Dedicated to
Professor L. Lusternik
on his 60-th birthday

On the disconnection of Banach spaces

by

A. Granas (Toruń)

1. Introduction. For arbitrary metric spaces X and Y, we denote by Y^X the set of all continuous mappings of X into Y, and by $X\times Y$ — the Cartesian product of X and Y. If a mapping $f\in Y^X$, then we shall write also $f\colon X\to Y$.

If $X_0\subset X$ and $f\in Y^X$ then $f|X_0$ will denote the *partial* mapping of f, i. e. the mapping f_0, defined in X_0 by the formula $f_0(x)=f(x)$; we shall say that f is an *extension* of f_0 over X and then we shall write $f_0\subset f$.

Two mappings $f,\ g\in Y^X$ are called *homotopic* (written $f\simeq g$) if there exists a mapping $h\in Y^{X\times I}$ (I denotes the closed interval $[0,1]$) such that for each $x\in X$

$$h(x,0)=f(x)\,,\qquad h(x,1)=g(x)\,.$$

If $f\in Y^X$ is homotopic to a constant mapping (i. e. a mapping onto a single-point set in Y) then we shall write $f\simeq 1$.

If the points x_1, x_2 belong to the same component of the space X, then we shall write: $x_1\sim x_2$ *in* X.

Let X be a closed and bounded subset of the n-dimensional Euclidean space R^n and let 0 denote the origin of R^n.

In 1931 K. Borsuk proved in the paper [1] the following theorem:

The set $R^n\setminus X$ is connected if and only if the functional space $(R^n\setminus\{0\})^X$ is connected or, which is the same, if any two mappings $f,g\in(R^n\setminus\{0\})^X$ are homotopic.

The same author also gave a criterion concerning the separation of the Euclidean space between two points (see, for instance, [4], p. 302):

$$(x_1\sim x_2\ \ in\ \ R^n\setminus X)\equiv(x-x_1)|X\sim(x-x_2)|X\ \ in\ \ (R^n\setminus\{0\})^X\,;$$

that is to say:

The set X does not separate the space R^n between two points $x_1,x_2\in R^n\setminus X$ if and only if the mappings $(x-x_1)|X$ and $(x-x_2)|X$ *are homotopic in* $(R^n\setminus\{0\})^X$.

In this paper we shall give an extension of Borsuk's theorems to the case of arbitrary Banach spaces (Theorems 2 and 3).[1]

In this case the space $(E^n\setminus\{0\})^X$ is replaced by the space $\mathfrak{C}(P_\infty^X)$ consisting of all non-vanishing compact fields on X, where X is a bounded closed subset of the Banach space and the homotopy of two mappings $f, g \in (E^n\setminus\{0\})^X$ is replaced by a homotopy of two elements of the space $\mathfrak{C}(P_\infty^X)$.

The proof of Theorems 2 and 3 is based on Theorem 1, which is Borsuk's Extension Homotopy Theorem[2] formulated for Banach spaces.

The invariance of the disconnection property of Banach spaces under a certain class of homeomorphisms is deduced directly from Theorem 3. The proof of this does not refer to the Leray-Schauder notion of the degree of a mapping [8]; it is, as a matter of fact, a consequence of the well-known Schauder Fixed Points Theorem.

2. Preliminaries. We shall use the following notation: E_∞ — infinite-dimensional Banach space, E_n — a subspace of E_∞ of dimension n, P_∞ — the space E_∞ without the origin 0, P_n — the space E_n without 0. If Z is a subset of E_∞, we denote the closure of Z by $\bar{Z}$ and the convex closure (i. e. the smallest convex closed set containing Z) by $\mathrm{conv}(Z)$. We shall denote by $V_\infty(x_0, \varrho)$ an open spherical region in the space E_∞ with centre x_0 and radius ϱ and by $S_\infty(x_0, \varrho)$ its boundary; if $x_0 \in E_n$, then we shall put

$$V_n(x_0, \varrho) = V_\infty(x_0, \varrho) \cap E_n, \quad S_{n-1}(x_0, \varrho) = S_\infty(x_0, \varrho) \cap E_n.$$

In the sequel we shall use the following lemma:

2.1. *Let X be a closed bounded separable convex subset of E_∞. Then X is a retract of E_∞, i. e. there exists a mapping $r: E_\infty \to X$ such that $r(x) = x$ for every $x \in X$.*

Proof. For $x \in E_\infty \setminus X$ and $y \in E_\infty$ the function

$$p(x, y) = \min\left\{2 - \frac{\|x-y\|}{\inf\limits_{z\in X}\|x-z\|}, 0\right\}$$

is continuous on the set $E_\infty\setminus X$ and we have $0 \leqslant p(x, y) \leqslant 2$. Hence if $\{y_k\}$ is a dense sequence of points in X, then the function

$$r(x) = \begin{cases} x, & x \in X, \\ \left(\sum_{k=1}^{\infty} 2^{-k} p(x, y_k)\right)^{-1} \left(\sum_{k=1}^{\infty} 2^{-k} p(x, y_k)\, y_k\right), & x \notin X, \end{cases}$$

is the required retraction of E_∞ onto X.

[1] These theorems were announced in [5].

[2] For Borsuk's Theorem, see [2] and [7], p. 86.

Let X be an arbitrary space. A mapping $F: X \to E_\infty$ is said to be *compact on* X if the image $F(X)$ is contained in some compact set.

Compact mappings will be denoted in the sequel by capital letters F, G, H.

A compact mapping $F: X \to E_\infty$ is said to be *finite-dimensional on* X if its values lie in some finite-dimensional subspace $E_n \subset E_\infty$ depending on F, i. e. $F: X \to E_n$.

The following theorem is due to J. Schauder and J. Leray [8]:

2.2. Approximation theorem. *Let $F: X \to E_\infty$ be a compact mapping on X. For every $\varepsilon > 0$, there exists a finite-dimensional mapping $F_\varepsilon: X \to E_n$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon \quad \textit{for each} \quad x \in X. \tag{1}$$

Proof. For a given $\varepsilon > 0$, we can find a finite subset $\{y_1, y_2, \ldots, y_k\}$ of E_∞ such that every point of the compact set $\overline{F(X)}$ is at a distance less than ε from at least one of the y_i. Let E_n be a finite-dimensional subspace of E_∞ which contains all the points y_i $(i = 1, 2, \ldots, k)$.

Let us put

$$F_\varepsilon(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) y_i}{\sum_{i=1}^{k} \lambda_i(x)} \quad \text{for} \quad x \in X, \tag{2}$$

where

$$\lambda_i(x) = \max\{0, \varepsilon - \|F(x) - y_i\|\} \quad \text{for} \quad x \in X \ (i = 1, 2, \ldots, k). \tag{3}$$

The mapping F_ε defined by (2) is finite-dimensional on X, $F_\varepsilon: X \to E_n$, and satisfies inequality (1); thus the proof is complete.

2.3. *Every compact mapping $F: X \to E_\infty$ can be represented in the form*

$$F(x) = \sum_{n=0}^{\infty} F_n(x), \tag{4}$$

where the mappings F_n are finite-dimensional on X $(n = 0, 1, \ldots)$ and

$$\|F_n(x)\| \leqslant \frac{1}{2^n} \quad \textit{for every} \quad x \in X \textit{ and } n = 1, 2, \ldots \tag{5}$$

Proof. This is a simple consequence of the Approximation Theorem 2.2.

In the sequel we shall use the following theorem, which is a very special case of the theorem of Dugundji concerning extensions of continuous transformations [3]:

2.4. Extension of compact mappings theorem. *Let X_0 be a closed subset of a metric space X. Then every compact mapping $F: X_0 \to E_\infty$ can be extended to a compact mapping $\bar{F}: X \to \mathrm{conv}(F(X_0))$.*

Proof. In the case when the mapping F is finite-dimensional our theorem is a simple consequence of lemma 2.1 and the well-known Tietze Extension Theorem ([7], p. 80). For the proof of our theorem in the general case let us consider the representation of F on X_0 given by formulas (4) and (5). Let $\overline{F}_n$ $(n = 0, 1, 2, ...)$ be an extension of the finite-dimensional mapping F_n from X_0 over X such that

$$\|\overline{F}_n(x)\| \leqslant \frac{1}{2^n} \quad \text{for} \quad x \in X \text{ and } n = 1, 2, \ldots$$

Denote by r a retraction of E_∞ on the set $\mathrm{conv}(F(X_0))$, which is obviously bounded and separable.

The mapping $\overline{F}$ defined on X by the formula

$$\overline{F}(x) = r\Big(\sum_{n=0}^{\infty} \overline{F}_n(x)\Big)$$

is the required extension of F from X_0 over X.

As a simple consequence of the Approximation Theorem 2.2 we shall prove the well-known Schauder Fixed Point Theorem [10], which will be used in the sequel:

2.5. *If X is a closed convex subset of E_∞ and F a compact mapping of X into itself, then F has a fixed point.*

Proof. By 2.1 for each $k = 1, 2, \ldots$ there exists a finite-dimensional mapping $F_{1/k}: X \to X \cap E_{n(k)}$ such that

$$\|F(x) - F_{1/k}(x)\| \leqslant \frac{1}{k} \quad \text{for each} \quad x \in X. \tag{6}$$

By the Brouwer Fixed Point Theorem ([4]) the mapping $F_{1/k}$ has a fixed point $x_k = F_{1/k}(x_k)$ and hence by (6) we have

$$\|F(x_k) - x_k\| \leqslant \frac{1}{k}. \tag{7}$$

Since F is a compact mapping, we can assume, without loss of generality, that there exists $\lim_{k=\infty} F(x_k) = x^*$. On account of (7) we have $\lim_{k=\infty} x_k = x^*$ and hence $\lim_{k=\infty} F(x_k) = F(x^*)$, i. e. $x^* = F(x^*)$, which completes the proof.

3. The space $\mathfrak{C}(E_\infty^X)$ of compact fields in E_∞. Now let X be a subset of the Banach space E_∞.

A mapping $f: X \to E_\infty$ is said to be a *compact vector field on* X if it can be represented in the form

$$f(x) = x - F(x), \tag{8}$$

where $F: X \to E_\infty$ is a compact mapping on the set X.

The set of all compact vector fields on X will be denoted by $\mathfrak{C}(E_\infty^X)$.

A compact vector field $f \in \mathfrak{C}(E_\infty^X)$ is said to be *finite-dimensional* if the mapping F of formula (8) is finite-dimensional. The set of all finite-dimensional vector fields on X will be denoted by $\mathfrak{C}_0(E_\infty^X)$.

In the sequel we shall consider the set $\mathfrak{C}(E_\infty^X)$ as a metric space and define the distance $\varrho(f, g)$ by setting

$$\varrho(f, g) = \sup_{x \in X} \|f(x) - g(x)\| \quad \text{for each} \quad f, g \in \mathfrak{C}(E_\infty^X). \tag{9}$$

From the Approximation theorem 2.2 we obtain:

3.1. *The set $\mathfrak{C}_0(E_\infty^X)$ is dense in the space $\mathfrak{C}(E_\infty^X)$.*

3.2. *If X is closed in E_∞ and $f \in \mathfrak{C}(E_\infty^X)$ then the set $f(X)$ is also closed in E_∞.*

Proof. Let $y_n \in f(X)$, $\lim_{n=\infty} y_n = y_0$, $y_n = f(x_n) = x_n - F(x_n)$; without loss of generality, we can assume that there exists $\lim_{n=\infty} F(x_n) = y^*$, $y^* \in E_\infty$. We have $\lim_{n=\infty} x_n = y_0 + y^*$, $\lim_{n=\infty} f(x_n) = f(y_0 + y^*)$, i. e. $y_0 = f(y_0 + y^*)$, $y_0 + y^* \in X$.

If X and Y are subsets of E_∞ then we shall put

$$\mathfrak{C}(Y^X) = \mathfrak{C}(E_\infty^X) \cap Y^X, \quad \mathfrak{C}_0(Y^X) = \mathfrak{C}_0(E_\infty^X) \cap Y^X.$$

In the sequel we shall consider the space $\mathfrak{C}(P_\infty^X)$ of *non-vanishing compact fields* on X.

Let X be a closed subset of E_∞. From 3.1, 3.2 we infer that:

3.3. *The set $\mathfrak{C}_0(P_\infty^X)$ is dense in the space $\mathfrak{C}(P_\infty^X)$.*

4. Notion of homotopy in the space $\mathfrak{C}(P_\infty^X)$. Two non-vanishing compact vector fields $f, g \in \mathfrak{C}(P_\infty^X)$ are called *homotopic* in the space $\mathfrak{C}(P_\infty^X)$ (we shall write $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$) if there exists a mapping $h \in P_\infty^{X \times I}$ which satisfies the following conditions:

1° $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for each $x \in X$;

2° a mapping h can be represented in the form

$$h(x, t) = x - H(x, t),$$

where the mapping $H: X \times I \to E_\infty$ is compact on $X \times I$.

The relation of homotopy established in the space $\mathfrak{C}(P_\infty^X)$ is a relation of equivalence and thus the set of all non-vanishing compact vector fields $f \in \mathfrak{C}(P_\infty^X)$ decomposes into disjoint classes of homotopic fields.

4.1. *Let X_0 be a subset of X and $f, g \in \mathfrak{C}(P_\infty^X)$; then $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$ implies $f|X_0 \simeq g|X_0$ in $\mathfrak{C}(P_\infty^{X_0})$.*

4.2. *For a given* $f \in \mathfrak{C}(P_\infty^X)$, *if a positive number* ε *is less than the distance* $\operatorname{dist}(f(X), 0)$ *then for every* $g \in \mathfrak{C}(P_\infty^X)$ *the condition* $\varrho(f, g) < \varepsilon$ *implies* $f \simeq g$ *in* $\mathfrak{C}(P_\infty^X)$.

Proof. Let $f(x) = x - F(x)$ and $g(x) = x - G(x)$. For each $x \in X$ we have $\|x - F(x)\| > \varepsilon$, $\|F(x) - G(x)\| < \varepsilon$. From these inequalities we infer that, for each $x \in X$ and $t \in I$, $x \neq H(x, t) = tF(x) + (1-t)G(x)$ and thus the mapping $h \in \mathfrak{C}(P_\infty^{X\times I})$ defined by $h(x, t) = x - H(x, t)$ (since H is a compact mapping on $X \times I$) is a homotopy between f and g.

Properties 3.3 and 4.2 imply:

4.3. *Every compact field* $f \in \mathfrak{C}(P_\infty^X)$ *is homotopic to some finite-dimensional field* $g \in \mathfrak{C}_0(P_\infty^X)$.

4.4. *Let* $X = \overline{V_\infty(x_0, \varrho)}$ *be a closed spherical region in* E_∞. *Then any two compact fields* $f, g \in \mathfrak{C}(P_\infty^X)$ *are homotopic* $f \simeq g$ *in* $\mathfrak{C}(P_\infty^X)$.

Proof. By 4.3 we can assume that the compact fields $f, g \in \mathfrak{C}(P_\infty^X)$ are finite-dimensional. Let $f(x) = x - F(x)$, $g(x) = x - G(x)$; we can assume that the values of F and G lie in the same finite-dimensional subspace $E_n \subset E_\infty$ and that the point x_0 belongs to E_n. Put $V_n = X \cap E_n$, $f_0 = f|V_n$, $F_0 = F|V_n$, $g_0 = g|V_n$, $G_0 = G|V_n$. We have $f_0, g_0: V_n \to P_n$ and thus $f_0 \simeq g_0$. Let $h_0(x, t) = x - H_0(x, t)$ be a homotopy joining f_0 with g_0 in the space $P_n^{V_n}$; we have $H_0(x, 0) = F_0(x)$, $H_0(x, 1) = G_0(x)$ for each $x \in V_n$. We shall extend the mapping $H_0: V_n \times I \to E_n$ over $X \times I$ to a compact mapping $H: X \times I \to E_\infty$ which satisfies the following conditions:

$$x \neq H(x, t) \quad \text{for each} \quad x \in X \text{ and } t \in I,$$
$$H(x, 0) = F(x), \; H(x, 1) = G(x) \quad \text{for each} \quad x \in X.$$

For this denote by $\{e_1, e_2, \ldots, e_n\}$, $e_k \in E_n$, a basis of E_n and by $\{l_1, l_2, \ldots, l_n\}$ the dual basis in the conjugate space to E_n; thus every element $z \in E_n$ can be written in the form

$$z = \sum_{i=1}^{n} l_i(z) e_i. \tag{10}$$

Let us consider the following closed subset of $X \times I$:

$$T_0 = (X \times \{0\}) \cup (V_n \times I) \cup (X \times l)$$

and define on T_0 a real-valued functions φ_i $(i = 1, 2, \ldots, n)$ as follows:

$$\varphi(x, t) = \begin{cases} l_i(F(x)) & \text{for} \quad x \in X \text{ and } t = 0, \\ l_i(G(x)) & \text{for} \quad x \in X \text{ and } t = 1, \\ l_i(H_0(x, t)) & \text{for} \quad x \in V_n \text{ and } t \in I. \end{cases} \tag{11}$$

Tietze Extension Theorem yields an extension $\tilde{\varphi}_i(x, t)$ of $\varphi_i(x, t)$ over $X \times I$; since each function φ_i is bounded, we can assume that also each $\tilde{\varphi}_i$ is bounded and thus the mapping $H: X \times I \in E_n$ defined by

$$H(x, t) = \sum_{i=1}^{n} \tilde{\varphi}_i(x, t) e_i \tag{12}$$

is compact. From (10) and (11) it follows that the mapping H defined by (12) is the desired extension of H_0 over $X \times I$ and thus the proof is complete.

5. Extension Homotopy Theorem. We shall consider the question of extending a non-vanishing compact field defined on a closed subset X_0 of $X \subset E_\infty$ to a non-vanishing compact field defined over the whole X. We shall prove that the existence of such extension depends only on the homotopy class of the given compact field.

THEOREM 1 (ON THE EXTENSION OF HOMOTOPY [5]). *Let* X_0 *be a closed subset of* $X \subset E_\infty$ *and* $f_0, g_0 \in \mathfrak{C}(P_\infty^{X_0})$ *two homotopic in* $\mathfrak{C}(P_\infty^{X_0})$ *compact fields. Then if there is an extension* $f \in \mathfrak{C}(P_\infty^X)$ *of* f_0 *over* X, *there is also an extension* $g \in \mathfrak{C}(P_\infty^X)$ *of* g_0 *over* X *with* f *and* g *homotopic in* $\mathfrak{C}(P_\infty^X)$.

Proof. The homotopy of the non-vanishing compact fields

$$f_0(x) = x - F_0(x), \quad F_0: X_0 \to E_\infty,$$
$$g_0(x) = x - G_0(x), \quad G_0: X_0 \to E_\infty,$$

means that there exists a compact mapping $H_0: X \times I \to E_\infty$ satisfying the following conditions:

$$x \neq H_0(x, t) \quad \text{for each} \quad x \in X_0 \text{ and } t \in I,$$
$$H_0(x, 0) = F_0(x), \quad H_0(x, 1) = G_0(x) \quad \text{for each} \quad x \in X_0.$$

There exists, by hypotesis, an extension $f \in \mathfrak{C}(P_\infty^X)$, $f(x) = x - F(x)$, of f_0 over X; thus $F_0 \subset F: X \to E_\infty$.

Denote by T_0 the following subset of the Cartesian product $X \times I$:

$$T_0 = (X_0 \times I) \cup (X \times \{0\}),$$

and define the following mapping $H_0^*: T_0 \to E_\infty$:

$$H_0^*(x, 0) = F(x) \quad \text{for} \quad x \in X \text{ and } t = 0,$$
$$H_0^*(x, t) = H_0(x, t) \quad \text{for} \quad x \in X_0 \text{ and } 0 \leqslant t \leqslant 1.$$

The mapping H_0^* is compact on T_0 and hence by 2.4 it can be extended to a compact mapping $H^*: X \times I \to E_\infty$ over $X \times I$.

Let us define the set $X_1 \subset X$ by the condition:

$$(x \in X_1) \text{ if and only if } (x - H_0(x,t) = 0 \text{ for some } t \in I).$$

X_1 and X_0 are obviously disjoint closed subsets of X. Hence there is a continuous real-valued function $\lambda(x)$ defined over X whose range is between 0 and 1 and which is 0 on X_1 and 1 on X_0.

Now consider the mapping

$$H(x,t) = H^*(x, \lambda(x)t) \quad \text{for} \quad x \in X \text{ and } t \in I.$$

It is clear that H is a compact mapping on $X \times I$ and for each $x \in X$ and $t \in I$

$$x \neq H(x,t).$$

If we define $g(x)$ by

$$g(x) = x - G(x), \quad \text{where} \quad G(x) = H(x,1), \; x \in X,$$

it is clear that $g(x)$ is an extension of $g_0(x)$ over X, and likewise that $H(x,0) = F(x)$ for $x \in X$. Since $H(x,1) = G(x)$ by definition, we conclude that the non-vanishing compact fields

$$f(x) = x - F(x) \quad \text{and} \quad g(x) = x - G(x) \quad (x \in X)$$

are homotopic in $\mathfrak{C}(P_\infty^X)$. The proof of Theorem 1 is complete.

6. Separation of the space between two points. Let X be a closed and bounded subset of E_∞.

THEOREM 2. *The set X does not separates the Banach space E_∞ between two points $x_1, x_2 \in E_\infty \setminus X$ if and only if the non-vanishing compact fields $(x-x_1)|X$, $(x-x_2)|X$ are homotopic in the space $\mathfrak{C}(P_\infty^X)$.*

The proof of theorem 1 is based on the following

LEMMA 1. *Let U be a bounded open set in E_∞, x_1 a point in U and Y the boundary of U. Then the non-vanishing compact field $(x-x_1)|Y$ cannot be extended to a non-vanishing compact field over $\overline{U} = U \cup Y$.*

Proof. Suppose it were possible to extend $(x-x_1)|Y$ over $\overline{U}$ to a non-vanishing compact field f, say, $f(x) = x - F(x)$, $f(x) = x - x_1$ for each $x \in Y$.

Let ϱ be so large that $\overline{U}$ and $F(\overline{U})$ are contained in the spherical region $\overline{V}_\infty = \overline{V_\infty(x_1, \varrho)}$ of radius ϱ and x_1 as centre.

The formulas

$$F^*(x) = x_1 \quad \text{for} \quad x \in \overline{V} \setminus U,$$
$$F^*(x) = F(x) \quad \text{for} \quad x \in U$$

would then define $F^*: \overline{V}_\infty \to \overline{V}_\infty$ as a compact mapping of $\overline{V}_\infty$ into itself without fixed points. This is a contradiction of the Schauder Fixed-Points Theorem 2.5. The proof is complete.

Proof of Theorem 2. Assuming first that x_1 and x_2 are not separated by X we shall prove that the compact fields $(x-x_1)|X$ and $(x-x_2)|X$ are not homotopic. We are given

$$E_\infty \setminus X = U \cup V,$$

U, V being disjoint sets which are open in E_∞, and $x_1 \in U$, $x_2 \in V$. One of the sets U, V, say U, is bounded. The non-vanishing compact field $(x-x_2)|X$ can be extended over $U \cup X$, in fact over $E_\infty \setminus \{x_2\} \supset U \cup X$. On the other hand, according to Lemma 1, it is not possible, in view of the boundary of U being contained in X, to extend $(x-x_1)|X$ over $U \cup X$ to a non-vanishing compact field. Hence $(x-x_1)|X$ and $(x-x_2)|X$ are not homotopic in the space $\mathfrak{C}(P_\infty^X)$ since Theorem 1 would be contradicted if they were.

Now let us assume that $x_1 \sim x_2$ in $E_\infty \setminus X$. Then one can join x_1 and x_2 by a continuous arc in $E_\infty \setminus X$, i. e. one can find a continuous function $r(t)$ of the real parameter t, $0 \leqslant t \leqslant 1$, with values in $E_\infty \setminus X$ such that

$$r(0) = x_1, \quad r(1) = x_2.$$

The mapping $h: X \times I \to P_\infty$, defined by

$$h(x,t) = x - r(t), \quad x \in X \text{ and } t \in I,$$

is obviously a homotopy joining $(x-x_1)|X$ and $(x-x_2)|X$ in $\mathfrak{C}(P_\infty^X)$. Hence Theorem 2 is proved.

7. The main theorem. For the proof of the main theorem we shall use the following lemmas.

LEMMA 2. *Let $V_\infty = V_\infty(x_0, \varrho)$, $x_0 \in E_n \subset E_\infty$, $V_n = V_\infty \cap E_n$. Suppose that the mapping $F: \overline{V}_n \to E_n$ has a finite number of fixed points $x_1, x_2, \ldots, x_k \in \overline{V}_n$. Then there is a compact mapping $\overline{F}: \overline{V}_\infty \to E_n$ which has the same fixed points as F and which is an extension of F over $\overline{V}_\infty$.*

Proof. By 2.1 it follows that $\overline{V}_n$ is a retract of $\overline{V}_\infty$, i. e. there is a mapping $r: \overline{V}_\infty \to \overline{V}_n$ such that $r(x) = x$ for $x \in \overline{V}_n$.

Putting for each $x \in \overline{V}_\infty$

$$\overline{F}(x) = F(r(x))$$

we obviously obtain the desired compact mapping $\overline{F}: \overline{V}_\infty \to E_n$.

LEMMA 3. *Let X be a closed bounded subset of E_n and $f_0 \in P_n^X$. Then there exists a mapping $f \in E_n^{E_n}$ such that:*

1° *the set N of all roots of the equation*

$$f(x) = 0$$

is finite; if $x_1, x_2 \in N$ then x_1, x_2 belong to different components of $E_n \setminus X$,

2° *$f(x) = f_0(x)$ for each $x \in X$.*

The proof is a slight modification of the proof of a similar Lemma given in [4], p. 300.

LEMMA 4. *Suppose that a bounded and closed subset X of E_∞ does not disconnect E_∞ and that $V_\infty = V_\infty(x_0, \varrho)$ is a spherical region which contains X. Then every non-vanishing compact field $f_0 \in \mathfrak{C}(P_\infty^X)$ can be extended over $\overline{V}_\infty$ to a compact non-vanishing field $f \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$.*

Proof. By Theorem 1 and 4.3 we can assume, without loss of generality, that a compact field $f_0(x) = x - F_0(x)$ is finite-dimensional, i. e. $F_0: X \to E_n$.

Suppose that S_∞ is the boundary of V_∞ and that a point $x^* \in E_n$ does not belong to $\overline{V}_\infty$. Define the mapping $f_1 \in P_\infty^{X_1}$, $X_1 = S_\infty \cup X$ by

$$f_1(x) = x - F_1(x), \quad \text{where} \quad F_1(x) = \begin{cases} x^* & \text{for} \quad x \in S_\infty, \\ F_0(x) & \text{for} \quad x \in X. \end{cases}$$

Putting $X_1^* = X_1 \cap E_n$, $f_1^* = f_1 | X_1^*$ we have $f_1^* \in P_n^{X_1^*}$.

By Lemma 3 there exists a mapping $f_2^* \in E_n^{\overline{V}_n}$ such that the set N of all roots of the equation $f_2^*(x) = 0$ is finite, $N = \{x_1, x_2, \ldots, x_k\}$, and $f_2^*(x) = f_1^*(x)$ for every $x \in X_1^*$.

By Lemma 2 the mapping $f_2^* | (\overline{V}_n \setminus N)$ can be extended to a finite-dimensional field $f_2 \in \mathfrak{C}(P_\infty^{\overline{V}_\infty \setminus N})$.

Since the set $\overline{V}_\infty \setminus N$ is connected, the points $x_1, x_2, \ldots, x_k$ can be joined by a chain $V_\infty^1, V_\infty^2, \ldots, V_\infty^l \subset V_\infty \setminus X$ of open spherical regions such that $\overline{V}_\infty^i$ intersects $\overline{V}_\infty^j$ if and only if $|i-j| = 1$ $(i, j = 1, 2, \ldots, l)$.

Let E_m be a finite-dimensional subspace of E_∞ spanned by the centres of $\overline{V}_\infty^i$ $(i = 1, 2, \ldots, l)$ and containing E_n.

Let us put $T = \bigcup_{i=1}^{l} V_\infty^i$, $T^* = T \cap E_m$, $\overline{V}_m = \overline{V}_\infty \cap E_m$, $X_2 = \overline{V}_\infty \setminus T$, $X_2^* = \overline{V}_m \setminus T^*$, $f_2^* = f_2 | X_2^*$.

Since X_2^* is connected, it follows that the mapping $f^* \in P_m^{X_2^*}$ can be extended to a mapping $f_3^* \in P_m^{\overline{V}_m \setminus V_m'}$ over $\overline{V}_m \setminus V_m'$, where V_m' is a certain open spherical region contained in T^*. Since $f_3^*(x) = x - x^*$ for $x \in S_{m-1} = E_m \cap S_\infty$, we have $f_3 | S_{m-1} \simeq 1$ and consequently $f_3 | S_{m-1}' \simeq 1$, where S_{m-1}' is the boundary of V_m'. This implies that $f_3^* \subset f_4^* \in P_m^{\overline{V}_m}$.

By Lemma 2 we extend the mapping $f_4^*(x) = x - F_4^*(x)$, $x \in \overline{V}_m$, to a compact field $f \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$ and thus the proof of Lemma 4 is complete.

The main result of this paper is the following

THEOREM 3 (ON THE DISCONNECTION OF BANACH SPACES). *Let X be a bounded closed subset of the Banach space E_∞. The set $E_\infty \setminus X$ is connected if and only if any two non-vanishing compact fields $f, g \in \mathfrak{C}(P_\infty^X)$ are homotopic in the space $\mathfrak{C}(P_\infty^X)$.*

Proof. *Necessity.* Suppose that X does not disconnect E_∞, and $f, g \in \mathfrak{C}(P_\infty^X)$. By Lemma 4 compact fields f and g can be extended to non-vanishing compact fields $\bar{f}, \bar{g} \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$ over a closed spherical region $\overline{V}_\infty$ which contains X.

By 4.4 we have $\bar{f} \simeq \bar{g}$ in $\mathfrak{C}(P_\infty^{\overline{V}_\infty})$ and hence by 4.1 the compact fields f and g are homotopic in $\mathfrak{C}(P_\infty^X)$.

Sufficiency. Suppose that X disconnects E_∞. Then there certainly exist two points x_1 and x_2 separated by X. By Theorem 2 the non-vanishing compact fields $(x - x_1)|X$ and $(x - x_2)|X$ are not homotopic in $\mathfrak{C}(P_\infty^X)$ and thus the proof of Theorem 3 is complete.

8. Jordan separation theorem in Banach spaces. We shall say that two bounded and closed subsets X and Y of E_∞ are *homeomorphic in the narrow sense* if there exists a homeomorphism $h \in \mathfrak{C}(E_\infty^X)$ such that $Y = h(X)$.

It is clear that *if the closed and bounded subsets X and Y of E_∞ are homeomorphic in the narrow sense then the space $\mathfrak{C}(P_\infty^X)$ consists of one homotopy class if and only if the space $\mathfrak{C}(P_\infty^Y)$ consists of one homotopy class.*

From this we obtain the following, due to J. Leray [9]:

CORROLARY 1. *If a closed and bounded subset X of the Banach space E_∞ disconnects E_∞, then so does every subset of E_∞ which is homeomorphic to X in the narrow sense.*

As an obvious application of Corrolary 1 we obtain the following:

CORROLARY 2 (JORDAN SEPARATION THEOREM). *A subset of E_∞ which is homeomorphic in the narrow sense to a sphere $S_\infty(x_0, \varrho)$ disconnects E_∞.*

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On completion of proximity spaces by local clusters *

by

S. Leader (Rutgers)

1. Introduction. In [4] the concept of a "cluster" of subsets from a proximity space X was introduced and used to construct the compactification $\bar{X}$ of X by identifying each point x in $\bar{X}$ with the cluster of all subsets of X which are close to x. Viewing the completion X^* of X as a subspace of $\bar{X}$, the present paper characterizes those clusters, the "local" clusters, from X which are determined by points in X^*. These clusters can be used to construct a completion theory for proximity spaces along the same lines as the compactification theory in [4].

The key concept in any completion theory for proximity spaces is that of "small" sets, since X^* consists of just those points in $\bar{X}$ which are close to small subsets of X. The concept of small sets can be introduced through various devices: uniform structures, uniform coverings, or pseudo-metrics. Smirnov uses the second device in [11], [12], [13] and [14]. We shall use the third device here making use of the ideas and results of [4].

In the last two sections of the paper two conjectures are posed for consideration by the interested reader.

2. Gauges. A *gauge* ϱ on a proximity space X is a real-valued function $\varrho(x,y)$ on $X\times X$ satisfying the following two conditions:

(2.1) $\varrho(x,y)\leqslant\varrho(x,z)+\varrho(y,z)$ for all x,y,z in X.

(2.2) Given A close to B in X and $\varepsilon>0$, there exists a in A and b in B such that $\varrho(a,b)<\varepsilon$.

We define $\varrho(A,B)$ to be the infimum of $\varrho(a,b)$ for all a in A and b in B. We define $\varrho[A]$, the ϱ-diameter of A, to be the supremum of $\varrho(x,y)$ for all x and y in A.

That $\varrho(y,y)\leqslant 0$ follows from (2.2) for all y in X. The reversed inequality follows from (2.1) if we set $z=y$. So $\varrho(y,y)=0$. Thus, setting $z=x$ in (2.1), we find $\varrho(x,y)\leqslant\varrho(y,x)$ for all x and y in X. So $\varrho(x,y)=\varrho(y,x)$. Finally, setting $y=x$ in (2.1) gives $0\leqslant\varrho(x,z)$.

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