On completion of proximity spaces by local clusters *

by

S. Leader (Rutgers)

1. Introduction. In [4] the concept of a "cluster" of subsets from a proximity space X was introduced and used to construct the compactification $\bar{X}$ of X by identifying each point x in $\bar{X}$ with the cluster of all subsets of X which are close to x. Viewing the completion X^* of X as a subspace of $\bar{X}$, the present paper characterizes those clusters, the "local" clusters, from X which are determined by points in X^*. These clusters can be used to construct a completion theory for proximity spaces along the same lines as the compactification theory in [4].

The key concept in any completion theory for proximity spaces is that of "small" sets, since X^* consists of just those points in $\bar{X}$ which are close to small subsets of X. The concept of small sets can be introduced through various devices: uniform structures, uniform coverings, or pseudo-metrics. Smirnov uses the second device in [11], [12], [13] and [14]. We shall use the third device here making use of the ideas and results of [4].

In the last two sections of the paper two conjectures are posed for consideration by the interested reader.

2. Gauges. A *gauge* ϱ on a proximity space X is a real-valued function $\varrho(x, y)$ on $X \times X$ satisfying the following two conditions:

(2.1) $\varrho(x, y) \leqslant \varrho(x, z) + \varrho(y, z)$ for all x, y, z in X.

(2.2) Given A close to B in X and $\varepsilon > 0$, there exists a in A and b in B such that $\varrho(a, b) < \varepsilon$.

We define $\varrho(A, B)$ to be the infimum of $\varrho(a, b)$ for all a in A and b in B. We define $\varrho[A]$, the ϱ-diameter of A, to be the supremum of $\varrho(x, y)$ for all x and y in A.

That $\varrho(y, y) \leqslant 0$ follows from (2.2) for all y in X. The reversed inequality follows from (2.1) if we set $z = y$. So $\varrho(y, y) = 0$. Thus, setting $z = x$ in (2.1), we find $\varrho(x, y) \leqslant \varrho(y, x)$ for all x and y in X. So $\varrho(x, y) = \varrho(y, x)$. Finally, setting $y = x$ in (2.1) gives $0 \leqslant \varrho(x, z)$.

* Research supported by a grant from the Research Council of Rutgers — The State University.

A proximity space is *discrete* if disjoint subsets are remote. A *pseudo-metric* is just a gauge on a discrete space.

An immediate result of (2.2) and the Urysohn lemma (Theorem 6 of [4]) is the following.

COROLLARY 1. *A is close to B in X if and only if $\varrho(A, B) = 0$ for every gauge ϱ on X.*

A proximity space X is *metrizable* if there exists a gauge ϱ on X such that $\varrho(A, B) > 0$ for every pair of remote subsets A and B. Any such gauge is called a *metric* on X.

The following properties (2.3)-(2.6) of any gauge ϱ are easy consequences of (2.1) and (2.2) and will be used in proofs throughout this paper.

(2.3) $\varrho(A, B) \leqslant \varrho(A, C) + \varrho[C] + \varrho(B, C)$ for all subsets A, B, C of X with C non-void.

(2.4) $\varrho(P, Q) \leqslant \varrho[R] + \varrho(R, S) + \varrho[S]$ if $R \supseteq P \supset \emptyset$ and $S \supseteq Q \supset \emptyset$.

(2.5) $\varrho[A \cup B] \leqslant \varrho[A] + \varrho[B]$ if A is close to B.

(2.6) $\varrho[A] = \varrho[\bar{A}]$ for all A, where $\bar{A}$ is the closure of A in X.

A gauge σ is *totally bounded* on X if for arbitrary positive ε there exists a finite subset E of X such that $\sigma(E, x) < \varepsilon$ for every x in X. Equivalently, σ is totally bounded on X if for arbitrary positive ε X can be covered by finitely many subsets of σ-diameter less than ε.

LEMMA 1. *If ϱ and σ are gauges on a proximity space X and σ is totally bounded on X, then $\varrho + \sigma$ is a gauge on X.*

Proof. The triangular inequality (2.1) for $\varrho + \sigma$ results from addition of the corresponding inequalities for ϱ and σ.

Let A be close to B in X and let ε be any positive number. To prove (2.2) for $\varrho + \sigma$ we must find a in A and b in B such that

$$\varrho(a, b) + \sigma(a, b) < \varepsilon . \tag{2.7}$$

Since σ is totally bounded, there exists a finite covering $\{S_1, S_2, \ldots, S_n\}$ of X with $\sigma[S_k] < \frac{1}{3}\varepsilon$ for $k = 1, \ldots, n$. Since A is close to B, there exist j and k such that $A \cap S_j$ is close to $B \cap S_k$. By (2.2), we can choose a in $A \cap S_j$ and b in $B \cap S_k$ with

$$\varrho(a, b) < \tfrac{1}{3}\varepsilon . \tag{2.8}$$

By (2.5),

$$\sigma[S_j \cup S_k] \leqslant \sigma[S_j] + \sigma[S_k] < \tfrac{2}{3}\varepsilon .$$

Hence

$$\sigma(a, b) < \tfrac{2}{3}\varepsilon . \tag{2.9}$$

Adding (2.8) and (2.9) gives (2.7).

LEMMA 2. *Let X be a non-empty subspace of a proximity space X'. Let a gauge ϱ on X have an extension to a gauge ϱ' on X'. Then ϱ' is unique if and only if X is dense in X'.*

Proof. Let ϱ' be an extension of ϱ. If there exists a point z in X' with z remote from X we can construct an extension τ of ϱ which is distinct from ϱ' as follows. By the Urysohn lemma there exists a proximity mapping f of X' into the closed unit interval $[0, 1]$ with $fX = 0$ and $fz = 1$. Define a gauge σ on X' by letting $\sigma(x, y) = |fx - fy|$. Let $\tau = \varrho' + \sigma$. Since σ is totally bounded, τ is a gauge on X' by Lemma 1. Moreover, $\sigma(x, y) = 0$ for all x and y in X. So τ agrees with ϱ on X. But for x in X we have $\tau(x, z) = \varrho'(x, z) + 1$. So τ is distinct from ϱ' on X'. Thus, if ϱ' is unique, X must be dense in X'.

To prove the converse let X be dense in X'. Let σ and τ be any extensions of ϱ to gauges on X'. We contend $\sigma = \tau$. It suffices to prove $\sigma \leqslant \tau$ since interchanging σ and τ throughout the proof will yield the reversed inequality.

Consider any points x and y in X'. For an arbitrary positive ε let A be the union of all points a in X such that $\sigma(a, x) < \varepsilon$, and B the union of all b in X such that $\sigma(b, y) < \varepsilon$. Then $\varrho[A] \leqslant 2\varepsilon$ and $\varrho[B] \leqslant 2\varepsilon$. Moreover since $\varrho(X - A, x) \geqslant \varepsilon$, $X - A$ is remote from x, by Corollary 1. A must therefore be close to x since X is dense in X'. Similarly B must be close to y. By repeated use of (2.3) we have, since σ and τ are extensions of ϱ,

$$\sigma(x, y) \leqslant \sigma(x, A) + \varrho[A] + \tau(A, B) + \varrho[B] + \sigma(B, y) . \tag{2.10}$$

Since the first and last terms on the right side of (2.10) vanish *via* Corollary 1, we have

$$\sigma(x, y) \leqslant 4\varepsilon + \tau(A, B) . \tag{2.11}$$

Applying (2.3) again,

$$\tau(A, B) \leqslant \tau(A, x) + \tau(x, y) + \tau(B, y) \leqslant \tau(x, y) . \tag{2.12}$$

From (2.11) and (2.12),

$$\sigma(x, y) \leqslant 4\varepsilon + \tau(x, y) . \tag{2.13}$$

Since ε is arbitrary, (2.13) implies $\sigma(x, y) \leqslant \tau(x, y)$.

We can now prove the fundamental theorem on extensions of gauges.

THEOREM 1. *Let X be a non-empty subspace of a proximity space X' and ϱ be a gauge on X. The following are then equivalent:*

(1) *ϱ has a unique extension to a gauge on X'.*

(2) *Every point in X' is close to subsets of X of arbitrarily small ϱ-diameter.*

Proof. Given (2), define $\varrho'(x,y)$ for x and y in X' to be the supremum of $\varrho(A,B)$ for all subsets A and B of X with A close to x and B close to y. Consider any y in X'. By (2) there exists for arbitrary positive ε a subset A of X close to y with $\varrho[A]<\varepsilon$. Choose a in A and let R be the union of all x in X such that $\varrho(a,x)<2\varepsilon$. Since R contains A, R is close to y. Since $\varrho(X-R,A)\geqslant\varepsilon$, $X-R$ is remote from A, hence from y. Thus every subset of X which is close to y has its intersection with R close to y. Therefore, in the definition of $\varrho'(x,y)$ we can impose the additional restriction that $\varrho[A]<\varepsilon$ and $\varrho[B]<\varepsilon$.

Given x and y in X' choose subsets R and S of X having finite ϱ-diameter such that R is close to x, $X-R$ is remote from x, S is close to y, and $X-S$ is remote from y. Then, if A and B are subsets of X which are close to x and y respectively, (2.4) implies, upon setting $P=A\cap R$ and $Q=B\cap S$, that $\varrho'(x,y)$ is finite.

Let x and y be points in X and A and B be subsets of X such that A is close to x and B is close to y. By Corollary 1, $\varrho(A,x)=\varrho(B,y)=0$. Thus by (2.3), $\varrho(A,B)\leqslant\varrho(A,x)+\varrho(x,y)+\varrho(y,B)\leqslant\varrho(x,y)$. Therefore $\varrho'(x,y)\leqslant\varrho(x,y)$. Since the reverse of the latter inequality is trivial, equality holds. Hence ϱ' agrees with ϱ on X.

Given x,y,z in X', let A and B be subsets of X with A close to x and B close to y. For arbitrary positive ε there exists, by (2), a subset C of X close to z with $\varrho[C]<\varepsilon$. Thus (2.3) implies $\varrho(A,B)\leqslant\varrho(A,C)+{}$ ${}+\varepsilon+\varrho(B,C)$. Therefore $\varrho'(x,y)\leqslant\varrho'(x,z)+\varepsilon+\varrho'(y,z)$. Since ε is arbitrary, ϱ' satisfies (2.1).

Let P and Q be non-empty subsets of X'. Suppose that there exists a positive ε such that $\varrho'(p,q)>3\varepsilon$ for every p in P and q in Q. We must show that P is remote from Q. Let A be the union of all subsets E of X such that $\bar{E}$ intersects P and $\varrho[E]<\varepsilon$. Similarly, let B be the union of all subsets E of X such that $\bar{E}$ intersects Q and $\varrho[E]<\varepsilon$. By (2), $\bar{A}$ contains P and $\bar{B}$ contains Q. To prove P is remote from Q we shall prove A is remote from B by showing that $\varrho(A,B)\geqslant\varepsilon$.

Given any point a in A, there exists a subset R of X such that R contains a, $\varrho[R]<\varepsilon$, and R is close to some point p in P. Let E be any subset of X close to p. Then E is close to R, so $\varrho(E,R)=0$. Thus by (2.1), $\varrho(E,a)\leqslant\varrho(E,R)+\varrho[R]+\varrho(R,a)\leqslant\varrho[R]<\varepsilon$. Hence $\varrho'(p,a)\leqslant\varepsilon$. Similarly, given any b in B, there exists q in Q such that $\varrho'(q,b)\leqslant\varepsilon$. Since ϱ' satisfies (2.1), $3\varepsilon<\varrho'(p,q)\leqslant\varrho'(p,a)+\varrho'(q,b)+\varrho'(a,b)$. Hence $3\varepsilon<2\varepsilon+\varrho(a,b)$. That is, $\varepsilon<\varrho(a,b)$ for all a in A and all b in B. We have thus shown that ϱ' satisfies (2.2). So ϱ' is a gauge on X'.

The uniqueness of ϱ' follows from (2) *via* Lemma 2, completing the proof that (2) implies (1).

To prove (1) implies (2) let ϱ' be the unique extension of ϱ. Given y in X' and a positive ε, let B be the union of all x in X such that $\varrho'(x,y)<\frac{1}{2}\varepsilon$. Then $\varrho'(X-B,y)\geqslant\varepsilon$. So $X-B$ is remote from y. Since, by Lemma 2, X is dense in X', X is close to y. Hence B is close to y. Moreover $\varrho[B]\leqslant\varepsilon$.

LEMMA 3. *Let σ be a gauge on a proximity space X and $\{x_a,y_a\}$ a net* [3] *in $X\times X$. If either* (a) *σ is totally bounded, or* (b) *the net is a sequence, then the following conditions are equivalent*:

(1) *If Q and R are subsets of X such that (x_a,y_a) is in $Q\times R$ for arbitrarily large a, then $\sigma(Q,R)=0$.*

(2) $\lim\limits_{a\to\infty}\sigma(x_a,y_a)=0$.

Proof. That (2) always implies (1) is trivial.

To prove (1) implies (2) under either of the hypotheses (a) or (b), suppose (2) fails to hold. Then there exists a positive ε such that

$$\sigma(x_a,y_a)>3\varepsilon \tag{2.14}$$

for arbitrarily large a. Let J be the set of all a satisfying (2.14). If (a) holds we immediately have

Case i. There exists a subset S of X such that $\sigma[S]\leqslant2\varepsilon$ and either x_a or y_a is in S for arbitrarily large a in J.

Given Case i, we may assume without loss of generality that x_a is in S for arbitrarily large a in J. Let Q be the union of all x_a in S and R be the union of the corresponding y_a. By (2.14) and (2.3),

$$3\varepsilon<\sigma(x_a,y_a)\leqslant\sigma(x_a,S)+\sigma[S]+\sigma(S,y_a)\leqslant2\varepsilon+\sigma(S,y_a)\,. \tag{2.15}$$

From (2.15) we have $\varepsilon\leqslant\sigma(S,y_a)$ for all y_a in R. Since Q is contained in S,

$$\sigma(Q,R)\geqslant\varepsilon\,. \tag{2.16}$$

So (1) is violated.

If Case i does not hold we have

Case ii. Given any subset S of X with $\sigma[S]\leqslant2\varepsilon$, both x_a and y_a are eventually in $X-S$.

Given Case ii and condition (b), we can, by considering spheres S of σ-diameter 2ε centered about each x_a and each y_a, inductively choose a subsequence $\{x_{a_n},y_{a_n}\}$ with a_n in J so that

$$\sigma(x_{a_m},y_{a_n})\geqslant\varepsilon \tag{2.17}$$

for all m and n. Taking Q to be the union of all x_{a_m} and R the union of all y_{a_n}, (2.17) gives (2.16) which violates (1).

THEOREM 2. *Let ϱ and σ be gauges on a proximity space X. The following conditions are equivalent*:

(1) $\varrho(Q, R) = 0$ *implies* $\sigma(Q, R) = 0$.

(2) *For every positive ε there exists a positive δ such that* $\varrho(x, y) < \delta$ *implies* $\sigma(x, y) < \varepsilon$.

(3) $\lim\limits_{\varrho[E]\to 0} \sigma[E] = 0$.

Proof. We shall prove only that (1) implies (2), since the other implications are trivial. Consider any sequence $\{x_n, y_n\}$ in $X \times X$ such that $\lim\limits_{n\to\infty} \varrho(x_n, y_n) = 0$. To prove (2) we have the following chain of implications: (1) implies (1) of Lemma 3, which implies (2) of Lemma 3, which in turn implies (2).

If any, hence all, of the conditions of Theorem 2 hold, σ is said to be *uniformly continuous* with respect to ϱ.

3. Small sets. A class of subsets of a proximity space is said to have *small* members if for every gauge ϱ and every positive ε there exists a member of the class with ϱ-diameter less than ε. The following connection between proximity and small sets is an immediate consequence of Corollary 1.

COROLLARY 2. *A is close to B in X if and only if X has small subsets which intersect both A and B.*

We can now give several characterizations of proximity mappings.

THEOREM 3. *Let f map a proximity space X into a proximity space Y. Then the following are equivalent*:

(1) *f preserves proximity*: *fA is close to fB whenever A is close to B.*

(2) *f is uniformly continuous*: *Given an arbitrary gauge σ on Y and a positive ε, there exists a gauge ϱ on X and a positive δ such that*

$$\sigma(fx, fy) < \varepsilon \quad \textit{whenever} \quad \varrho(x, y) < \delta. \tag{3.1}$$

(3) *f carries small sets into small sets*: *If a class $\{A_a\}$ has small subsets A_a of X, then $\{fA_a\}$ has small subsets of Y.*

Proof. Given (1) and a gauge σ on Y, the equation $\varrho(x, y) = \sigma(fx, fy)$ defines a gauge ϱ on X. Taking $\delta = \varepsilon$, we obtain (2). So (1) implies (2).

Given (2), let Γ be a class of small subsets of X. For σ a gauge on Y and ε a positive number, choose ϱ and δ such that (3.1) holds. Choose a member A of Γ such that $\varrho[A] < \delta$. Then (3.1) implies $\sigma[fA] < \varepsilon$. So (2) implies (3).

Let A be close to B in X. Then, by Corollary 2, the class $\{C_a\}$ of all subsets of X which intersect both A and B has small members. Given (3), the class $\{fC_a\}$ has small members. Moreover, since C_a intersects both A and B, fC_a intersects both fA and fB. Hence, by Corollary 2, fA is close to fB. So (3) implies (1).

4. Local clusters and the completion of X. A cluster [4] from a proximity space X is defined to be *local* if it has small members. The class of all subsets of X close to a given point is clearly a local cluster, since a point has diameter zero for every gauge. A proximity space is defined to be *complete* if every local cluster has a point. (Note the analogy between this definition and Theorem 2 of [4]).

THEOREM 4. *A closed subspace of a complete proximity space is complete.*

Proof. Let X be a closed subspace of a complete proximity space Y. Let $\boldsymbol{c}$ be a local cluster from X. We must show $\boldsymbol{c}$ has a point. By Theorem 3 of [4], $\boldsymbol{c}$ is part of a cluster $\boldsymbol{d}$ from Y. Since $\boldsymbol{c}$ is local and since every gauge on Y is, by restriction, a gauge on X, $\boldsymbol{d}$ is local. Therefore, since Y is complete, there exists a point c in Y which belongs to $\boldsymbol{d}$. Since X belongs to $\boldsymbol{c}$, hence to $\boldsymbol{d}$, X is close to c in Y. But X is closed in Y. So X must contain c. Finally, since c is in X and belongs to $\boldsymbol{d}$, c belongs to $\boldsymbol{c}$.

Using our definition of completeness we next prove the completion theorem of Smirnov [11].

THEOREM 5. *Every proximity space X is a dense subspace of a minimal complete proximity space X^*. The completion X^* of X is the largest extension of X to which every gauge on X has a unique extension.*

Proof. By Theorem 4 of [4], X is a dense subspace of a compact Hausdorff space $\overline{X}$ in which two subsets are close if and only if their closures intersect. Let X^* be the subspace of $\overline{X}$ consisting of those points in $\overline{X}$ which are close to small subsets of X. Since X^* contains X and X is dense in $\overline{X}$, X is dense in X^*. Since, by Theorems 2 and 3 of [4], every cluster $\boldsymbol{c}$ from X is just the class of all subsets of X close to some point c in $\overline{X}$, every local cluster $\boldsymbol{c}$ from X is just the class of all subsets of X close to some point c in X^*. Thus, every complete extension of X must contain X^*. So X^* is minimal, hence unique. The second statement in Theorem 5 follows from Theorem 1 and the definition of X^*.

Because of the one-one correspondence between local clusters from X and points in X^*, X is complete if and only if $X = X^*$. Hence, to show X^* is complete we need only prove that $X^* = X^{**}$. But this follows immediately from the second statement in Theorem 5, since every gauge on X is uniquely extendable to X^* and from there to X^{**}.

5. Funnels, filters, and fundamental nets. A *funnel* $\boldsymbol{F}$ from a set X is a class of non-empty subsets of X directed by inclusion: Given A and B in $\boldsymbol{F}$, there exists some C in $\boldsymbol{F}$ contained in both A and B.

Every filter is a funnel. Every funnel $\boldsymbol{F}$ can be extended to a filter $\boldsymbol{F}'$ by taking $\boldsymbol{F}'$ to be the class of all subsets of X which contain any members of $\boldsymbol{F}$. A maximal funnel is clearly just an ultrafilter.

A funnel from a proximity space X is *local* if it has small members. The class of all neighborhoods of a point in X is clearly a local funnel.

We next investigate the connection between funnels and clusters.

THEOREM 6. *Every ultrafilter $\boldsymbol{F}$ from a proximity space X is a subclass of a unique cluster $\boldsymbol{c}$ from X.*

Proof. Let $\boldsymbol{c}$ be the class of all subsets of X which are close to every member of $\boldsymbol{F}$. Clearly $\boldsymbol{c}$ contains $\boldsymbol{F}$ and every cluster containing $\boldsymbol{F}$ is contained in $\boldsymbol{c}$. Hence we need only show $\boldsymbol{c}$ is a cluster.

(a) *If A and B belong to $\boldsymbol{c}$, then A is close to B.*

Since $\boldsymbol{F}$ is an ultrafilter, for every E in X either E or $X-E$ belongs to $\boldsymbol{F}$. So both A and B are close to either E or $X-E$. Therefore, for every E in X either A is close to E or B is close to $X-E$. By Axiom 4 of [4], A is close to B.

(b) *If A is close to every C in $\boldsymbol{c}$, then A belongs to $\boldsymbol{c}$.*

Property (b) is trivial since $\boldsymbol{F}$ is a subclass of $\boldsymbol{c}$.

(c) *If $A\cup B$ belongs to $\boldsymbol{c}$ and A does not belong to $\boldsymbol{c}$, then B belongs to $\boldsymbol{c}$.*

Given the hypothesis of (c), there exists a member E of $\boldsymbol{F}$ which is remote from A. Then every member F of $\boldsymbol{F}$ which is a subset of E is close to $A\cup B$, but remote from A. So F is close to B. Therefore F is close to B for every F belonging to $\boldsymbol{F}$. That is, B belongs to $\boldsymbol{c}$.

THEOREM 7. *Every funnel from a proximity space X is a subclass of some cluster from X.*

Proof. By the axiom of choice every funnel is a subclass of some maximal funnel, that is, of some ultrafilter. Apply Theorem 6.

THEOREM 8. *Every local funnel $\boldsymbol{F}$ from a proximity space X is a subclass of a unique cluster $\boldsymbol{c}$ from X. Moreover, $\boldsymbol{c}$ is local and consists of all subsets of X which are close to every member of $\boldsymbol{F}$.*

Proof. Let $\boldsymbol{c}$ consist of all subsets of X which are close to every member of the local funnel $\boldsymbol{F}$. Clearly, $\boldsymbol{c}$ contains $\boldsymbol{F}$. So $\boldsymbol{c}$ has small members. Moreover, every cluster which contains $\boldsymbol{F}$ must be contained in $\boldsymbol{c}$. We need only show that $\boldsymbol{c}$ is a cluster.

Given A and B in $\boldsymbol{c}$, let C be any member of $\boldsymbol{F}$ and ϱ be any gauge on X. Since $\varrho(A, C)=\varrho(B, C)=0$, (2.3) implies $\varrho(A, B)\leqslant\varrho[C]$. Since $\varrho[C]$ can be made arbitrarily small by a suitable choice of C, $\varrho(A, B)=0$. Hence, by Corollary 1, A is close to B. So (a) holds for $\boldsymbol{c}$.

Properties (b) and (c) follow exactly as in the proof of Theorem 6.

Let $\{x_\alpha\}$ be a net [3] in a proximity space X. Let E_α be the union of all x_β with $\beta\geqslant\alpha$. E_α will be called an *eventual range* of $\{x_\alpha\}$. Since for arbitrary indices α and β there exists γ such that γ is beyond both α and β, the class of all eventual ranges of a net is a funnel. If this funnel is local, we define the net to be *fundamental*. An obvious corollary of this definition follows.

COROLLARY 3. *A net $\{x_\alpha\}$ in a proximity space X is fundamental if and only if for every gauge ϱ on X, $\lim\limits_{\alpha\to\infty}\varrho[E_\alpha]=0$.*

Every convergent net is fundamental since E_α is eventually in any given neighborhood of the limit, and a point has small neighborhoods. Another trivial result is the following.

COROLLARY 4. *A net $\{x_\alpha\}$ converges to x if and only if for every gauge ϱ, $\lim\limits_{\alpha\to\infty}\varrho(x_\alpha, x)=0$.*

We now characterize completeness in terms of funnels and nets.

THEOREM 9. *The following conditions are equivalent:*

(1) *X is complete.*

(2) *Every local funnel of closed subsets of X has a non-empty intersection.*

(3) *Every fundamental net in X converges to some point in X.*

Proof. Given (1), let $\boldsymbol{F}$ be a local funnel of closed subsets of X. Let $\boldsymbol{c}$ be the local cluster which, according to Theorem 8, contains $\boldsymbol{F}$. By (1), there exists a point x which belongs to $\boldsymbol{c}$. Thus x is close to every member of $\boldsymbol{c}$, hence to every member of $\boldsymbol{F}$. Since the members of $\boldsymbol{F}$ are closed, x is contained in every member of $\boldsymbol{F}$. (Note that x is unique).

Given (2), let $\{x_\alpha\}$ be a fundamental net in X. Then the class $\{E_\alpha\}$ of all eventual ranges of this net is a local funnel. By (2.6), the closures also form a local funnel $\{\bar{E}_\alpha\}$. By (2), there exists some point x which is in every $\bar{E}_\alpha$. Given any neighborhood S of x, there exists a gauge ϱ such that $\varrho(x, X-S)>0$. Since $\{\bar{E}_\alpha\}$ is a local funnel, $\varrho[\bar{E}_\alpha]<\varrho(x, X-S)$ eventually. So E_α is eventually in S. Hence, x_α is eventually in S. That is, x_α converges to x.

Given (3), let x be any point in X^*. To prove (1) we need only show that x is in X. Let $\{S_\alpha\}$ be the class of all neighborhoods of x in X^*. Define $\alpha\geqslant\beta$ to be the *funnel direction* $S_\alpha\subseteq S_\beta$. Using the axiom of choice, choose x_α in $X\cap S_\alpha$. Because of Theorem 1, $\{x_\alpha\}$ is a fundamental net in X. Moreover, x_α converges to x. Since limits are unique in a Hausdorff space, (3) implies that x is in X. (Note that if the word "local" be deleted, then (2) becomes a familiar criterion for compactness.)

6. Local proximity and precontinuity.

THEOREM 10. *Let A and B be subsets of a proximity space X. The following conditions are then equivalent*:

(1) *There exists a local cluster* $\mathbf{c}$ *from X to which both A and B belong.*

(2) *The closures of A and B in X^* intersect.*

(3) *There exists a local funnel* $\mathbf{F}$ *from X such that every member of* $\mathbf{F}$ *intersects both A and B.*

Proof. Given (1), let c be the point in X^* corresponding to a local cluster to which both A and B belong. Then c is close to both A and B in X^*. Hence (2).

Given (2), let c be close to both A and B in X^*. Take $\mathbf{F}$ to be the class of all intersections with X of neighborhoods of c in X^*. Clearly $\mathbf{F}$ is a local funnel and satisfies (3).

Given (3), let $\mathbf{c}$ be the local cluster which, according to Theorem 8, contains $\mathbf{F}$. Then (1) follows from Theorem 8.

We define A to be *locally close* to B in X if any, hence all, of the conditions in Theorem 10 hold. Local proximity is a proximity relation whenever X^* is normal. A mapping f of a proximity space X into a proximity space Y is said to *preserve local proximity* if fA is locally close to fB in Y whenever A is locally close to B in X.

THEOREM 11. *Let f be a mapping of a proximity space X into a proximity space Y. The following conditions are then equivalent*:

(1) *f carries local clusters into local clusters*: *Given a local cluster* $\mathbf{c}$ *from X, the class* $\mathbf{d}$ *of all subsets of Y which are close to fA for every A in* $\mathbf{c}$ *is a local cluster from Y.*

(2) *f has an extension (necessarily unique) to a continuous mapping of X^* into Y^*.*

(3) *f carries local funnels into local funnels*: *Given a local funnel $\{F_a\}$ from X, the class $\{fF_a\}$ is a local funnel from Y.*

(4) *f carries fundamental nets into fundamental nets*: *Given a fundamental net $\{x_a\}$ in X, then $\{fx_a\}$ is a fundamental net in Y.*

Proof. Given (1) and any local cluster $\mathbf{c}$ from X, let $f\mathbf{c} = \mathbf{d}$. This agrees with f on X under the imbedding of X in X^* induced by identifying points with the clusters to which they belong. Let P be a subset of X^* and c a point in X^* such that fc is remote from fP in Y^*. To show f is continuous on X^* we must prove that c is remote from P.

Choose S in Y^* such that S is remote from fc and Y^*-S is remote from fP. Let A be the intersection of X with $f^{-1}S$. By (1), A is remote from c and $X-A$ is remote from every point in P. The latter statement implies that A is close to every point in P. That is, the closure of A in X^* contains P. Thus, since A is remote from c, P must be remote from c. So (1) implies (2).

Given (2), let $\{F_a\}$ be a local funnel from X. By (2) of Theorem 9, there exists a point x in X^* close to every F_a. By (2), every neighborhood S of fx in Y^* yields a neighborhood $f^{-1}S$ of x in X^*. Then, since F_a is eventually contained in $f^{-1}S$, fF_a is eventually contained in S. Since every point of Y^* has small neighborhoods relative to the gauges on Y, $\{fF_a\}$ is a local funnel from Y. So (2) implies (3).

Given (3), let $\{x_a\}$ be a fundamental net in X. Let $\{E_a\}$ be the local funnel of all eventual ranges of this net. By (3), $\{fE_a\}$ is a local funnel from Y. But $\{fE_a\}$ is just the class of all eventual ranges of $\{fx_a\}$. Hence, $\{fx_a\}$ is a fundamental net in Y. So (3) implies (4).

Given (4), let $\mathbf{c}$ be any local cluster from X and let $\{S_a\}$ be the funnel of all neighborhoods of c in X^*. Using the axiom of choice, choose x_a in the intersection of X with S_a. With the funnel direction $\{x_a\}$ is a fundamental net, since c has small neighborhoods. By (4), $\{fx_a\}$ is a fundamental net in Y. By Theorem 9, fx_a converges to some d in Y^*.

Now d is independent of the choice of x in $X \cap S_a$. For, given any other net $\{y_a\}$ with y_a in $X \cap S_a$, let $z_{a,1} = x_a$ and $z_{a,2} = y_a$. With lexicographical ordering $\{z_{a,k}\}$ is a fundamental net. By (4) and Theorem 9, $fz_{a,k}$ converges to some point in Y^*. Since Y^* is Hausdorff and $\{fx_a\}$ is a subnet of $\{fz_{a,k}\}$, this point must be d. Since $\{fy_a\}$ is also a subnet, fy_a converges to d. Given any subset A of X belonging to $\mathbf{c}$, choose any net $\{a_a\}$ with a_a in $A \cap S_a$. Then fa_a converges to d. So fA belongs to $\mathbf{d}$. Thus, (4) implies (1).

A mapping satisfying any, hence all, of the conditions of Theorem 11 will be called *precontinuous*. The precontinuous mappings are just the "s-mappings" of Smirnov [13], [14].

The following corollary is a consequence of (3) of Theorem 3 and (3) of Theorem 11.

COROLLARY 5. *Every proximity mapping is precontinuous.*

The next corollary is a consequence of (1) of Theorem 10 and (1) of Theorem 11.

COROLLARY 6. *Every precontinuous mapping preserves local proximity.*

7. Precompact spaces.

THEOREM 12. *The following conditions on a proximity space X are equivalent*:

(1) *The completion X^* of X is compact.*

(2) *Every cluster from X is local.*

(3) *Every mapping f on X into the real numbers which preserves local proximity is bounded.*

(4) *Every gauge on X is totally bounded.*

(5) *Every funnel from X is contained in some local funnel.*

(6) *Every net in X has a subnet which is fundamental.*

Proof. The equivalence of (1) and (2) follows from the identification of $\bar{X}$ with the set of all clusters from X, and X^* with the set of all local clusters from X.

Given (2), hence (1), let A be close to B in X. Then by Theorem 1 of [4], A and B belong to some cluster. By (2), this cluster is local. So A is locally close to B. Thus, (2) implies the equivalence of proximity and local proximity in X. Therefore, any mapping on X which preserves local proximity preserves proximity. Hence, by Corollary 5, f is precontinuous. So, by (2) of Theorem 11, f has a continuous extension mapping X^* into the real numbers. Since X^* is compact, its image fX^* is compact, hence bounded. So (2) implies (3).

Given (3), suppose that there exists a gauge ϱ on X which is not totally bounded. Then for some positive ε we can inductively choose an infinite sequence $\{x_0, x_1, x_2, \ldots\}$ in X such that $\varrho(x_i, x_j) > 3\varepsilon$ for $i > j$. For x in X^* define

$$fx = \operatorname*{Max}_{k=0,1,\ldots} k\big(\varepsilon - \varrho(x_k, x)\big)\,. \tag{7.1}$$

Then f is continuous on X^*, hence preserves local proximity on X. But $fx_k = k\varepsilon$, making f unbounded on X. This would contradict (3). So (3) implies (4).

Given (4) and a funnel from X, imbed the funnel in a maximal funnel (ultrafilter) by using the axiom of choice. We contend that such a maximal funnel $\boldsymbol{F}$ is local. By (4), for an arbitrary gauge ϱ on X and any positive ε, there exists a finite covering $\{E_1, \ldots, E_n\}$ of X with $\varrho[E_k] < \varepsilon$ for $k = 1, \ldots, n$. Since $\boldsymbol{F}$ is maximal at least one E_k must belong to $\boldsymbol{F}$. So $\boldsymbol{F}$ has small members. Hence (4) implies (5).

Let $\{x_a\}$ be any net in X and $\{E_a\}$ the funnel of eventual ranges of this net. Given (5), this funnel is part of some local funnel $\boldsymbol{F}$. Using the axiom of choice, choose for each member F_ν of $\boldsymbol{F}$ some x_{a_ν} in F_ν. This is possible because every F_ν intersects every E_a. With the funnel direction the net $\{x_{a_\nu}\}$ is a subnet of $\{x_a\}$. Since every F_ν contains an eventual range of this subnet and $\boldsymbol{F}$ is local, the subnet is fundamental. So (5) implies (6).

To prove (6) implies (1) we need only show that $\bar{X}$ is contained in X^*. Given x in $\bar{X}$, there exists a net in X converging to x, since X is dense in $\bar{X}$ by Theorem 4 of [4]. By (6), this net has a fundamental subnet. This subnet must also converge to x. By (3) of Theorem 9, x is in X^*. So (6) implies (1).

We call X *precompact* if any, hence all, of the conditions of Theorem 12 hold. (Smirnov [13] calls these spaces "totally bounded".) The following three corollaries are immediate consequences of Theorem 12.

COROLLARY 7. *Proximity and local proximity are equivalent in a precompact proximity space.*

COROLLARY 8. *Every mapping on a precompact space which preserves local proximity is a proximity mapping.*

COROLLARY 9. *Every cluster which has a precompact member is a local cluster.*

THEOREM 13. *Let X be a precompact proximity space and f be a mapping on X into a proximity space Y such that f preserves local proximity. Then fX is precompact.*

Proof. By Corollary 8, f is a proximity mapping. By Corollary 5, f is also precontinuous. Let $Y = fX$. Then f has a continuous extension which maps $\bar{X}$ onto $\bar{Y}$ and X^* into Y^*. But $\bar{X} = X^*$. So $\bar{Y} = f\bar{X} = fX^* \subseteq Y^* \subseteq \bar{Y}$. Hence $\bar{Y} = Y^*$. That is, Y is precompact.

LEMMA 4. *Let f be a mapping from a proximity space X into a proximity space Y which preserves local proximity. Then f carries local clusters into clusters. Moreover, f has a (necessarily unique) continuous extension $\bar{f}$ which maps X^* into $\bar{Y}$.*

Proof. The proof is just the proof of Theorem 5 of [4] with "proximity" in X replaced by "local proximity".

THEOREM 14. *Let X and Y be proximity spaces such that either of the following conditions holds:*

(1) *Y is precompact.*

(2) *Every local cluster from X has a precompact member.*

Then every mapping f on X into Y which preserves local proximity is precontinuous.

Proof. Given (1), the conclusion follows from Lemma 4.

Given (2) and a local cluster $\boldsymbol{c}$ from X, there exists a precompact set A which belongs to $\boldsymbol{c}$. By Lemma 4, $\boldsymbol{c}$ is mapped into a cluster $\boldsymbol{d}$. By Theorem 13, fA is precompact. Since fA belongs to $\boldsymbol{d}$, $\boldsymbol{d}$ is local, by Corollary 9. This gives (1) of Theorem 11, hence Theorem 14.

Since metrizable spaces and spaces with locally compact completions satisfy (2) in Theorem 14, we have two corollaries.

COROLLARY 10. *Every mapping which preserves local proximity on a metrizable proximity space is precontinuous.*

COROLLARY 11. *Every mapping which preserves local proximity on a proximity space with a locally compact completion is precontinuous.*

CONJECTURE 1. *There exist mappings which preserve local proximity, but are not precontinuous.*

8. Convergence of mappings. Let $\{f_a\}$ be a net of mappings on an abstract set X into a proximity space Y. f_a is said to *converge in proximity* to a mapping f if fA remote from R in Y implies f_aA is eventually remote from R. f_a is said to *converge uniformly* to f if, for every gauge σ on Y, $\sigma(f_ax, fx)$ converges to 0 uniformly for all x in X.

THEOREM 15. *Uniform convergence implies convergence in proximity.*

Proof. Let f_a converge uniformly to f. Let fA be remote from R in Y. Then there exists a gauge σ on Y such that $\sigma(fA, R) > 2\varepsilon$ for some positive ε. Thus for every a in A,

$$2\varepsilon < \sigma(fa, R) \leqslant \sigma(fa, f_a a) + \sigma(f_a a, R). \tag{8.1}$$

Since the first term on the right of (8.1) is eventually uniformly less than ε, $\varepsilon \leqslant \sigma(f_a a, R)$. That is, $\varepsilon \leqslant \sigma(f_a A, R)$ eventually. By Corollary 1, $f_a A$ is eventually remote from R. So f_a converges in proximity to f.

THEOREM 16. *Convergence in proximity is equivalent to uniform convergence under either of the following conditions:*

(1) *The image space Y is precompact.*

(2) *The net of mappings is a sequence.*

Proof. In view of Theorem 15 we need only prove that convergence in proximity implies uniform convergence, given either (1) or (2). To prove uniform convergence it suffices to prove that for every net $\{x_a\}$ in X corresponding to $\{f_a\}$ and gauge σ on Y,

$$\lim_{a\to\infty}\sigma(fx_a, f_a x_a) = 0. \tag{8.2}$$

Let Q and R be any subsets of Y such that $(fx_a, f_a x_a)$ is in $Q\times R$ for arbitrarily large a. We contend that Q is close to R. For, if Q were remote from R, $f_a f^{-1}Q$ would be remote from R eventually, since f_a converges in proximity to f. But $f_a f^{-1}Q$ intersects R for arbitrarily large a. Since Q is close to R, $\sigma(Q, R) = 0$. Since (1) and (2) give (a) and (b) of Lemma 3, respectively, (8.2) follows from (2) of Lemma 3.

CONJECTURE 2. *Convergence in proximity need not imply uniform convergence.*

THEOREM 17. *Let $\{f_a\}$ be a net of proximity mappings on a proximity space X into a proximity space Y. Let f_a converge in proximity to a mapping f. Then f is a proximity mapping.*

Proof. Let A be close to B in X. We must prove fA is close to fB in Y. Suppose that fA were remote from fB. Then there would exist a subset R of Y with fA remote from R and fB remote from $Y-R$. Since f_a converges in proximity to f, f_aA would be eventually remote from R and f_aB eventually remote from $Y-R$. Hence, f_aA and f_aB would be remote eventually, contradicting the hypothesis that f_a preserves proximity.

THEOREM 18. *Let $\{f_a\}$ be a net of continuous mappings on a topological space X into a proximity space Y. Let f_a converge in proximity to a mapping f. Then f is continuous.*

Proof. Apply the proof of Theorem 17, taking B to be a point x and using the fact that f_a preserves proximity between points and sets.

THEOREM 19. *Let $\{f_a\}$ be a net of precontinuous mappings on a proximity space X into a proximity space Y. Let f_a converge uniformly to a mapping f. Then f is precontinuous.*

Proof. Let $\boldsymbol{F}$ be a local funnel from X. By (3) of Theorem 11 we must show that f maps $\boldsymbol{F}$ onto a local funnel, given that each f_a does so.

Given a gauge σ on Y and a positive ε, we need only show there exists a member S of $\boldsymbol{F}$ with

$$\sigma[fS] < \varepsilon. \tag{8.3}$$

Since f_a converges uniformly to f, we can choose a such that

$$\sigma(f_a x, fx) < \tfrac{1}{3}\varepsilon \tag{8.4}$$

for all x in X. Since f_a is precontinuous, there exists a member S of F such that

$$\sigma[f_a S] < \tfrac{1}{3}\varepsilon. \tag{8.5}$$

By (2.1) we have for all x and y in S,

$$\sigma(fx, fy) \leqslant \sigma(fx, f_a x) + \sigma(f_a x, f_a y) + \sigma(f_a y, fy). \tag{8.6}$$

Since, by (8.4) and (8.5), each term on the right side of (8.6) is less than $\frac{1}{3}\varepsilon$, (8.6) gives (8.3).

References

[1] V. A. Efremovič, *The geometry of proximity I*, Mat. Sbornik N. S. 31 (73) (1952), p. 189-200.

[2] — and A. S. Švarc, *A new definition of uniform spaces. Metrization of proximity spaces*, Dok. Akad. Nauk SSSR (N. S.) 89 (1953), p. 393-396.

[3] J. L. Kelley, *Convergence in topology*, Duke Math. Jour. 17 (1950), p. 277-283.

[4] S. Leader, *On clusters in proximity spaces*, Fund. Math. 47 (1959), p. 205-213.

[5] S. Mrówka, *On complete proximity spaces*, Dok. Akad. Nauk SSSR (N. S.) 108 (1956), p. 587-590.

[6] — *On the notion of completeness in proximity spaces*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), p. 477-478.

[7] — *On the uniform convergences in proximity spaces*, Bull. Acad. Polon. Sci., Cl. III, 5 (1957), p. 225-257.

[8] N. S. Ramm and A. S. Švarc, *Geometry of proximity, uniform geometry and topology*, Mat. Sbornik N. S. 33 (75) (1953), p. 157-180.

[9] Yu. M. Smirnov, *On proximity spaces in the sense of V. A. Efremovič*, Dok. Akad. Nauk SSSR (N. S.) 84 (1952), p. 895-898.

[10] — *On proximity spaces*, Mat. Sbornik N. S. 31 (73) (1952), p. 543-574.

[11] — *On the completeness of proximity spaces*, Dok. Akad. Nauk SSSR (N. S.) 88 (1953), p. 761-764.

[12] — *On completeness of uniform spaces and proximity spaces*, Dok. Akad. Nauk SSSR (N. S.) 91 (1953), p. 1281-1284.

Reçu par la Rédaction le 30. 3. 1959

On superpositions of simple mappings

by

K. Sieklucki (Warszawa)

1. Introduction. K. Borsuk and R. Molski considered in [4] a class of continuous mappings called *simple mappings*. A continuous mapping f of a space X onto the space Y *is of order* $\leqslant k$ if for every point $y \in Y$ the set $f^{-1}(y)$ contains at most k points (cf. [8], p. 52). The mappings of order $\leqslant 2$ are said to be *simple mappings*. In [4] the authors raise the following question (p. 92, No 4): does there exist a continuous mapping of a finite order which is not a superposition of a finite number of simple mappings?

The purpose of this paper is to prove that every continuous mapping f of a finite order defined on the compact space X of a finite dimension is a superposition of a finite number of simple mappings. On the other hand, we shall construct a compact infinite dimensional space X and a continuous mapping of a finite order f defined on X which will not be a superposition of a finite number of simple mappings.

2. Auxiliary definitions and notations.

Definition 2.1. A collection of subsets of a space X constitutes a *decomposition* $\mathfrak{W}$ of X if the sets of $\mathfrak{W}$ are disjoint and non-empty, and if they fill up X. The decomposition $\mathfrak{W}$ is said to be *upper semi-continuous* if for every closed subset A of X the union of all sets of $\mathfrak{W}$ intersecting A is closed in X (see [8], p. 42).

P. Alexandroff ([1] and [2]; cf. also [8], p. 42) has proved the following theorem: *In order that a decomposition $\mathfrak{W}$ of a compact space X be upper semicontinuous, it is sufficient and necessary that there exist a space Y and a continuous mapping f of X onto Y such that the sets belonging to $\mathfrak{W}$ are the same as the sets $f^{-1}(y)$ where $y \in Y$.*

Let $\{A_i\}$ $(i = 1, 2, \ldots)$ denote a sequence of subsets of the space X and let $\underset{i\to\infty}{\mathrm{Lim}}\, A_i$ be defined as in [7], p. 241-245. We shall use the following important properties of the notion of this limit:

(i) The generalized Bolzano-Weierstrass theorem. *If the space is separable, then from every sequence of its subsets we can choose a convergent subsequence (may be to the empty set)* (see [7], p. 246).