty} x_{j_i}$. Hence by (1) we obtain

$$\varphi_{l_0-1}(\xi_0) = x_0. \tag{4}$$

Since $\varphi_{l_0} = f_{l_0} \circ \varphi_{l_0-1}$, it follows from (2) that

$$\lim_{i\to\infty} \varphi_{l_0}(\xi_{j_i}) = \lim_{i\to\infty} f_{l_0}(x_{j_i}) \tag{5}$$

and from (4)

$$\varphi_{l_0}(\xi_0) = f_{l_0}(x_0). \tag{6}$$

Combining (5), (6) and (3) in view of the continuity of φ_{l_0} we obtain $\lim_{i\to\infty} f_{l_0}(x_{j_i}) = f_{l_0}(x_0)$ contrary to (1). This proves the sufficiency.

LEMMA 3.2. *In order that the continuous mapping f of order $\leqslant k$ defined on the compact X and determining the upper semicontinuous decomposition $\mathfrak{W}$ be a superposition of r mappings of order $\leqslant m$ $(2 \leqslant m \leqslant k)$ it is necessary and sufficient that there exists a sequence $\mathfrak{W}^l$ $(l = 0, 1, \ldots, r)$ of upper semicontinuous decompositions of X such that*

1° *$\mathfrak{W}^0$ consists of the points of X,*

2° $\mathfrak{W}^0 \underset{m}{\subset} \mathfrak{W}^1 \underset{m}{\subset} \ldots \underset{m}{\subset} \mathfrak{W}^{r-1} \underset{m}{\subset} \mathfrak{W}^r = \mathfrak{W}$.

Proof. To prove necessity let us suppose that $f = f_r \circ f_{r-1} \circ \ldots \circ f_2 \circ f_1$ where f_l $(l = 1, 2, \ldots, r)$ is the continuous mapping of order $\leqslant m$ defined on X_l onto X_{l+1}. The functions $\varphi_0 =$ identity on $X = X_1$, $\varphi_l = f_l \circ f_{l-1} \circ \ldots \circ f_2 \circ f_1$ $(l = 1, 2, \ldots, r)$ are obviously continuous. Denoting by $\mathfrak{W}^l$ the upper semicontinuous decomposition of X corresponding to φ_l $(l = 0, 1, \ldots, r)$ we obtain the sequence of upper semicontinuous decompositions satisfying 1° and 2°. To prove sufficiency let us suppose that the sequence $\mathfrak{W}^l$ $(l = 0, 1, \ldots, r)$ of upper semicontinuous decompositions of X satisfies conditions 1° and 2°. By Alexandroff's theorem there exists a sequence of continuous functions φ_l mapping X_l onto X_{l+1} $(l = 0, 1, \ldots, r)$ and such that $\varphi_0 =$ identity on $X = X_1$. Condition 2° implies that for every $l = 1, 2, \ldots, r$ there exists a function f_l of X_l onto X_{l+1} such that $\varphi_l = f_l \circ \varphi_{l-1}$. In view of lemma 3.1 the functions $f_1, f_2, \ldots, f_r$ are continuous. By condition 2° they are also of order $\leqslant m$. Since $f = \varphi_r = f_r \circ f_{r-1} \circ \ldots \circ f_2 \circ f_1$, the proof is complete.

4. Main theorem. We have the following

Definition 4.1. Let $\mathfrak{W} = \{W_\tau\}$ denote an upper semicontinuous decomposition of the space X. Moreover let us suppose that in X there exists a binary relation $\prec$. The relation $\prec$ is said to be *closed relative to* $\mathfrak{W}$ if

1° X is partially ordered by $\prec$. This means that
(a) If $x^1 \prec x^2$, then not $x^2 \prec x^1$;
(b) if $x^1 \prec x^2$ and $x^2 \prec x^3$, then $x^1 \prec x^3$.

2° Each set $W_\tau \in \mathfrak{W}$ is completely ordered by $\prec$. This means that (besides 1°) for each $x^1, x^2 \in W_\tau$ such that $x^1 \neq x^2$ either $x^1 \prec x^2$ or $x^2 \prec x^1$ holds.

3° Let $\{W_{\tau_i}\}$ be an arbitrary convergent sequence of sets of $\mathfrak{W}$. By property (iii) (p. 218) there exists $W_{\tau_0} \in \mathfrak{W}$ such that $\operatorname*{Lim}_{i\to\infty} W_{\tau_i} \subset W_{\tau_0}$. We require that if $x_i^1, x_i^2 \in W_{\tau_i}$, $x_i^1 \prec x_i^2$ for $i = 1, 2, \ldots$, $\lim_{i\to\infty} x_i^1 = x_0^1 \in W_{\tau_0}$, $\lim_{i\to\infty} x_i^2 = x_0^2 \in W_{\tau_0}$ then either $x_0^1 \prec x_0^2$ or $x_0^1 = x_0^2$.

Definition 4.2. Let $\mathfrak{W} = \{W_\tau\}$ be an upper semicontinuous decomposition of X and let $\prec$ be a relation closed relative to $\mathfrak{W}$. Writing $\Pi(W_\tau) = W_\tau \times W_\tau$ we can introduce in $\bigcup_\tau \Pi(W_\tau)$ a topology induced by the imbedding in the Cartesian product $X \times X$ and a relation $<$ defined by the method of first differences (see for example [9], p. 159).

Lemma 4.3. *Let there exist in the compact metric space X an upper semicontinuous decomposition $\mathfrak{W} = \{W_\tau\}$ and a relation $\prec$ closed relative to $\mathfrak{W}$. Let us suppose that the convergent sequence $\{W_{\tau_i}\}$ satisfies*

1. $\overline{\overline{W_{\tau_i}}} = p$ $(i = 1, 2, \ldots)$ *where p is a natural number;*

2. *there exists $\delta > 0$ such that if $W_{\tau_i} = (x_i^1, x_i^2, \ldots, x_i^p)$, then, for every i, $\min_{1\leqslant\nu<\mu\leqslant p} \varrho(x_i^\nu, x_i^\mu) \geqslant \delta$ holds.*

Then

1. $\overline{\overline{\operatorname*{Lim}_{i\to\infty} W_{\tau_i}}} = p$.

2. *For every two sequences $\{\Pi_i^1\}, \{\Pi_i^2\}$ such that $\Pi_i^1, \Pi_i^2 \in \Pi(W_{\tau_i})$, $\Pi_i^1 < \Pi_i^2$ $(i = 1, 2, \ldots)$, $\lim_{i\to\infty} \Pi_i^\nu = \Pi_0^\nu \in \Pi(W_{\tau_0})$ $(\nu = 1, 2)$, we have the relation $\Pi_0^1 < \Pi_0^2$.*

Proof. By property (ii), p. 218, we conclude that $\lim_{i\to\infty} \operatorname{dist}(\operatorname*{Lim}_{i\to\infty} W_0, W_{\tau_i}) = 0$. Hence in each sphere $K(x, \varepsilon)$, where $x \in \operatorname*{Lim}_{i\to\infty} W_{\tau_i}$, for almost all i there exist $x_i \in W_{\tau_i}$. So $\overline{\overline{\operatorname*{Lim}_{i\to\infty} W_{\tau_i}}} \leqslant \overline{\overline{W_{\tau_i}}} = p$ holds. If $\overline{\overline{\operatorname*{Lim}_{i\to\infty} W_{\tau_i}}} < \overline{\overline{W_{\tau_i}}}$, then for almost all i at least two points $x_i^1, x_i^2 \in W_{\tau_i}$ would belong to a sphere $K(x, \varepsilon)$ where $x \in \operatorname*{Lim}_{i\to\infty} W_{\tau_i}$. Hence we should have $\varrho(x_i^1, x_i^2) < \varepsilon$ contrary to the suppositions. By the above remarks and in view of our suppositions we can assume that $W_{\tau_i} = (x_i^1, x_i^2, \ldots, x_i^p)$ for $i = 0, 1, \ldots$ where $\lim_{i\to\infty} x_i^\nu = x_0^\nu$ for $\nu = 1, 2, \ldots, p$.

By assumption 2 we infer that for almost all i the pair Π_i^1 can be written as $\langle x_i^{\nu_1}, x_i^{\mu_1}\rangle$ where ν_1, μ_1 do not depend on i. Similarly $\Pi_i^2 = \langle x_i^{\nu_2}, x_i^{\mu_2}\rangle$ where ν_2, μ_2 do not depend on i. Then $\Pi_0^1 = \langle x_0^{\nu_1}, x_0^{\mu_1}\rangle$, $\Pi_0^2 = \langle x_0^{\nu_2}, x_0^{\mu_2}\rangle$. In view of assumption 2 we conclude that either $x_i^{\nu_1} \prec x_i^{\mu_2}$ or $x_i^{\nu_1} = x_i^{\nu_2}$ and $x_i^{\mu_1} \prec x_i^{\mu_2}$ for $i = 1, 2, \ldots$ In the first case by the closeness of $\prec$ we have $x_0^{\nu_1} \prec x_0^{\nu_2}$. In the second case $\nu_1 = \nu_2$, $\mu_1 \neq \mu_2$ and we have $x_0^{\nu_1} = x_0^{\nu_2}$, $x_0^{\mu_1} \prec x_0^{\mu_2}$, which proves that $\Pi_0^1 < \Pi_0^2$.

Lemma 4.4. *If in the metric compact space X there exist an upper semicontinuous decomposition $\mathfrak{W} = \{W_\tau\}$ such that $\overline{\overline{W_\tau}} \leqslant k$ $(k \geqslant 3)$ and the relation $\prec$ closed relative to $\mathfrak{W}$, then there exists a finite sequence $\{\mathfrak{W}^l\}$, $l = 1, 2, \ldots, r, r+1$, $r = k(k+1)$, of upper semicontinuous decompositions of X satisfying:*

1° *The sets of $\mathfrak{W}^0$ are the same as points of X.*

2° $\mathfrak{W}^0 \underset{k-1}{\subset} \mathfrak{W}^1 \underset{k-1}{\subset} \ldots \underset{k-1}{\subset} \mathfrak{W}^r \underset{k-1}{\subset} \mathfrak{W}^{r+1} = \mathfrak{W}$.

Proof. A pair $\Pi^0 = \langle x^1, x^2\rangle \in \Pi(W_\tau)$ is said to be a *minimal pair* if $x^1 \prec x^2$ and the diameter of Π^0 is equal to the minimum of diameters (different from zero) of all pairs $\Pi \in \Pi(W_\tau)$. We shall say that a set $W \subset W_\tau$ is *minimally connected* if for every $x, y \in W$ there exists a sequence $x_i \in W_\tau$ $(i = 1, 2, \ldots, t)$ such that $x = x_1$, $y = x_t$ and the pair $\langle x_i, x_{i+1}\rangle$ is the minimal one for $i = 1, 2, \ldots, t-1$. We shall say that a set $W \subset W_\tau$ is the *minimal component* of W_τ if W is minimally connected and there is no minimally connected set $W \neq W' \subset W_\tau$. In each W_τ there exists at least one minimal component and they are all disjoint.

Let $\mathfrak{V} \subset \mathfrak{W}$ denote the family of sets of $\mathfrak{W}$ consisting of exactly k points. Let $\mathfrak{V}_\nu$ denote the subfamily of $\mathfrak{V}$ consisting of those sets which possesse exactly ν minimal pairs.

We shall define the sequence $\{\mathfrak{W}^l\}$ $(l = 1, 2, \ldots,\ r = k(k+1))$ of decompositions of X as follows:

A. For an odd $l = 2\alpha+1$ the sets of $\mathfrak{W}^l$ are

(a) the first (in the sense of definition 4.2) minimal pairs in the minimal components of sets of the family $\mathfrak{V}_{\binom{k}{2}-\alpha}$,

(b) the minimal components of sets of the families $\mathfrak{V}_{\binom{k}{2}-\beta}$ where $0 \leqslant \beta < \alpha$,

(c) the remaining points of X.

B. For an even $l = 2\alpha$ the sets of $\mathfrak{W}^l$ are

(a) the minimal components of sets of the families $\mathfrak{V}_{\binom{k}{2}-\beta}$ where $0 \leqslant \beta < \alpha$,

(b) the remaining points of X.

To prove the upper semicontinuity of this decomposition let us take an arbitrary sequence of its sets $\{W_i\}$ convergent to W. By property (i), p. 217, we can suppose that $W_i \subset V_i$ $(i = 1, 2, \ldots)$ where $\{V_i\}$ is a sequence of sets of the family $\mathfrak{V}$ convergent to V. Moreover, choosing again a suitable subsequence we can suppose that W_i are given (for $i = 1, 2, \ldots$) by the same definition. In cases A (c) and B (b) the set W contains at most one point and then it is contained in a set of the family $\mathfrak{W}^l$.

Let us suppose that W_i $(i = 1, 2, \ldots)$ are defined by A (a). If W contains at most one point it is contained in a set of the family $\mathfrak{W}^l$. Let us suppose that W contains two different points. We shall prove that $\overline{\overline{V}} = k$ (it means that $V \in \mathfrak{V}$). Indeed, in the opposite case in almost every V_i there would exist two arbitrarily near points. Then almost every W_i, being the minimal pair, would have an arbitrarily small diameter, contrary to $\overline{\overline{W}} = 2$. We observe that $V \in \mathfrak{V}_{\binom{k}{2}-\beta}$ where $0 \leqslant \beta \leqslant \alpha$. If $V \in \mathfrak{V}_{\binom{k}{2}-\beta}$ where $0 \leqslant \beta < \alpha$, then by $W \subset V$ and by the point A (b) of our construction, the set W is contained in a set of the family $\mathfrak{W}^l$. If $V \in \mathfrak{V}_{\binom{k}{2}-\alpha}$ then the minimal components of V are convergent to the corresponding minimal components of V, and in view of lemma 4.3 the same holds for their first minimal pairs. Hence W is a set of thefamily $\mathfrak{W}_l$.

The cases A (b) and B (a) can be considered together. We suppose as above that $\overline{\overline{W}} \geqslant 2$. Then we shall prove that $\overline{\overline{V}} = k$ (it means that $V \in \mathfrak{V}$). Indeed, in the opposite case in almost every V_i there would exist two arbitrarily near points. Then in almost every W_i each minimal pair would have an arbitrarily small diameter. By the definition of the minimal component we conclude that almost every W_i would have an arbitrarily small diameter, contrary to $\overline{\overline{W}} \geqslant 2$. We observe that $V \in \mathfrak{V}_{\binom{k}{2}-\beta}$ where

$0 \leqslant \beta \leqslant \alpha$. If $V \in \mathfrak{V}_{\binom{k}{2}-\beta}$ where $0 \leqslant \beta < \alpha$, then by $W \subset V$ and by the minimal connexity, the set W is contained in a set of the family $\mathfrak{W}^l$. If $V \in \mathfrak{V}_{\binom{k}{2}-\alpha}$ then the minimal components of V_i are convergent to the corresponding minimal components of V and the set W belongs to the family $\mathfrak{W}^l$.

Defining $\mathfrak{W}^0$ as the upper semicontinuous decomposition consisting of the points of X and putting $\mathfrak{W}^{r+1} = \mathfrak{W}$ we easily verify that

$$\mathfrak{W}^0 \underset{k-1}{\subset} \mathfrak{W}^1 \underset{k-1}{\subset} \ldots \underset{k-1}{\subset} \mathfrak{W}^{r-1} \underset{k-1}{\subset} \mathfrak{W}^r \underset{k-1}{\subset} \mathfrak{W}^{r+1} = \mathfrak{W}.$$

Thus the proof is finished.

Lemma 4.5. *Every continuous mapping f of order $\leqslant k$ $(k \geqslant 3)$ defined on a compact n-dimensional space X is a superposition of $s(k, n) = (2n+1)k(k+1)$ mappings of order $\leqslant k-1$.*

Proof. By the theorem of Menger-Nöbeling ([10], [11]) we can suppose that X is a subset of an m-dimensional Euclidean space, where $m = 2n+1$. Let $\mathfrak{E}^\mu$ $(0 \leqslant \mu \leqslant m)$ denote the decomposition of X consisting of the intersections of sets $f^{-1}(p)$ with the hyperplanes given by the system of equations: $x_{\mu+1} = c_{\mu+1}, x_{\mu+2} = c_{\mu+2}, \ldots, x_m = c_m$. In view of lemma 2.3 the decomposition $\mathfrak{E}^\mu$ is upper semicontinuous for $\mu = 0, \ldots, m$. Evidently $\mathfrak{E}^0 \underset{k}{\subset} \mathfrak{E}^1 \underset{k}{\subset} \ldots \underset{k}{\subset} \mathfrak{E}^m$. Using lemma 3.2 we conclude that there exists a sequence of spaces $X = X_1, X_2, \ldots, X_{m+1} = f(X)$ and continuous mappings of order $\leqslant k$: $f_0 =$ identity on X_1; f_i of X_i onto X_{i+1} $(i = 1, 2, \ldots, s(k, n))$ such that the mapping $f_l \circ f_{l-1} \circ \ldots \circ f_0$ $(l = 0, 1, \ldots, m)$ determines the decomposition $\mathfrak{E}^l$ of X.

In each of the spaces X_j $(j = 1, 2, \ldots, m+1)$ we define the relation $\prec_j$ as follows: $p' \prec_j p''$ if and only if the j-th coordinate of the set $(f_{j-1} \circ \ldots \circ f_0)^{-1}(p')$ is less than the j-th coordinate of the set $(f_{j-1} \circ \ldots \circ f_0)^{-1}(p'')$. It can easily be verified that the relation $\prec_j$ $(j = 1, 2, \ldots, m+1)$ is closed relative to the decomposition determined by f_{j-1}. By lemma 4.4 we conclude that the mapping f_i $(i = 0, 1, \ldots, m)$ is a superposition of $r = k(k+1)$ mappings of order $\leqslant k$. Hence f is a superposition of $s(k, n) = (2n+1)(k)(k+1)$ mappings of order $\leqslant k-1$.

Theorem 1. *Every continuous mapping f of order $\leqslant k$ $(k \geqslant 2)$ defined on a compact n-dimensional space X is a finite superposition of $z(k, n)$ simple mappings* (1).

Proof. By the lemma 4.5, f is a superposition of $s(k, n)$ mappings $f_{1,k-1}, \ldots, f_{s,k-1}$ of order $\leqslant k-1$. The theorem of Hurewicz [5] states that

(1) It can easily be verified that the following inequality holds:

$$z(k, n) \leqslant \prod_{i=1}^{k} \left\{2\left[n + (i-1)k - \frac{i(i-1)}{2}\right] + 1\right\}.$$

for every continuous mapping h of order $\leqslant t$ defined on the compact Y we have $\dim h(Y) \leqslant \dim(Y)+t-1$. Hence the space $f_{i,k-1}, \ldots, f_{1,k-1}(X)$ is of finite dimension for $i = 1, 2, \ldots, s(k, n)$.

In this manner we can repeat our reasoning $k-2$ times, which completes the proof.

5. Counter-example. Let U denote a continuous mapping of the sphere S_{n-1} onto itself such that $U, U^2, \ldots, U^{p-1}$ have no fixed points but $U^p = \text{identity}$.

Definition 5.1. We shall say that the *set* $Z \subset S_{n-1}$ *has the property* (U) if 1° $U(Z) = Z$ and 2° in every component of Z there is no pair of points of the form $x, U^\nu(x)$ where $\nu = 1, 2, \ldots, p-1$.

Lemma 5.2. *Besides the aforesaid suppositions let U be an isometry. If the closed set $Z \subset S_{n-1}$ has the property (U), then there exists an open set Y such that $Z \subset Y$ and $\overline{Y}$ has the property (U).*

Proof. In the contrary case let $Y_i = \{x \in S_{n-1} | \operatorname{dist}(x, Z) < 1/i\}$ where $\operatorname{dist}(x, Z) = \inf_{z \in Z} \varrho(x, z)$. Then $U(Y_i) = Y_i$ for $i = 1, 2, \ldots$ Moreover for every i there exist points $U^{\nu_i}(x_i)$, $U^{\mu_i}(x_i)$ $(0 \leqslant \nu_i < \mu_i \leqslant p-1)$ and a connected set P_i such that $U^{\nu_i}(x_i), U^{\mu_i}(x_i) \in P_i \subset Y$. Using the Bolzano-Weierstrass theorem (usual and generalized) we can suppose that $\nu^i = \nu$, $\mu^i = \mu$ $(\nu \neq \mu)$, $\lim_{i\to\infty} U^\nu(x_i) = U^\nu(x)$, $\lim_{i\to\infty} U^\mu(x_i) = U^\mu(x)$, $\lim_{i\to\infty} P_i = P$. It is easy to see that P is connected and $U^\nu(x) \in P$, $U^\mu(x) \in P$. On the other hand, $U^\nu(x), U^\mu(x) \in Z$ contrary to the supposition.

Definition 5.3. For the closed $Z \subset S_{n-1}$ possessing the property (U) the set whose existence is asserted by lemma 5.2 will be denoted by $[Z]^*$.

We shall use the following theorem due to Krasnosjelski [6]:

Theorem of Krasnosjelski. *Let U denote a continuous mapping of the sphere S_{n-1} into itself such that $U, U^2, \ldots, U^{p-1}$ have no fixed points but $U^p = \text{identity}$. Let the family of closed sets $F_1, F_2, \ldots, F_r$ cover S_{n-1} and let each set F_l $(l = 1, 2, \ldots, r)$ possess the property (U). Then $r \geqslant n$.*

In the special case $p = 2$, $U = \text{identity}$ we obtain the well-known theorem of K. Borsuk [3].

Definition 5.4. Let n be even. The isometry $U_\varphi\colon E_n \to E_n$ given by the orthogonal matrix

$$\left\| \begin{array}{cc|c} \cos\varphi, & -\sin\varphi & \\ \sin\varphi, & \cos\varphi & 0 \\ \hline & & \ddots \\ & 0 & \begin{array}{cc} \cos\varphi, & -\sin\varphi \\ \sin\varphi, & \cos\varphi \end{array} \end{array} \right\|$$

is said to be a *paratactic rotation* (see [12], p. 91, 92). In [12] it is proved that every plane determined by the vectors x, $U_\varphi(x)$ $(\varphi \neq 0)$ is mapped by U_φ onto itself. In this manner the mapping U_φ $(\varphi \neq 0)$ considered on the sphere S_{n-1} divides it into the family of disjoint great circles.

Let n be even and let $\varphi = \frac{2}{3}\pi$. The paratactic rotation $U_{2\pi/3}$ will be denoted simply by U. Then U and U^2 have no fixed points and $U^3 = \text{identity}$. We shall write $U(x) = x'$, $U^2(x) = x''$. Let the continuous mapping f be determined by the upper semicontinuous decomposition consisting of all triads (x, x', x'').

Theorem 2. *If the mapping f defined above is a superposition of z simple mappings, then $z \geqslant n+1$.*

Proof. Let $f_1, f_2, \ldots, f_z$ denote those simple mappings. By lemma 3.2 there exists a sequence $\mathfrak{W}^l$ $(l = 1, 2, \ldots, z)$ of upper semicontinuous decompositions of S_{n-1} satisfying conditions 1 and 2 of that lemma.

We shall define the sequence of sets $G_l \subset S_{n-1}$ $(l = 1, 2, \ldots, z)$ as follows:

$x \in G_l$ if and only if there exist $m < l$ and $W \in \mathfrak{W}^m$ such that $\{x'\} \cup \{x''\} = W$ and l is the least of numbers m for which if $x \in W \in \mathfrak{W}^m$ then $\overline{\overline{W}} = 3$.

Roughly speaking G_l consists of those points x for which x' and x'' are matched by a mapping f_m $(m < l)$ while f_l subjoins x to the matched (but still different from x) pair $x' = x''$.

Let $H_l = G_l \cup U(G_l) \cup U^2(G_l)$ $(l = 1, 2, \ldots, z)$. The sets H_l defined above are subject to the following conditions:

1. $H_1 = 0$,
2. H_l possesses the property (U) for $l = 1, 2, \ldots, z$,
3. $\bigcup_{l=1}^{m} H_l$ $(1 \leqslant m \leqslant z)$ is closed in S_{n-1},
4. $\bigcup_{l=1}^{z} H_l = S_{n-1}$.

Property 1 is immediate. To prove 2 and 3 let us observe that if $x_i \in G_l$ $(1 \leqslant l \leqslant z;\ i = 1, 2, \ldots)$ and $\lim_{i\to\infty} x_i = x_0$, then $x_0 \in G_q$ where $1 \leqslant q \leqslant l$. Hence follows property 3. We shall state that for each l $(1 \leqslant l \leqslant z)$ the sets G_l, $U(G_l)$ and $U^2(G_l)$ have disjoint closures. Indeed, if $x_i \in G_l$ for $i = 1, 2, \ldots$, $\lim_{i\to\infty} x_i = x_0 \in U(G_l)$, then also $x_0 \in G_q$ for certain $1 \leqslant q \leqslant l$. Using the definition of the decomposition $\mathfrak{W}^q$ we infer that $q = l$ and $U(G_l) \cap G_l \neq 0$ contrary to the definition of G_l. Hence we immediately obtain property 2. Property 4 is obvious.

We shall now define a sequence F_l $(l = 1, 2, \ldots, z)$ of sets on S_{n-1} satisfying the following conditions:

1. F_l is closed in S_{n-1} $(l = 1, 2, \ldots, z)$,
2. F_l possesses the property (U) for $l = 1, 2, \ldots, z$,
3. $\bigcup_{l=1}^{m} H_l \subset \operatorname{Int}(\bigcup_{l=1}^{m} F_l)$ $(1 \leqslant m \leqslant z)$.

It will be defined by induction. We put $F_1 = H_1 = 0$. Of course properties 1-3 are satisfied. Let us suppose that the sets $F_1, F_2, \ldots, F_m$ $(0 \leqslant m \leqslant z)$ on S_{n-1} satisfy properties 1, 2 and 3. Let us consider the set $\Phi = \overline{H_{m+1} - \bigcup_{l=1}^{m} F_l}$. By the closeness of $\bigcup_{l=1}^{m} H_l$ we have $\overline{H_{m+1}} \subset \bigcup_{l=1}^{m+1} H_l$. Using the set-theoretical rule: $\overline{A-B} \subset \bar{A} - \operatorname{Int}(B)$ we infer that

$$\Phi = \overline{H_{m+1} - \bigcup_{l=1}^{m} F_l} \subset \overline{H_{m+1}} - \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \subset \bigcup_{l=1}^{m+1} H_l - \operatorname{Int}(\bigcup_{l=1}^{m} F_l)\,. \tag{1}$$

Since by assumption

$$\bigcup_{l=1}^{m} H_l \subset \operatorname{Int}(\bigcup_{l=1}^{m} F_l)\,, \tag{2}$$

we have

$$\bigcup_{l=1}^{m+1} H_l - \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \subset \bigcup_{l=1}^{m+1} H_l - \bigcup_{l=1}^{m} H_l \subset H_{m+1}\,. \tag{3}$$

From (1) and (3) we obtain $\Phi \subset H_{m+1}$. In this manner we have concluded that Φ is contained in a set possessing property (U). Since $H_{m+1}, F_1, F_2, \ldots, F_m$ satisfy the first condition of property (U), we have $U(\Phi) = \Phi$. Hence Φ also possesses property (U). Now let $F_{m+1} = \overline{[\Phi]^*}$. By its definition F_{m+1} is closed and possesses property (U). In order to prove that $\bigcup_{l=1}^{m+1} H_l \subset \operatorname{Int}(\bigcup_{l=1}^{m+1} F_l)$ let us observe that

$$\Phi \subset \operatorname{Int}(F_{m+1})\,. \tag{4}$$

Using the set-theoretical rule: $A - \operatorname{Int}(B) \subset \overline{A-B}$ and the definition of Φ we have

$$H_{m+1} - \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \subset \Phi\,. \tag{5}$$

Combining (4) with (5) we obtain

$$H_{m+1} - \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \subset \operatorname{Int}(F_{m+1})\,. \tag{6}$$

Adding inclusions (2) and (6) we have

$$\bigcup_{l=1}^{m} H_l \cup H_{m+1} - \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \subset \operatorname{Int}(\bigcup_{l=1}^{m} F_l) \cup \operatorname{Int}(F_{m+1}) \subset \operatorname{Int}(\bigcup_{l=1}^{m+1} F_l)\,. \tag{7}$$

In view of (2) we obtain from (7) $\bigcup_{l=1}^{m+1} H_l \subset \operatorname{Int}(\bigcup_{l=1}^{m+1} F_l)$, which completes the construction of the sets F_l $(l = 1, 2, \ldots, z)$. By property 3 of the sets F_l and by property 4 of the sets H_l we infer that $S_{n-1} = \bigcup_{l=1}^{z} F_l$. Using the theorem of Krasnosjelski we obtain $z \geqslant n$. Since $F_1 = 0$, we have $z \geqslant n+1$. Thus the proof of the theorem is finished.

Definition 5.5. The *Hilbert space* $\mathfrak{H}$ is a family of real sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{i=1}^{m} x_i^2 < \infty$ with distance $\varrho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$.

Example 1. Let $p_\nu = (1/2^\nu, 0, 0, \ldots)$ for $\nu = 1, 2, \ldots$ Let $Z_\nu = \{x = (x_1, x_2, \ldots) \in \mathfrak{H} \mid x_i = 0 \text{ for } i > 2\nu;\ \varrho(x, p_\nu) = 1/2^{\nu+2}\}$. It can easily be seen that Z_ν is homeomorphic with the sphere $S_{2\nu-1}$. Let f_ν denote the continuous mapping defined on $S_{2\nu-1}$ considered in theorem 2. Let $Z = \bigcup_{\nu=1}^{\infty} Z_\nu \cup p_0$ where $p_0 = (0, 0, \ldots)$. In this compact set we define a continuous mapping f as follows: $f|Z_\nu = f_\nu$ $(\nu = 1, 2, \ldots)$; $f(p_0) = p_0$.

If f were a superposition of z simple mappings, then for $\nu = \frac{1}{2}(z-1)$ the mapping f would be a superposition od z simple mappings where $z < 2\nu+1$, contrary to theorem 2.

Definition 5.6. The *Hilbert elipsoid* $\mathfrak{E}$ is a subset of the space $\mathfrak{H}$ consisting of those points $x = (x_1, x_2, \ldots)$ for which $\sum_{i=1}^{\infty} 2^{i-1}(x_{2i-1}^2 + x_{2i}^2) \leqslant 1$.

Example 2. Let U denote the isometry of $\mathfrak{H}$ onto itself given by the infinite matrix

$$\left\| \begin{array}{cc|cc|c} \cos\varphi, & -\sin\varphi & & & \\ \sin\varphi, & \cos\varphi & & & 0 \\ \hline & & \cos\varphi, & -\sin\varphi & \\ & & \sin\varphi, & \cos\varphi & \\ & 0 & & & \ddots \end{array} \right\| \quad \text{where } \varphi = \tfrac{2}{3}\pi\,.$$

It can easily be proved that $U(\mathfrak{E}) = \mathfrak{E}$ and $U^3 = $ identity. We define in $\mathfrak{E}$ a decomposition consisting of all triads $(x, U(x), U^2(x))$ and the point $(0, 0, \ldots)$. In the finite dimensional case we have defined such a decomposition only on the surface of the sphere but here it is not compact. Let us take a convergent sequence $(x_i, U(x_i), U^2(x_i))$ of sets of our decomposition. Since $\mathfrak{E}$ is compact, we can assume that $\lim_{i\to\infty} x_i = x^0$, $\lim_{i\to\infty} U(x_i) = x^1$, $\lim_{i\to\infty} U^2(x_i) = x^2$. By the continuity of U we obtain $x^1 = U(x^0)$, $x^2 = U^2(x^0) = U(x^1)$, which proves that our decomposition is upper semi-continuous. The mapping f determined by it is not a finite superposition of simple mappings because crossing $\mathfrak{E}$ with the hyperplane of sufficiently large dimension (and making an affine mapping) we should obtain a contradiction of theorem 2.

References

[1] P. Alexandroff, *Über stetige Abbildungen kompakter Räume*, Proc. Acad. Amsterdam 28 (1925), p. 997.

[2] — *Über stetige Abbildungen kompakter Räume*, Math. Ann. 96 (1926), p. 555.

[3] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), p. 177-190.

[4] K. Borsuk and R. Molski, *On a class of continuous mappings*, Fund. Math. 45 (1957), p. 84-98.

[5] W. Hurewicz, *Über stetige Bilder von Punktmengen*, Proc. Acad. Amsterdam 30 (1927), p. 164.

[6] М. А. Красносельский, *О специальных покрытиях конечномерной сферы*, ДАН 103, No 6 (1955) (in Russian).

[7] C. Kuratowski, *Topologie I*, Warszawa-Wrocław 1948.

[8] — *Topologie II*, Warszawa 1952.

[9] K. Kuratowski and A. Mostowski, *Teoria mnogości*, Warszawa-Wrocław 1952.

[10] K. Menger, *Über umfassendste n-dimensionale Mengen*, Proc. Acad. Amsterdam 29 (1926), p. 1125.

[11] G. Nöbeling, *Über eine n-dimensionale Universalmenge im R_{2n+1}*, Math. Ann. 104 (1930).

[12] Б. А. Розенфельд, *Неевклидовы геометрии*, Москва (in Russian).

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