On the existence of conjugate functions of higher order*

by

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1. In this paper we investigate properties of certain extensions of the notion of conjugate function. Before we formulate these extensions we recall some known facts about generalized derivatives. Proofs and bibliographic references can be found e. g. in [5], vol. II, Chapter XI, §§ 1-5 (see the References at the end of the paper).

A function $f(x)$ defined in the neighbourhood of a point x_0 is said to have at x_0 a generalized derivative of order r $(r = 1, 2, \ldots)$ if

$$f(x_0+t) = a_0+a_1t+\ldots+\frac{a_r}{r!}t^r+o(t^r)$$

for $t\to 0$, the a_j denoting constants. The number a_r is called the rth *generalized derivative* of f at x_0 and will be denoted by $f_{(r)}(x_0)$. Clearly, if an ordinary derivative $f^{(r)}(x_0)$ exists so does $f_{(r)}(x_0)$ and both have the same value $(r = 1, 2, \ldots)$; the existence of $f_{(r)}(x_0)$ implies that of $f_{(r-1)}(x_0)$; finally, if $f_{(r)}(x_0)$ exists and F is the indefinite integral of f, then $F_{(r+1)}(x_0)$ exists and equals $f_{(r)}(x_0)$ (F is defined near x_0 since the hypothesis implies that f is bounded near x_0).

Suppose that f, defined in the neighborhood of x_0, has a generalized derivative $f_{(r-1)}(x_0)$. Writing a_j for $f_{(j)}(x_0)$ we define the function $\delta_r(x_0, t)$ by the formula

$$(1.1)\quad \frac{1}{2}[f(x_0+t)+f(x_0-t)] = a_0+\frac{a_2}{2!}t^2+\ldots+\frac{a_{r-1}}{(r-1)!}t^{r-1}+\frac{1}{2}\delta_r(x_0,t)\frac{t^r}{r!}$$

if r is odd, and by

$$(1.2)\quad \frac{1}{2}[f(x_0+t)-f(x_0-t)] = a_1t+\frac{a_3}{3!}t^3+\ldots+\frac{a_{r-1}}{(r-1)!}t^{r-1}+\frac{1}{2}\delta_r(x_0,t)\frac{t^r}{r!}$$

if r is even. If

$$\delta_r(x_0) = \lim_{t\to+0}\delta_r(x_0,t)$$

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exists, we may consider $\delta_r(x_0)$ as the jump of the rth derivative at x_0, though we do not assume the existence of the rth derivative. Correspondingly, the condition $\delta_r(x_0) = 0$, which we shall call *smoothness of order r* of f at x_0, may be interpreted as a sort of continuity of the rth derivative at x_0. For example,

$$\delta_1(x_0, t) = \frac{f(x_0+t)+f(x_0-t)-2f(x_0)}{t},$$

and smoothness of order 1 is what is usually called *smoothness of the function* (see [5], vol. I, p. 42) and is a substitute for the existence and continuity of the first derivative. It is easy to see that if $f_{(r)}(x_0)$ exists, then $\delta_r(x_0) = 0$.

Let now $f(x)$ be periodic (i. e. of period 2π) and integrable (over a period). It is very well known that the conjugate function

$$(1.3)\qquad \tilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan\frac{1}{2}t}\,dt = -\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_\varepsilon^\pi$$

exists almost everywhere. The fact is of importance and there are several proofs of it; some use pure real-variable methods, others — complex variable. Using the classical development $\frac{1}{2}\cot\frac{1}{2}x = \sum(x+2\pi\nu)^{-1}$ we can represent (1.3) in the form

$$(1.4)\qquad \tilde{f}(x) = -\frac{1}{\pi}\int_0^\infty \frac{f(x+t)-f(x-t)}{t}\,dt,$$

which will be more convenient for us.

Let $F(x)$ be the indefinite integral of $f(x)$. We may assume that the mean value of f over a period is 0 (subtracting a constant from f does not affect either the existence or the value of $\tilde{f}$), so that F is periodic and bounded. At each point where F' exists (it is enough to assume that F is smooth of order 1 at the point) integrating by parts we can write the right hand side of (1.3) in the form

$$(1.5)\qquad -\frac{1}{\pi}\int_0^\infty \frac{F(x+t)+F(x-t)-2F(x)}{t^2}\,dt = -\frac{1}{\pi}\int_0^\infty \frac{\delta_1(x_0, t; F)}{t}\,dt,$$

where the integral is improper at $t = 0$. The integral (1.5) exists, however, under much more general conditions: it is known that if F is merely periodic and integrable, and is differentiable in a set E, then (1.5) exists at almost all points of E. It is therefore natural to ask what are the necessary and sufficient conditions for (1.5) to exist almost everywhere

in a set E if we merely assume that F is periodic and integrable. A special case of Theorem 1 below asserts that (1.5) *exists almost everywhere in E if and only if the indefinite integral of F has a generalized second derivative almost everywhere in E.*

Return to (1.4) and replace there f by φ. Integrate by parts r times selecting the successive primitives of φ in such a way that their mean values over a period are 0, and denote the rth primitive of φ by f. A simple computation shows (cf. [5], vol. II, p. 65) that the right-hand side of (1.4) can be written

$$(1.6)\qquad -\frac{1}{\pi}\int_0^\infty \frac{\delta_r(x_0, t)}{t}\,dt,$$

where $\delta_r(x, t)$ is the function defined above, relative to f. We shall call (1.6) the *conjugate of order r of f* and denote it by $\tilde{f}_r(x)$. Thus (1.4) is the conjugate of order 0 of f, and (1.5) the conjugate of order 1 of F. The conjugate of order r can be formally defined if f is periodic, integrable, and differentiable up to order $r-1$ at a given point (this is a prerequisite for defining $\delta_r(x, t)$). The integral (1.6) is improper at $t = 0$ but converges absolutely near $t = \infty$ if $r > 0$. The latter follows from the fact that for $t\to\pm\infty$ we have

$$t^{-1}\delta_r(x_0, t) = O(t^{-2}) + O\left\{\frac{|f(x_0+t)|+|f(x_0-t)|}{t^{r+1}}\right\}$$

(cf. (1.1) and (1.2)). We have just indicated that if f is an rth integral then $\tilde{f}_r(x)$ is the (ordinary) conjugate function of $f^{(r)}(x)$.

It is well known (see [5], vol. II, Chapter XI, § 5) that if f has a generalized rth derivative in a set E, then $\tilde{f}_r(x)$ exists almost everywhere in E. The theorem which follows generalizes this result and is the main result of the paper.

THEOREM 1. *Let $f(x)$ be periodic and integrable and suppose that $f_{(r-1)}(x)$ exists at each point of a set E of positive measure. Then a necessary and sufficient condition for the r-th conjugate $\tilde{f}_r(x)$ to exist almost everywhere in E is that the indefinite integral of f has a generalized derivative of order $r+1$ almost everywhere in E.*

The proof of the sufficiency of the condition is comparatively simple if one uses known results. It can be based either on purely real or complex methods; we give the real-variable proof. The proof of the necessity of the condition lies deeper and is accessible to us only by complex methods. One of the difficulties of the situation is that in investigating the behavior of successive differentiations of trigonometric series we cannot use Abel's method of summation since it is known that Abel summability of a trigonometric series in a set of positive measure does not necessarily imply Abel summability of the conjugate series almost everywhere in the set. The situation can be straightened out by using Cesàro summability (see Lemma 3 below), but this complicates the computation.

In section 2 we prove the sufficiency of the condition. Sections 3 and 4 contain the proof of the necessity. The concluding section 5 contains an additional observation.

2. Given any closed set P we shall denote by $\chi_P(x)$, or simply $\chi(x)$, the distance of the point x from P. Thus the function χ vanishes in P and over each interval d contiguous to P the graph of χ is an isosceles triangle of height $\frac{1}{2}|d|$.

The proof of the following two lemmas, and bibliographic references, may e. g. be found in [5], Chapter IV, § 2 and Chapter XI, § 4.

Lemma 1. *If P is a closed set contained in a finite interval (a, b) and $\lambda > 0$, the integral*

$$\int_a^b \frac{\chi^\lambda(t)}{|t-x|^{\lambda+1}}\,dt \tag{2.1}$$

is finite for almost all x in P.

Lemma 2. *Suppose that $f(x)$, defined in a finite interval, is measurable and has a k-th generalized derivative in a measurable set E. Then we can find a closed set $P\subset E$ of measure arbitrarily close to that of E, and a decomposition*

$$f(x) = g(x)+h(x)$$

with the following properties:

(i) $g(x)\in C^k$ (¹);

(ii) $h(x) = 0$ *in* P;

(iii) *except possibly for a finite number of intervals contiguous to P we have*

$$|h(x)| \leqslant C\chi^k(x),$$

where C is independent of x and $\chi(x) = \chi_P(x)$. If f is periodic, g and h can also be made periodic.

Let now $f(x)$ be periodic, integrable over a period, and suppose that the indefinite integral F of f has an $(r+1)$th generalized derivative at each point of $E\subset(-\pi,\pi)$. We want to show that $\tilde{f}_{(r)}$ exists almost everywhere in E.

(¹) We denote by C^k the class of functions having a continuous kth derivative.

Applying Lemma 2 to F, with $r+1$ for k, we have for a suitable closed $P\subset E$, with $|E-P|$ arbitrarily small, a decomposition

$$F(x) = G(x)+H(x),$$

where $G\in C^{r+1}$ and

$$|H(x)| \leqslant C\chi^{r+1}(x), \tag{2.2}$$

except possibly for a finite number of intervals $I_1, I_2, \ldots, I_m$ contiguous to P. If $f = F'$, $g = G'$, $h = H'$, then g exists everywhere, f and h almost everywhere, and almost everywhere we have

$$f = g+h. \tag{2.3}$$

The function h is integrable. Since $G\in C^{r+1}$, we have $g\in C^r$. Hence $\tilde{g}_r$ exists almost everywhere and it is enough to show that $\tilde{h}_r$ exists almost everywhere in P.

The inequality (2.2) implies that at each x which is a point of density of P we have

$$H(x+t) = o(t^{r+1}) \quad \text{as} \quad t\to 0.$$

Hence

$$H(x) = H_{(1)}(x) = H_{(2)}(x) = \ldots = H_{(r+1)}(x) = 0$$

at each point of density of P. Since H is the integral of h and, by (2.3), h has an $(r-1)$th generalized derivative at each point of E, this implies that

$$h(x) = h_{(1)}(x) = \ldots = h_{(r-1)}(x) = 0 \tag{2.4}$$

at each point of density of P. We will show that $\tilde{h}_r$ exists at each point $x\in P$ which is not an end-point of the intervals I_j mentioned above and at which the integral of Lemma 1 with $r+1$ for λ is finite. In view of (2.4) this amounts to showing that the integral

$$\int_{-\pi}^{\pi} \frac{h(x+t)}{t^{r+1}}\,dt \tag{2.5}$$

exists for such x. Without loss of generality we may suppose that $x = 0$. Finally, instead of taking the whole interval $(-\pi,\pi)$ it is enough to consider the part of the integral extended over an interval $(-\varepsilon', \varepsilon)$, where $-\varepsilon'$ and ε are in P and $(-\varepsilon', \varepsilon)$ is so small that it does not overlap with any of the I_j.

Let (a_i, b_i) $(i = 1, 2, \ldots)$ be all the intervals contiguous to P and situated in $(0, \varepsilon)$. Integrating by parts we have

$$\left|\int_{a_i}^{b_i} \frac{h(t)}{t^{r+1}}\,dt\right| = (r+1)\left|\int_{a_i}^{b_i} \frac{H(t)}{t^{r+2}}\,dt\right| \leqslant C(r+1)\int_{a_i}^{b_i} \frac{\chi^{r+1}(t)}{t^{r+2}}\,dt,$$

by (2.2). By hypothesis, the sum of the integrals on the right extended over all intervals (a_i, b_i) is finite. Since for $a_i < \xi < b_i$ the second mean-value theorem gives

$$\left|\int_{a_i}^{\xi} \frac{h(t)}{t^{r+1}}\,dt\right| = \frac{1}{a_i^{r+1}}\left|\int_{a_i}^{\xi'} h(t)\,dt\right| = \frac{H(\xi')}{a_i^{r+1}} \leqslant C\left(\frac{b_i - a_i}{a_i}\right)^{r+1} = o(1) \quad (a_i < \xi < \xi' < b_i)$$

(the last equation being a consequence of the fact that $x = 0$ is a point of density of P) and since $\chi = 0$ in P, we see that the integral

$$\int_0^{\varepsilon} \frac{h(t)}{t^{r+1}}\,dt = \lim_{\eta\to+0}\int_{\eta}^{\varepsilon}$$

exists. In the same way we prove the existence of the integral extended over $(-\varepsilon', 0)$. Hence $\tilde{h}_r(0)$ exists and the sufficiency of the condition in Theorem 1 is established.

3. We now pass to the proof of the necessity of the condition. The proof uses complex methods.

LEMMA 3. *Suppose that the trigonometric series*

$$(3.1)\qquad \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx) = \sum_{k=0}^{\infty} A_k(x)$$

is summable (C, α), $\alpha = 0, 1, 2, \ldots$, *in a set* E *of positive measure. Then the conjugate series*

$$(3.2)\qquad \sum_{k=1}^{\infty}(a_k\sin kx - b_k\cos kx) = \sum_{k=1}^{\infty} B_k(x)$$

is summable (C, α) *almost everywhere in* E.

The result holds for all $\alpha > -1$ but only the case of integral α is of interest to us. For the proof of Lemma 3, see [4], or [3], or [2].

The Fourier series of a function f will be denoted by $S[f]$, and the series conjugate to $S[f]$ by $\tilde{S}[f]$. The series obtained from these by termwise differentiation k times will be denoted by $S^{(k)}[f]$ and $\tilde{S}^{(k)}[f]$, respectively. The (C, α) means of $S[f]$ and $\tilde{S}[f]$ will be denoted by $\sigma_\omega^\alpha(x)$ and $\tilde{\sigma}_\omega^\alpha(x)$.

LEMMA 4. *Suppose that* (3.1) *and* (3.2) *are both summable* (C, α), $\alpha = 0, 1, 2, \ldots$, *at a point* x_0, *the former to sum* s. *Let* $k \geqslant \alpha + 2$ *and let* $F(x)$ *be the sum of the series* (3.1) *integrated termwise* k *times. Then* $F_{(k)}(x_0)$ *exists and has value* s.

For the proof see e. g. [5], vol. II, p. 69.

If $\sum A_k(x)$ is the Fourier series of a periodic f, the partial sums of the conjugate series $\sum B_k(x)$ are given by the formula

$$(3.3)\qquad {\sum_{k\leqslant\omega}}' B_k(x) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+t)\,\frac{1-\cos\omega t}{t}\,dt,$$

where ω is any positive number, not necessarily an integer, and the prime $'$ indicates that when ω is an integer $B_\omega(x)$ is to be multiplied by $\frac{1}{2}$. In considering (C, β) means, $\beta > 0$, of $\sum B_k(x)$ it will be convenient to use the Riesz form

$$(3.4)\qquad \tilde{\sigma}_\omega^\beta(x) = \sum_{k\leqslant\omega} B_k(x)\left(1-\frac{k}{\omega}\right)^{\beta} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x+t)\,\tilde{K}_\omega^\beta(t)\,dt,$$

where $\tilde{K}_\omega^\beta(t)$ are the (C, β) means of the expression

$$\frac{1-\cos\omega t}{t} = \int_0^{\omega} \sin ut\,du$$

qua function of ω. We are interested in integral values of β only. We have

$$\tilde{K}_\omega^\beta(t) = \beta\omega^{-\beta}\int_0^{\omega}(\omega-u)^{\beta-1}\frac{1-\cos ut}{t}\,du,$$

and the formula can also be written

$$(3.5)\qquad \tilde{K}_\omega^\beta(t) = \omega^{-\beta}\int_0^{\omega}(\omega-u)^{\beta}\sin ut\,du = \omega^{-\beta}\Im\int_0^{\omega}(\omega-u)^{\beta}e^{itu}\,du.$$

Integrating by parts β times we obtain

$$(3.6)\qquad \tilde{K}_\omega^\beta(t) = \Im\left\{-\frac{1}{it} - \frac{\beta}{\omega(it)^2} - \frac{\beta(\beta-1)}{\omega^2(it)^3} - \ldots - \frac{\beta!}{\omega^{\beta-1}(it)^{\beta}} + \beta!\,\frac{e^{it\omega}-1}{\omega^{\beta}(it)^{\beta+1}}\right\}.$$

From this we easily deduce that

$$(3.7)\qquad \frac{d^k}{dt^k}\tilde{K}_\omega^\beta(t) = O(t^{-k-1}) \qquad (t\to+\infty;\ 0\leqslant k\leqslant\beta),$$

and the first equation (3.5) also gives

$$(3.8)\qquad \left|\frac{d^k}{dt^k}\tilde{K}_\omega^\beta(t)\right| \leqslant \omega^{k+1} \qquad (k = 0, 1, 2, \ldots).$$

We write

$$\widetilde{K}^{\beta}_{\omega}(t) = \frac{1}{t} + H^{\beta}_{\omega}(t)\,. \tag{3.9}$$

We need estimates for $H^{\beta}_{\omega}(t)$ in the range $1/\omega \leqslant t < \infty$. From (3.6) (since the contribution of the second term on the right is 0) we easily obtain

$$\left.\begin{aligned} &\left|\frac{d^{\beta-2}}{dt^{\beta-2}} H^{\beta}_{\omega}(t)\right| \leqslant \frac{C_\beta}{\omega^2 t^{\beta+1}} &&(3.10)\\ &\left|\frac{d}{dt}\left(t^{\beta-2}\frac{d^{\beta-2}}{dt^{\beta-2}} H^{\beta}_{\omega}(t)\right)\right| \leqslant \frac{C_\beta}{\omega t^3} &&(3.11)\end{aligned}\right\}\ \beta \geqslant 2,\ \omega t \geqslant 1\,.$$

Finally, using integration by parts, the estimates (3.7) and the fact that $\widetilde{K}^{\beta}_{\omega}(t)$ is an odd function of t (and so is 0 at $t=0$ together will all derivatives of even order), we find that

$$\int_0^\infty \{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = \int_0^\infty t^2\{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = \ldots = \int_0^\infty t^{r-1}\{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = 0 \quad (r\text{—odd}), \tag{3.12}$$

$$\int_0^\infty t\{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = \int_0^\infty t^3\{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = \ldots = \int_0^\infty t^{r-1}\{\widetilde{K}^{\beta}_{\omega}(t)\}^{(r)}dt = 0 \quad (r\text{—even}), \tag{3.13}$$

provided $r \leqslant \beta+1$.

LEMMA 5. *If $\widetilde{f}_r$ exists at the point x, then $\widetilde{S}^{(r)}[f]$ is summable $(C, r+2)$ at x to sum $\widetilde{f}_r(x)$.*

Consider the right-hand side of (3.4) with $\beta = r+2$ and suppose for example that r is odd. The $(C, r+2)$ mean of $\widetilde{S}^{(r)}[f]$ is

$$\frac{(-1)^{r+1}}{\pi}\int_{-\infty}^{+\infty} f(x+t)\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\,dt =$$

$$= \frac{2}{\pi}(-1)^{r+1}\int_0^\infty \tfrac{1}{2}[f(x+t)+f(x-t)]\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\,dt\,.$$

The hypothesis of Lemma 5 implies that $f_{(2)}(x) = a_2,\ f_{(4)}(x) = a_4, \ldots, f_{(r-1)}(x) = a_{r-1}$ exist. Using (3.12) and (1.1) we find that the last integral equals

$$\frac{(-1)^{r+1}}{\pi}\int_0^\infty \delta_r(x,t)\,\frac{t^r}{r!}\,\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\,dt = \frac{(-1)^{r+1}}{\pi}\int_0^{1/\omega} + \frac{(-1)^{r+1}}{\pi}\int_{1/\omega}^{\infty} = P+Q\,,$$

say.

The convergence of the integral

$$-\frac{1}{\pi}\int_0^\infty \frac{\delta_r(x,t)}{t}\,dt = -\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_\varepsilon^\infty \tag{3.14}$$

implies that

$$\int_0^t \delta_r(x,s)\,ds = o(t) \qquad (t\to+0)\,. \tag{3.15}$$

Denote the left-hand side here by $\Delta(t)$. Integrating by parts and using (3.8) we find that, except for a numerical factor, P is

$$\int_0^{1/\omega} \Delta'(t)\,t^r\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\,dt =$$

$$= \left[\Delta(t)\,t^r\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\right]_0^{1/\omega} - \int_0^{1/\omega}\Delta(t)\frac{d}{dt}\left\{t^r\frac{d^r}{dt^r}\widetilde{K}^{r+2}_{\omega}(t)\right\}dt$$

$$= o(1) + \int_0^{1/\omega} o(1/\omega)\,O(\omega^2)\,dt = o(1)\,.$$

On account of (3.9),

$$Q = -\frac{1}{\pi}\int_{1/\omega}^\infty \frac{\delta_r(x,t)}{t}\,dt + \frac{(-1)^{r+1}}{\pi r!}\int_{1/\omega}^\infty \frac{\delta_r(x,t)}{t}\,t^{r+1}\frac{d^r}{dt^r}H^{\beta}_{\omega}(t)\,dt = Q_1+Q_2\,,$$

say, and the lemma will be established if we show that $Q_2\to 0$ as $\omega\to\infty$.

We split the integral Q_2 into two parts, extended over $1/\omega \leqslant t \leqslant 1$ and $1 \leqslant t < \infty$. That the second integral tends to 0 is immediate since, as we have already observed, $\delta_r(x,t)/t$ is absolutely integrable away from $t=0$ and, in view of (3.10) with $\beta = r+2$, the cofactor of this ratio tends uniformly to 0 away from $t=0$.

Consider now (3.11) for $\beta = r+2$ and denote the expression in parenthesis on the left by $L(t)$. Then, except for a numerical factor, the integral which remains is

$$\int_{1/\omega}^1 \Delta'(t)L(t)\,dt = \big[\Delta(t)L(t)\big]_{1/\omega}^1 - \int_{1/\omega}^1 \Delta(t)L'(t)\,dt$$

$$= \big[o(t)\,O(\omega^{-2}t^{-3})\big]_{1/\omega}^1 + \int_{1/\omega}^1 o(t)\,O(\omega^{-1}t^{-3})\,dt = o(1)\,.$$

This completes the proof of Lemma 5.

4. Return to Theorem 1 and suppose that $\tilde{f}_r$ exists at each point of a set E. By Lemma 5, $\tilde{S}^{(r)}[f]$ is summable $(C, r+2)$ in E to sum $\tilde{f}_r$. By Lemma 3, $S^{(r)}[f]$ is summable $(C, r+2)$ in a subset E_1 of E of the same measure as E. By Lemma 4, the function F obtained by integrating $S^{(r)}[f]$ termwise $r+4$ times has at each point of E_1 a generalized derivative of order $r+4$. Clearly F is a fourth integral of f. By Lemma 4, with $k = r+4$, we can find a closed subset P of E, with $|E-P|$ arbitrarily small, and a decomposition $F = G+H$, where $H \in C^{r+4}$ and $|H(x)| \leqslant C\chi^{r+4}(x)$, except in a finite number of intervals contiguous to P. It follows that $H(x+t) = o(t^{r+4})$ at each point of density of P, and in particular that

$$H(x) = H_{(1)}(x) = \ldots = H_{(r+4)}(x) = 0 \tag{4.1}$$

at such points.

We write

$$G^{(4)}(x) = g(x), \quad H'^{(4)}(x) = h(x).$$

Since F is a fourth integral of f and G is in C^{r+4}, it follows that in the decomposition

$$f = g+h$$

the function g is in C^r and h is integrable. Hence the indefinite integral of g has a derivative of order $r+1$, and if we show that the indefinite integral of h has a generalized derivative of order $r+1$ almost everywhere in P, it will follow that the indefinite integral of f has a generalized derivative of order $r+1$ almost everywhere in P, and so almost everywhere in E, and Theorem 1 will be established.

Now the function h has, like f, generalized derivatives up to order $r-1$ at each point of E. It follows from (4.1) that

$$h(x) = h_1(x) = \ldots = h_{(r-1)}(x) = 0$$

almost everywhere in P. Also, since g is in C^r, $\tilde{g}_r$ exists almost everywhere, and since by hypothesis $\tilde{f}_r$ exists in E, it follows that $\tilde{h}_r$ exists almost everywhere in E. Thus Theorem 1 in the general form reduces to the special case when (replacing h by f),

$$f(x) = f(x) = \ldots = f_{(r-1)}(x) = 0 \tag{4.2}$$

in E. In this case

$$-\frac{1}{\pi}\int_0^\infty \frac{\delta_r(x,t)}{t}\,dt = -\frac{r!}{\pi}\int_0^\infty \frac{f(x+t) \pm f(x-t)}{t^{r+1}}\,dt, \tag{4.3}$$

where the signs "+" and "−" correspond to the cases of r odd and even. These cases require somewhat different treatments and we consider them separately.

1. *r — odd.*

Denote the indefinite integral of f by Φ. At each point where the integral (4.3) with the sign "+" exists we have

$$\int_0^t [f(x+s)+f(x-s)]\,ds = o(t^{r+1}) \quad (t \to +0),$$

that is,

$$\Phi(x+t) - \Phi(x-t) = o(t^{r+1}). \tag{4.4}$$

Let Q be a closed subset of E in which this relation holds uniformly in x; the measure of Q can differ arbitrarily little from that of E. If we show that for each x_0 which is a point of density of Q we have

$$\Phi(x_0+h) - \Phi(x_0) = o(h^{r+1}), \tag{4.5}$$

it will follow that $\Phi_{(r+1)}$ exists (and equals 0) almost everywhere in Q, and so also almost everywhere in E.

Without loss of generality we may suppose that $x_0 = 0$ is a point of density of Q. We note that if h is positive and sufficiently small, then in $(0, h)$ we can find a point ξ such that both $\frac{1}{2}\xi$ and $\frac{1}{2}(\xi+h)$ are in Q. For if $\gamma(t)$ is the characteristic function of Q, then the measure of the $\xi \in (0, h)$ with $\frac{1}{2}\xi \in Q$ is

$$\int_0^h \gamma(\tfrac{1}{2}u)\,du = 2\int_0^{h/2} \gamma(v)\,dv \simeq h,$$

the measure of the set of $\xi \in (0, h)$ such that $\frac{1}{2}(\xi+h) \in Q$ is

$$\int_0^h \gamma\big(\tfrac{1}{2}(u+h)\big)\,du = 2\int_{h/2}^h \gamma(v)\,dv \simeq h,$$

and the existence of the required ξ follows.

Now, (4.4) applied to the points $x = \frac{1}{2}\xi$ $(t = \frac{1}{2}\xi)$ and $x = \frac{1}{2}(\xi+h)$ $(t = \frac{1}{2}(h-\xi))$ gives

$$\Phi(\xi) - \Phi(0) = o(\xi^{r+1}) = o(h^{r+1}),$$

$$\Phi(h) - \Phi(\xi) = o\left\{\left(\frac{h-\xi}{2}\right)^{r+1}\right\} = o(h^{r+1}).$$

Hence $\Phi(h) - \Phi(0) = o(h^{r+1})$ and (4.5) follows for $h \to +0$. The case $h \to -0$ is treated similarly.

2. *r — even.*

At each point x where $\tilde{f}_r$ exists we now have

$$\int_0^t [f(x+s)-f(x-s)]\,ds = o(t^{r+1}),$$

or

$$\Phi(x+t)+\Phi(x-t)-2\Phi(x) = o(t^{r+1}), \tag{4.6}$$

where Φ is the indefinite integral of f. This implies, as we will show, that

$$\Phi(x+2h)-2\Phi(x+h)+\Phi(x) = o(h^{r+1}) \qquad (h\to 0) \tag{4.7}$$

at almost all points of E. For let Q be a set of positive measure in which (4.6) holds uniformly, and let x_0 be a point of density of Q. Without loss of generality we may suppose that $x_0 = 0$. Let h be, say, positive and sufficiently small. Then, arguing as in the preceding case, we can show that there is a ξ in $(0, h)$ such that the midpoints of the intervals $(0,\xi)$, (ξ, h) and $(\xi, 2h)$ are all in Q. We write

$$R_1 = \Phi(\xi)+\Phi(0)-2\Phi\left(\frac{\xi}{2}\right) = o(\xi^{r+1}) = o(h^{r+1}),$$

$$R_2 = \Phi(\xi)+\Phi(h)-2\Phi\left(\frac{\xi+h}{2}\right) = o\left\{\left(\frac{h-\xi}{2}\right)^{r+1}\right\} = o(h^{r+1}),$$

$$R_3 = \Phi(\xi)+\Phi(2h)-2\Phi\left(\frac{\xi+2h}{2}\right) = o\left\{\left(\frac{2h-\xi}{2}\right)^{r+1}\right\} = o(h^{r+1}),$$

and so also $R_1-2R_2+R_3 = o(h^{r+1})$. But

$$\begin{aligned}R_1-2R_2+R_3 &= \Phi(0)-2\Phi(h)+\Phi(2h)-2\left[\Phi\left(\frac{\xi}{2}\right)-2\Phi\left(\frac{\xi+h}{2}\right)+\Phi\left(\frac{\xi+2h}{2}\right)\right]\\ &= \Phi(0)-2\Phi(h)+\Phi(2h)+o(h^{r+1}),\end{aligned}$$

and (4.7) with $x = x_0 = 0$ follows. Hence (4.7) is valid almost everywhere in E.

At each point where (4.7) holds with $r > 0$ we also have

$$\Phi(x+h)-\Phi(x) = o(h^{r+1}), \tag{4.8}$$

by a familiar argument. Suppose for example that $x = 0$ and $\Phi(0) = 0$, and suppose that the left-hand side of (4.7) does not exceed $\varepsilon|h|^{r+1}$ for $|h| \leqslant \eta$. Multiplying the inequalities

$$\left|\Phi(h)-2\Phi\left(\frac{h}{2}\right)\right| \leqslant \varepsilon\left(\frac{|h|}{2}\right)^{r+1}, \quad \left|\Phi\left(\frac{h}{2}\right)-2\Phi\left(\frac{h}{2^2}\right)\right| \leqslant \varepsilon\left(\frac{|h|}{2^2}\right)^{r+1},$$
$$\left|\Phi\left(\frac{h}{2^2}\right)-2\Phi\left(\frac{h}{2^3}\right)\right| \leqslant \varepsilon\left(\frac{|h|}{2^3}\right)^{r+1}, \ldots$$

by $1, 2, 2^2, \ldots$ we obtain by adding the first n of them

$$\left|\Phi(h)-2^n\Phi\left(\frac{h}{2^n}\right)\right| \leqslant \varepsilon|h|^{r+1}. \tag{4.9}$$

But the hypothesis (4.2) at $x = 0$ implies that $\Phi(t) = o(t^r) = o(t)$, and so, making $n\to\infty$ in (4.9) we obtain $|\Phi(h)| \leqslant \varepsilon|h|^{r+1}$ for $|h| \leqslant \eta$. This proves (4.8) for almost all x in E and completes the proof of Theorem 1.

5. THEOREM 2. *Suppose that $f_{(1)}(x), f_{(2)}(x), \ldots, f_{(r-1)}(x)$ exist in a set E and that at each $x \in E$ the integral*

$$-\frac{1}{\pi}\int_\varepsilon^\infty \frac{\delta_r(x,t)}{t}\,dt \tag{5.1}$$

remains bounded as $\varepsilon\to+0$. Then $\tilde{f}_r(x)$ exists almost everywhere in E. In particular, the integral (5.1) *tends to a limit almost everywhere in E as $\varepsilon\to+0$.*

An argument parallel to the proof of the necessity of the condition in Theorem 1 shows that almost everywhere in E the integral Φ of f satisfies the condition

$$\Phi(x+t) = \Phi(x)+\Phi_{(1)}(x)t+\ldots+\Phi_{(r)}(x)\frac{t^r}{r!}+O(t^{r+1}),$$

and it is known that this implies the existence of $\Phi_{(r+1)}$ almost everywhere in E (see [1] or [2]). It remains to apply Theorem 1.

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