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A representation theorem for Marczewski's algebras

by

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I. Let us consider an arbitrary non-empty set A. Let $\boldsymbol{A}$ be the class of A-valued functions of finitely many variables running over A such that 1° the functions defined by the formula $f(x_1, \ldots, x_n) = x_k$ ($n = 1, 2, \ldots$; $k = 1, 2, \ldots, n$) belong to $\boldsymbol{A}$, 2° $\boldsymbol{A}$ is closed with respect to the superposition of functions. The system $\mathfrak{A} = \langle A, \boldsymbol{A}\rangle$ will be called an *algebra*. The properties of the class $\boldsymbol{A}$ are given in papers [1] and [2].

By $\boldsymbol{A}^{(0)}$ we shall denote the class of all values of constant functions belonging to $\boldsymbol{A}$. Further, by $\boldsymbol{A}^{(n)}$ ($n \leqslant 1$) we shall denote the class of all functions of n variables belonging to $\boldsymbol{A}$. If $1 \leqslant k \leqslant n$, then $\boldsymbol{A}^{(n,k)}$ will denote the subclass of $\boldsymbol{A}^{(n)}$ containing all functions depending on at most k variables, i. e. $f \in \boldsymbol{A}^{(n,k)}$ if there is a function $g \in \boldsymbol{A}^{(k)}$ such that $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_k})$ for a system of indices $i_1, \ldots, i_k$ and for every $x_1, \ldots, x_n \in A$. By $\boldsymbol{A}^{(n,0)}$ we shall denote the subclass of $\boldsymbol{A}^{(n)}$ containing all constant functions.

Let $f, g \in \boldsymbol{A}^{(n)}$ ($n \geqslant 1$). We say that the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

depends on the variable x_j ($1 \leqslant j \leqslant n$) if there exists a system $a_1, \ldots, a_n, a'_j$ of elements belonging to A for which

$$f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) = g(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$f(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n) \neq g(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_n)\,.$$

An algebra $\mathfrak{A}$ is called a *Marczewski algebra* if for every pair of integers j, n ($1 \leqslant j \leqslant n$) and for every pair of functions $f, g \in \boldsymbol{A}^{(n)}$ for which the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{1}$$

depends on x_j there exists a function $h \in \boldsymbol{A}^{(n-1)}$ such that equality (1) is equivalent to the equality

$$x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\,.$$

The study of these algebras was initiated by E. Marczewski (see [3]), who proved that the notion of independence in this class of algebras has the property of linear independence.

Now we shall give some examples of Marczewski algebras.

1. Let A be a linear space over a field $\mathcal{K}$ and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all functions f defined as

$$f(x_1, \dots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \dots, \lambda_n \in \mathcal{K}$ and $a \in A_0$, then $\mathfrak{A} = \langle A, \boldsymbol{A} \rangle$ is a Marczewski algebra. In this case we have the relations

$$\boldsymbol{A}^{(0)} \neq 0, \quad \boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)} \quad \text{for} \quad n \geqslant 2.$$

This example is due to E. Marczewski.

2. Let A be a linear space over a field $\mathcal{K}$ and let A_0 be a linear subspace of A. If $\boldsymbol{A}$ is the class of all functions f defined as

$$f(x_1, \dots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \dots, \lambda_n \in \mathcal{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$, then $\mathfrak{A} = \langle A, \boldsymbol{A} \rangle$ is a Marczewski algebra. In this case we have the relations

$$\boldsymbol{A}^{(0)} = 0, \quad \boldsymbol{A}^{(n)} \neq \boldsymbol{A}^{(n,1)} \quad \text{for} \quad n \geqslant 3.$$

(Let us remark that $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(2,1)}$ in the case where the field $\mathcal{K}$ contains two elements only.)

3. Let $\mathcal{G}$ be a group of transformations of a non-empty set A. We suppose that every transformation that is not the identity has at most one fixed point in A.

We say that a subset $B \subset A$ is *normal with respect to the group* $\mathcal{G}$ if B contains fixed points of all transformations that are not the identity belonging to $\mathcal{G}$ and $g(B) \subset B$ for every $g \in \mathcal{G}$. We remark that if transformations belonging to $\mathcal{G}$ have no fixed point, then the empty set is normal with respect to $\mathcal{G}$.

Let A_0 be a subset of A normal with respect to $\mathcal{G}$. If $\boldsymbol{A}$ is the class of all functions f defined as

$$f(x_1, \dots, x_n) = g(x_j) \quad (1 \leqslant j \leqslant n),$$

or

$$f(x_1, \dots, x_n) = a,$$

where $g \in \mathcal{G}$ and $a \in A_0$, then $\mathfrak{A} = \langle A, \boldsymbol{A} \rangle$ is a Marczewski algebra. In this case we have the relation

$$\boldsymbol{A}^{(n)} = \boldsymbol{A}^{(n,1)} \quad \text{for} \quad n \geqslant 1.$$

In the present paper we shall prove the following representation theorem, which is an answer to a problem raised by E. Marczewski.

THEOREM. *Let $\mathfrak{A} = \langle A, \boldsymbol{A} \rangle$ be a Marczewski algebra.*

(i) *If $\boldsymbol{A}^{(0)} \neq 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and further, there exists a linear subspace A_0 of A such that $\boldsymbol{A}$ is the class of all functions f defined as*

$$f(x_1, \dots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \dots, \lambda_n \in \mathcal{K}$ and $a \in A_0$.

(ii) *If $\boldsymbol{A}^{(0)} = 0$ and $\boldsymbol{A}^{(3)} \neq \boldsymbol{A}^{(3,1)}$, then there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$ and further, there exists a linear subspace A_0 of A such that $\boldsymbol{A}$ is the class of all functions f defined as*

$$f(x_1, \dots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \dots, \lambda_n \in \mathcal{K}$, $\sum_{k=1}^{n} \lambda_k = 1$ and $a \in A_0$.

(iii) *If $\boldsymbol{A}^{(3)} = \boldsymbol{A}^{(3,1)}$, then there is a group $\mathcal{G}$ of transformations of the set A such that every transformation that is not the identity has at most one fixed point in A. Moreover, there is a subset $A_0 \subset A$ normal with respect to the group $\mathcal{G}$ such that $\boldsymbol{A}$ is the class of all functions f defined as*

$$f(x_1, \dots, x_n) = g(x_j) \quad (1 \leqslant j \leqslant n),$$

or

$$f(x_1, \dots, x_n) = a,$$

where $g \in \mathcal{G}$ and $a \in A_0$.

II. Before proving the Theorem we shall prove some lemmas. We assume that all algebras considered in this part of the paper are Marczewski algebras.

For any $f \in \boldsymbol{A}$ we denote be $\hat{f}$ the function belonging to $\boldsymbol{A}^{(1)}$ defined as

$$\hat{f}(x) = f(x, \dots, x). \tag{2}$$

$\tilde{\boldsymbol{A}}^{(n)}$ $(n \geqslant 1)$ will denote the subclass of $\boldsymbol{A}^{(n)}$ containing all functions f for which $\hat{f}(x) = x$. $\tilde{\boldsymbol{A}}^{(n,k)}$ will denote the intersection $\tilde{\boldsymbol{A}}^{(n)} \cap \boldsymbol{A}^{(n,k)}$. The following assertion is a direct consequence of the definition of Marczewski algebras: if $f, g \in \widetilde{A}^{(n)}$ $(n \geqslant 2)$ and if the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

depends on x_j $(1 \leqslant j \leqslant n)$, then there is a function $h \in \widetilde{A}^{(n-1)}$ such that the last equality is equivalent to the equality

$$x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

LEMMA 1. *Let $f, g \in A^{(n)}$ and $1 \leqslant j \leqslant n$. If there exist two functions $h_1, h_2 \in A^{(n-1)}$ such that $h_1 \neq h_2$ and the equality $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ holds for $x_j = h_1(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$, $x_j = h_2(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and for each $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \in A$, then $f = g$.*

Proof. Let us suppose that the equality

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{3}$$

depends on x_j. Then there is a function $h \in A^{(n-1)}$ such that equality (3) is equivalent to the equality $x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Hence follows the equality $h_1 = h = h_2$, which is impossible. Thus equality (3) does not depend on x_j, which implies the assertion of our Lemma.

LEMMA 2. *If $A^{(n)} \neq A^{(n,1)}$ for an index $n \geqslant 3$, then $\widetilde{A}^{(n)} \neq \widetilde{A}^{(n,1)}$.*

Proof. First we shall prove that there is a function $g \in A^{(n)} \setminus A^{(n,1)}$ for which $\hat{g}$ non $\in A^{(1,0)}$. Let $f \in A^{(n)} \setminus A^{(n,1)}$. If $\hat{f}$ non $\in A^{(1,0)}$, then we put $g = f$. Now let us assume that $\hat{f} \in A^{(1,0)}$.

If $f \in A^{(n,2)}$, then there is a function $f_0 \in A^{(2)}$ such that $f(x_1, \ldots, x_n) = f_0(x_i, x_j)$ for a pair of indices i, j $(1 \leqslant i, j \leqslant n)$ and for any $x_1, \ldots, x_n \in A$. Moreover, the function f_0 depends on both variables. Consequently, the equality

$$f_0(x_1, x_2) = a,$$

where $a = \hat{f}(x)$, depends on x_1. Then there is a function $h_0 \in A^{(1)}$, such that the last equality is equivalent to the equality $x_1 = h_0(x_2)$. From the equality

$$f_0(x, x) = \hat{f}(x) = a \tag{4}$$

we find $\hat{h}_0(x) = x$. Hence we get the inequality

$$f_0(x_1, x_2) \neq a \quad \text{for} \quad x_1 \neq x_2. \tag{5}$$

Put $g(x_1, \ldots, x_n) = f_0(x_1, f_0(x_2, x_3))$. In virtue of (4) and (5) we have the formulas

$$\hat{g}(a) = f_0(a, f_0(a, a)) = f_0(a, a) = a,$$

$$\hat{g}(x) = f_0(x, f_0(x, x)) = f_0(x, a) \neq a \quad \text{for} \quad x \neq a,$$

which imply $\hat{g}$ non $\in A^{(1,0)}$. Further, according to (4) and (5), the following formulas are true:

$$g(x_1, a, a, \ldots, a) = f_0(x_1, f_0(a, a)) = f_0(x_1, a) \neq a \quad \text{for} \quad x_1 \neq a,$$

$$g(a, a, \ldots, a) = f_0(a, f_0(a, a)) = f_0(a, a) = a,$$

$$g(a, x_2, a, \ldots, a) = f_0(a, f_0(x_2, a)) = f_0(a, x_2) \neq a \quad \text{for} \quad x_2 \neq a.$$

Hence it follows that $g(x_1, \ldots, x_n)$ depends on x_1 and x_2. Thus $g \in A^{(n)} \setminus A^{(n,1)}$.

If f non $\in A^{(n,2)}$, there is a triplet of indices i, j, k $(1 \leqslant i, j, k \leqslant n)$ such that $f(x_1, \ldots, x_n)$ depends on x_i, x_j and x_k. Consequently, the equality

$$f(x_1, \ldots, x_n) = a,$$

where $a = \hat{f}(x)$, depends on x_i, x_j and x_k. Then there are functions $h_i, h_j, h_k \in A^{(n-1)}$ such that the last equality is equivalent to each of the equalities

$$x_i = h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

$$x_j = h_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

$$x_k = h_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

Hence it follows that $h_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ depends on x_i and x_j. Thus, setting $g(x_1, \ldots, x_n) = h_k(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$, we have $g \in A^{(n)} \setminus A^{(n,1)}$. Since $f(x, \ldots, x) = a$, we have the equality $x = h_k(x, \ldots, x)$, which implies $\hat{g}(x) = x$ and, consequently, $\hat{g}$ non $\in A^{(1,0)}$. Our statement is thus proved.

Now let us consider a function $g \in A^{(n)} \setminus A^{(n,1)}$ for which $\hat{g}$ non $\in A^{(1,0)}$. Then the equality

$$\hat{g}(x_1) = x_2 \tag{6}$$

depends on x_1 and, consequently, there is a function $h \in A^{(1)}$ such that equality (6) is equivalent to the equality

$$x_1 = h(x_2).$$

Hence in particular we obtain the equalities

$$h(\hat{g}(x)) = x = \hat{g}(h(x)). \tag{7}$$

Setting $g_0(x_1, \ldots, x_n) = h(g(x_1, \ldots, x_n))$ we have g_0 non $\in A^{(n,1)}$ and, according to (7),

$$\hat{g}_0(x) = h(\hat{g}(x)) = x,$$

which implies that $g_0 \in \widetilde{A}^{(n)} \setminus \widetilde{A}^{(n,1)}$. The Lemma is thus proved.

We say that a function $s \in \widetilde{A}^{(3)}$ is *quasi-symmetric* if

$$s(x_1, x_2, x_1) = s(x_2, x_1, x_1) = x_2 \tag{8}$$

for each $x_1, x_2 \in A$. Every quasi-symmetric function belongs to $\widetilde{A}^{(3)} \setminus \widetilde{A}^{(3,2)}$. In fact, according to (8), we have the equalities

$$s(x_1, x_2, x_1) = x_2, \quad s(x_1, x_2, x_2) = x_1,$$

which imply that $s(x_1, x_2, x_3)$ depends on x_1, x_2, x_3 and consequently $s \notin A^{(3,2)}$.

LEMMA 3. *If s is a quasi-symmetric function, then for all $x_1, x_2, x_3, x_4 \in A$ the following equalities are true:*

$$s(x_1, x_2, x_3) = s(x_2, x_1, x_3), \tag{9}$$

$$s\big(s(x_1, x_2, x_3), x_4, x_3\big) = s\big(x_1, s(x_2, x_4, x_3), x_3\big), \tag{10}$$

$$f\big(s(x_1, x_2, x_3), x_3\big) = s\big(f(x_1, x_3), f(x_2, x_3), x_3\big) \quad \textit{for any} \quad f \in \widetilde{A}^{(2)}, \tag{11}$$

$$f(x_1, x_2, x_3) = s\big(f(x_1, x_1, x_3), f(x_1, x_2, x_1), x_1\big) \quad \textit{for any} \quad f \in \widetilde{A}^{(3)}. \tag{12}$$

Proof. Replacing x_1 by x_2 and x_2 by x_1 in formula (8) we obtain the equality

$$s(x_1, x_2, x_2) = s(x_2, x_1, x_2) = x_1.$$

Hence and from (8) it follows that equality (9) holds for all x_1, x_2, $x_3 = x_1$ and $x_3 = x_2$. Thus, in view of Lemma 1, equality (9) holds for all x_1, x_2 and x_3.

Taking into account formula (8) we have the equalities

$$s\big(s(x_1, x_2, x_2), x_4, x_2\big) = s(x_1, x_4, x_2),$$

$$s\big(x_1, s(x_2, x_4, x_2), x_2\big) = s(x_1, x_4, x_2),$$

$$s\big(s(x_1, x_2, x_4), x_4, x_4\big) = s(x_1, x_2, x_4),$$

$$s\big(x_1, s(x_2, x_4, x_4), x_4\big) = s(x_1, x_2, x_4),$$

which imply that equality (10) holds for all x_1, x_2, x_4, $x_3 = x_2$ and $x_3 = x_4$. Hence, in virtue of Lemma 1, we get equality (10) for all x_1, x_2, x_3 and x_4.

Further, from the equalities

$$f\big(s(x_1, x_2, x_1), x_1\big) = f(x_2, x_1),$$

$$s\big(f(x_1, x_1), f(x_2, x_1), x_1\big) = s\big(x_1, f(x_2, x_1), x_1\big) = f(x_2, x_1),$$

$$f\big(s(x_1, x_2, x_2), x_2\big) = f(x_1, x_2),$$

$$s\big(f(x_1, x_2), f(x_2, x_2), x_2\big) = s\big(f(x_1, x_2), x_2, x_2\big) = f(x_1, x_2),$$

where $f \in \widetilde{A}^{(2)}$, it follows that equality (11) holds for all x_1, x_2, $x_3 = x_1$ and $x_3 = x_2$, which implies, in view of Lemma 1, that equality (11) holds for all x_1, x_2 and x_3.

Finally, taking into account formula (8), we have for every $f \in \widetilde{A}^{(3)}$ the following equalities:

$$s\big(f(x_3, x_3, x_3), f(x_3, x_2, x_3), x_3\big) = s\big(x_3, f(x_3, x_2, x_3), x_3\big) = f(x_3, x_2, x_3),$$

$$s\big(f(x_2, x_2, x_3), f(x_2, x_2, x_2), x_2\big) = s\big(f(x_2, x_2, x_3), x_2, x_2\big) = f(x_2, x_2, x_3).$$

Hence it follows that equality (12) holds for all x_2, x_3, $x_1 = x_2$ and $x_1 = x_3$, which implies, in virtue of Lemma 1, that equality (12) holds for all x_1, x_2 and x_3. The Lemma is thus proved.

LEMMA 4. *If $A^{(3)} \neq A^{(3,1)}$, then there is a quasi-symmetric function.*

Proof. By Lemma 2 we may assume that $\widetilde{A}^{(3)} \neq \widetilde{A}^{(3,1)}$. First we suppose that $\widetilde{A}^{(2)} \neq \widetilde{A}^{(2,1)}$. Let $f \in \widetilde{A}^{(2)} \setminus \widetilde{A}^{(2,1)}$. Then the equality $f(x_1, x_2) = x_3$ depends on x_1 and x_2. There are then two functions $g_1, g_2 \in \widetilde{A}^{(2)}$ such that the last equality is equivalent to each of the equalities

$$x_1 = g_1(x_2, x_3), \quad x_2 = g_2(x_1, x_3).$$

Hence, in particular, we obtain the formulas

$$f\big(x_1, g_2(x_1, x_2)\big) = x_2, \quad f\big(g_1(x_1, x_2), x_1\big) = x_2. \tag{13}$$

Setting $s(x_1, x_2, x_3) = f\big(g_1(x_3, x_1), g_2(x_3, x_2)\big)$ we have the equality $\hat{s}(x) = f\big(g_1(x, x), g_2(x, x)\big) = f(x, x) = x$. Thus $s \in \widetilde{A}^{(3)}$. Moreover, from (13) it follows that

$$s(x_1, x_2, x_1) = f\big(g_1(x_1, x_1), g_2(x_1, x_2)\big) = f\big(x_1, g_2(x_1, x_2)\big) = x_2,$$

$$s(x_2, x_1, x_1) = f\big(g_1(x_1, x_2), g_2(x_1, x_1)\big) = f\big(g_1(x_1, x_2), x_1\big) = x_2.$$

Consequently, s is a quasi-symmetric function.

Now let us suppose that

$$\widetilde{A}^{(2)} = \widetilde{A}^{(2,1)}. \tag{14}$$

We shall prove that every function belonging to $\widetilde{A}^{(3)} \setminus \widetilde{A}^{(3,1)}$ is quasi-symmetric. To prove this it suffices to show that for every $f \in \widetilde{A}^{(3)} \setminus \widetilde{A}^{(3,1)}$ we have the equality

$$f(x_1, x_1, x_3) = x_3 \tag{15}$$

for any $x_1, x_3 \in A$. Contrary to this statement let us suppose that $f(x_1, x_1, x_3) \neq x_3$ for a pair $x_1, x_3 \in A$. Hence and from (14) we obtain the equality

$$f(x_1, x_1, x_3) = x_1. \tag{16}$$

Since $f \notin \tilde{A}^{(3,1)}$, there is a triplet x_1, x_2, x_3 for which $f(x_1, x_2, x_3) \neq x_1$. Hence and from (16) it follows that the equality

$$f(x_1, x_2, x_3) = x_1$$

depends on x_2. Thus there is a function $g \in \tilde{A}^{(2)}$ such that the last equality is equivalent to the equality $x_2 = g(x_1, x_3)$. By formula (14), $g(x_1, x_3) = x_1$ or x_3, which implies, in virtue of (16), $g(x_1, x_3) = x_1$. Consequently, $f(x_1, x_2, x_3) \neq x_1$ for $x_2 \neq x_1$. Therefore, taking into account equality (14), we have

$$f(x_1, x_2, x_1) = x_2. \tag{17}$$

Since $f \notin A^{(3,1)}$, there is a triplet $x_1, x_2, x_3 \in A$ for which $f(x_1, x_2, x_3) \neq x_2$. Hence and from (16) it follows that the equality

$$f(x_1, x_2, x_3) = x_2$$

depends on x_2. Thus there is a function $h \in \tilde{A}^{(2)}$ such that the last equality is equivalent to the equality $x_2 = h(x_1, x_3)$. By formula (14), $h(x_1, x_3) = x_1$ or x_3, which implies, in virtue of (16), $h(x_1, x_3) = x_1$. Consequently $f(x_1, x_2, x_1) \neq x_2$ for $x_2 \neq x_1$, which contradicts equality (17). Formula (15) and, consequently, the Lemma are thus proved.

In the sequel we hall denote by $\mathcal{K}$ the class $\tilde{A}^{(2)}$. Elements of $\mathcal{K}$ will be denoted by small Greek letters: $\lambda, \mu, \nu, \ldots$

LEMMA 5. *If $A^{(3)} \neq A^{(3,1)}$, then $\mathcal{K}$ is a field with respect to the operations*

$$(\lambda+\mu)(x_1, x_2) = s\big(\lambda(x_1, x_2), \mu(x_1, x_2), x_2\big), \tag{18}$$

$$(\lambda\cdot\mu)(x_1, x_2) = \lambda\big(\mu(x_1, x_2), x_2\big), \tag{19}$$

where s is a quasi-symmetric function.

Proof. First of all we remark that the existence of a quasi-symmetric function follows from Lemma 4.

We define the zero-element and the unit element by following formulas:

$$0(x_1, x_2) = x_2, \quad 1(x_1, x_2) = x_1.$$

Obviously, $0 \neq 1$. From (8) and (18) it follows for every $\lambda \in \mathcal{K}$ that

$$(\lambda+0)(x_1, x_2) = s\big(\lambda(x_1, x_2), x_2, x_2\big) = \lambda(x_1, x_2).$$

Thus $\lambda+0 = \lambda$ for every $\lambda \in \mathcal{K}$. Further

$$(\lambda\cdot 1)(x_1, x_2) = \lambda(x_1, x_2), \quad (1\cdot\lambda)(x_1, x_2) = \lambda(x_1, x_2),$$

which implies $\lambda\cdot 1 = 1\cdot\lambda = \lambda$ for every $\lambda \in \mathcal{K}$.

The following formula is a direct consequence of definition (19):

$$\lambda\cdot(\mu\cdot\nu) = (\lambda\cdot\mu)\cdot\nu \quad (\lambda, \mu, \nu \in \mathcal{K}).$$

If $\lambda \neq 0$, i. e. $\lambda(x_1, x_2) \neq x_2$ for a pair $x_1, x_2 \in A$, then the equality

$$\lambda(x_1, x_2) = x_3$$

depends on x_1. Thus there is a function $\lambda^{-1} \in \mathcal{K}$ such that the last equality is equivalent to the equality $x_1 = \lambda^{-1}(x_3, x_2)$. Hence we obtain the equalities

$$\lambda\big(\lambda^{-1}(x_1, x_2), x_2\big) = x_1, \quad \lambda^{-1}\big(\lambda(x_1, x_2), x_2\big) = x_1,$$

which imply $\lambda\cdot\lambda^{-1} = \lambda^{-1}\cdot\lambda = 1$.

Taking into account assertions (9), (10) and (11) of Lemma 3, we have the equalities

$$(\lambda+\mu)(x_1, x_2) = s\big(\lambda(x_1, x_2), \mu(x_1, x_2), x_2\big) = s\big(\mu(x_1, x_2), \lambda(x_1, x_2), x_2\big) = (\mu+\lambda)(x_1, x_2),$$

$$\big((\lambda+\mu)+\nu\big)(x_1, x_2) = s\Big(s\big(\lambda(x_1, x_2), \mu(x_1, x_2), x_2\big), \nu(x_1, x_2), x_2\Big) = s\Big(\lambda(x_1, x_2), s\big(\mu(x_1, x_2), \nu(x_1, x_2), x_2\big), x_2\Big) = \big(\lambda+(\mu+\nu)\big)(x_1, x_2),$$

$$\big(\lambda\cdot(\mu+\nu)\big)(x_1, x_2) = \lambda\Big(s\big(\mu(x_1, x_2), \nu(x_1, x_2), x_2\big), x_2\Big) = s\Big(\lambda\big(\mu(x_1, x_2), x_2\big), \lambda\big(\nu(x_1, x_2), x_2\big), x_2\Big) = (\lambda\cdot\mu+\lambda\cdot\nu)(x_1, x_2),$$

which imply

$$\lambda+\mu = \mu+\lambda, \quad (\lambda+\mu)+\nu = \lambda+(\mu+\nu), \quad \lambda\cdot(\mu+\nu) = \lambda\cdot\mu+\lambda\cdot\nu$$

for every $\lambda, \mu, \nu \in \mathcal{K}$.

Further, the following equalities are a direct consequence of definitions (18) and (19)

$$\big((\mu+\nu)\cdot\lambda\big)(x_1, x_2) = s\Big(\mu\big(\lambda(x_1, x_2), x_2\big), \nu\big(\lambda(x_1, x_2), x_2\big), x_2\Big),$$

$$(\mu\cdot\lambda+\nu\cdot\lambda)(x_1, x_2) = s\Big(\mu\big(\lambda(x_1, x_2), x_2\big), \nu\big(\lambda(x_1, x_2), x_2\big), x_2\Big).$$

Thus $(\mu+\nu)\cdot\lambda = \mu\cdot\lambda+\nu\cdot\lambda$ for every $\lambda, \mu, \nu \in \mathcal{K}$.

Since, by formula (8), $s\big(\lambda(x_1, x_1), x_3, x_1\big) = s(x_1, x_3, x_1) = x_3$ for every $\lambda \in \mathcal{K}$, the equality

$$s\big(\lambda(x_1, x_2), x_3, x_2\big) = x_2$$

depends on x_3. Thus there is a function $-\lambda \in \mathcal{K}$ such that the last equality is equivalent to the equality $x_3 = -\lambda(x_1, x_2)$. Hence we get the equality

$$s\big(\lambda(x_1, x_2), -\lambda(x_1, x_2), x_2\big) = x_2,$$

which implies $\lambda+(-\lambda) = 0$ for every $\lambda \in \mathcal{K}$. The Lemma is thus proved.

LEMMA 6. *If $A^{(3)} \neq A^{(3,1)}$, then A is a linear space over $\mathcal{K}$ with respect to the operations*

$$x+y = s(x, y, \theta) \quad (x, y \in A),$$
$$\lambda \cdot x = \lambda(x, \theta) \quad (\lambda \in \mathcal{K},\ x \in A),$$

where θ is an element of $A^{(0)}$ if $A^{(0)} \neq 0$ and is an element of A if $A^{(0)} = 0$.

Proof. The element θ is the zero-element of A. In fact, according to (8), $x+\theta = s(x, \theta, \theta) = x$ for every $x \in A$.

Further we have, in virtue of Lemma 3, the following equalities:

$$x+y = s(x, y, \theta) = s(y, x, \theta) = y+x,$$
$$(x+y)+z = s\big(s(x, y, \theta), z, \theta\big) = s\big(x, s(y, z, \theta), \theta\big) = x+(y+z),$$
$$\lambda(x+y) = \lambda\big(s(x, y, \theta), \theta\big) = s\big(\lambda(x, \theta), \lambda(y, \theta), \theta\big) = \lambda \cdot x + \lambda y,$$

for any $x, y, z \in A$ and $\lambda \in \mathcal{K}$.

Moreover, the equalities

$$\lambda(\mu \cdot x) = \lambda\big(\mu(x, \theta), \theta\big) = (\lambda \cdot \mu)x,$$
$$1 \cdot x = x,$$
$$(\lambda+\mu)x = s\big(\lambda(x, \theta), \mu(x, \theta), \theta\big) = \lambda \cdot x + \mu \cdot x$$

are true for any $x \in A$ and $\lambda, \mu \in \mathcal{K}$. Hence, setting $-x = (-1)x$, we get the equality $x+(-x) = 0 \cdot x = \theta$. The Lemma is thus proved.

LEMMA 7. *Let $A^{(3)} \neq A^{(3,1)}$ and let the addition in $\mathcal{K}$ be defined by a function s. If the field $\mathcal{K}$ has the characteristic 2, then s is a symmetric function, i. e.*

$$s(x_1, x_2, x_3) = s(x_{i_1}, x_{i_2}, x_{i_3})$$

for every permutation i_1, i_2, i_3 of indices 1, 2, 3. Moreover, for every $f \in A^{(n)}$ $(n \geqslant 2)$ and every $x_1, \ldots, x_{n+1} \in A$ the equality

$$s\big(f(x_1, \ldots, x_n), s(x_{n-1}, x_n, x_{n+1}), x_{n+1}\big) = s\big(f(x_1, \ldots, x_n), x_{n-1}, x_n\big) \tag{20}$$

is true.

Proof. To prove the symmetry of s, in view of Lemma 3 (formula (9)), it suffices to show that for every triplet $x_1, x_2, x_3 \in A$ we have the equality

$$s(x_1, x_2, x_3) = s(x_1, x_3, x_2).$$

In other words, according to Lemma 3, it suffices to show that the function $s_0(x_1, x_2, x_3) = s(x_3, x_1, x_2)$ is quasi-symmetric. We have, according to the definition of addition in $\mathcal{K}$, the equality

$$s_0(x_1, x_2, x_1) = s(x_1, x_1, x_2) = (1+1)(x_1, x_2) = 0(x_1, x_2) = x_2,$$

and, according to (8), the equality

$$s_0(x_2, x_1, x_1) = s(x_1, x_2, x_1) = x_2,$$

which imply the quasi-symmetry of s_0 and, consequently, the symmetry of s.

From formula (8) and the symmetry of s we get for any $f \in A^{(n)}$ the equalities

$$s\big(f(x_1, \ldots, x_n), s(x_{n-1}, x_n, x_{n-1}), x_{n-1}\big) = s\big(f(x_1, \ldots, x_n), x_n, x_{n-1}\big) = s\big(f(x_1, \ldots, x_n), x_{n-1}, x_n\big),$$

$$s\big(f(x_1, \ldots, x_n), s(x_{n-1}, x_n, x_n), x_n\big) = s\big(f(x_1, \ldots, x_n), x_{n-1}, x_n\big),$$

which imply that equality (20) holds for every $x_1, \ldots, x_n$, $x_{n+1} = x_{n-1}$, $x_{n+1} = x_n$. Consequently, in view of Lemma 1, equality (20) holds for every $x_1, \ldots, x_{n+1}$. The Lemma is thus proved.

LEMMA 8. *If $A^{(3)} \neq A^{(3,1)}$, then all functions f defined as*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\lambda_1, \ldots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$, belong to $\tilde{A}^{(n)}$ $(n = 1, 2, \ldots)$.

Proof. We prove our Lemma by induction with respect to n. For $n = 1$ the assertion is obvious. To prove our assertion for $n = 2$ it suffices to show that for every $\lambda \in \mathcal{K}$ formula

$$\lambda(x_1, x_2) = \lambda \cdot x_1 + (1-\lambda)x_2 \tag{21}$$

is true. Setting $f(x_1, x_2, x_3) = \lambda(x_2, x_3)$ in formula (12) of Lemma 3 we infer that

$$\lambda(x_2, x_3) = s\big(\lambda(x_1, x_3), \lambda(x_2, x_1), x_1\big) \tag{22}$$

for every $x_1, x_2, x_3 \in A$. Replacing in the last formula x_2 and x_3 by x_1, x_1 by x_2 we obtain the equality

$$x_1 = s\big(\lambda(x_2, x_1), \lambda(x_1, x_2), x_2\big).$$

Hence, according to the definition of the unit element and addition in $\mathcal{K}$, we have the equality

$$\lambda(x_2, x_1) = (1-\lambda)(x_1, x_2).$$

Setting $x_1 = \theta$ in formula (22) and replacing x_2 by x_1 and x_3 by x_2 we infer that

$$\lambda(x_1, x_2) = s\big(\lambda(\theta, x_2), \lambda(x_1, \theta), \theta\big) = \lambda x_1 + (1-\lambda)x_2.$$

Formula (21) is thus proved for every $\lambda \in \mathcal{K}$.

Now let us suppose that $n \geqslant 3$ and that the assertion of our Lemma is true for indices less than n. Let us consider a function

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\sum_{k=1}^{n} \lambda_k = 1$.

First we assume that there is an index k_0 $(1 \leqslant k_0 \leqslant n)$ for which $\lambda_{k_0} \neq 1$. Put

$$g(x_1, x_2) = (1-\lambda_{k_0})x_1 + \lambda_{k_0}x_2,$$

$$h(x_1, \ldots, x_{k_0-1}, x_{k_0+1}, \ldots, x_n) = \sum_{\substack{k=1\\ k\neq k_0}}^{n} \lambda_k(1-\lambda_{k_0})^{-1}x_k.$$

By the induction assumption $g \in \widetilde{A}^{(2)}$ and $h \in \widetilde{A}^{(n-1)}$. It is easy to verify that $f(x_1, \ldots, x_n) = g\big(h(x_1, \ldots, x_{k_0-1}, x_{k_0+1}, \ldots, x_n), x_{k_0}\big)$, which implies $f \in \widetilde{A}^{(n)}$.

Now let us assume that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ and that the field $\mathcal{K}$ has a characteristic different from 2. Since $1 \neq 0$ and $n \cdot 1 = \sum_{k=1}^{n} \lambda_k = 1$, we have the inequality $(n-2)\cdot 1 \neq 0$. Put

$$g_1(x_1, x_2) = 2x_1 + (n-2)x_2,$$

$$g_2(x_1, x_2) = 2^{-1}x_1 + 2^{-1}x_2,$$

$$g_3(x_3, \ldots, x_n) = \sum_{k=3}^{n} (n-2)^{-1}x_k.$$

By the induction assumption, $g_1, g_2 \in \widetilde{A}^{(2)}$ and $g_3 \in \widetilde{A}^{(n-2)}$. Since $f(x_1, \ldots, x_n) = g_1\big(g_2(x_1, x_2), g_3(x_3, \ldots, x_n)\big)$, we have $f \in \widetilde{A}^{(n)}$.

Finally let us assume that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ and that the field $\mathcal{K}$ has the characteristic 2. Since $(n-2)\cdot 1 = n \cdot 1 = 1$, by the induction assumption the function

$$f_0(x_1, \ldots, x_{n-2}) = \sum_{k=1}^{n-2} x_k$$

belongs to $\widetilde{A}^{(n-2)}$. Using Lemma 7 we infer that

$$f(x_1, \ldots, x_n) = f_0(x_1, \ldots, x_{n-2}) + x_{n-1} + x_n$$
$$= s\big(f_0(x_1, \ldots, x_{n-2}), s(x_{n-1}, x_n, \theta), \theta\big) = s\big(f_0(x_1, \ldots, x_{n-2}), x_{n-1}, x_n\big),$$

which implies $f \in \widetilde{A}^{(n)}$. The Lemma is thus proved.

LEMMA 9. *If $A^{(3)} \neq A^{(3,1)}$, then all functions f belonging to $\widetilde{A}^{(n)}$ $(n \geqslant 1)$ are of the form*

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k,$$

where $\lambda_1, \ldots, \lambda_n \in \mathcal{K}$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Proof. We shall prove our Lemma by induction with respect to n. For $n = 1$ the assertion is obvious. For $n = 2$ it follows from formula (21). Now let us suppose that $n \geqslant 3$ and that our assertion is true for indices less than n. Let $f \in \widetilde{A}^{(n)}$. For every k $(1 \leqslant k \leqslant n)$ setting $x_k = x_1$ and $x_j = x_2$ $(j = 1, 2, \ldots, n$ and $j \neq k)$ in $f(x_1, \ldots, x_n)$ we obtain the expression $\nu_k(x_1, x_2)$, where obviously $\nu_k \in \mathcal{K}$.

First let us assume that there exists an index k_0 for which $\nu_{k_0} \neq 1$. Without loss of the generality of our considerations we may suppose that $\nu_n \neq 1$. By Lemma 8 the function g defined by the formula

$$g(x_1, x_2) = (1-\nu_n)^{-1}x_1 - \nu_n(1-\nu_n)^{-1}x_2$$

belongs to $\widetilde{A}^{(2)}$. Putting

$$h(x_1, \ldots, x_n) = g\big(f(x_1, \ldots, x_n), x_n\big) = (1-\nu_n)^{-1}f(x_1, \ldots, x_n) - \nu_n(1-\nu_n)^{-1}x_n, \tag{23}$$

$$h_j(x_1, \ldots, x_{n-1}) = h(x_1, \ldots, x_{n-1}, x_j) \quad (j = 1, 2, \ldots, n-1) \tag{24}$$

we infer that $h \in \widetilde{A}^{(n)}$ and $h_j \in \widetilde{A}^{(n-1)}$ $(j = 1, 2, \ldots, n-1)$. Consequently, by the induction assumption, there are functions $\mu_k^{(j)} \in \mathcal{K}$ $(j, k = 1, 2, \ldots, n-1)$ for which

$$\sum_{k=1}^{n-1} \mu_k^{(j)} = 1 \quad (j = 1, 2, \ldots, n-1)$$

and

$$h_j(x_1, \ldots, x_{n-1}) = \sum_{k=1}^{n-1} \mu_k^{(j)} x_k \quad (j = 1, 2, \ldots, n-1). \tag{25}$$

Setting $x_1 = x_2 = \ldots = x_{k-1} = x_{k+1} = \ldots = x_{n-1} = \theta$ in the last equality for $j \neq k$ $(1 \leqslant j, k \leqslant n-1)$ we obtain, according to (23), (24) and the definition of functions ν_k, the formula

$$\mu_k^{(j)} x_k = (1-\nu_n)^{-1}\nu_k(x_k, \theta) \quad (k \neq j;\ 1 \leqslant j, k \leqslant n-1). \tag{26}$$

Replacing $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1}$ by x_1, x_j by x_2 and x_n by x_3 $(1 \leqslant j \leqslant n-1)$ in $f(x_1, \ldots, x_n)$ we obtain the expression $f_j(x_1, x_2, x_3)$, where $f_j \in \widetilde{A}^{(3)}$. By formula (12) of Lemma 3 we have the equality

$$f_j(x_1, x_2, x_3) = s\big(f_j(x_1, x_1, x_3), f_j(x_1, x_2, x_1), x_1\big) \quad (1 \leqslant j \leqslant n-1).$$

Setting in the last equality $x_1=\theta$, $x_2=x_3=x_j$ and taking into account the definition of functions ν_k we infer that

$$f_j(\theta, x_j, x_j)=\nu_n(x_j,\theta)+\nu_j(x_j,\theta) \quad (1\leqslant j\leqslant n-1). \tag{27}$$

Further, setting $x_1=x_2=\ldots=x_{j-1}=x_{j+1}=\ldots=x_{n-1}=\theta$ in equality (25) we obtain, according to (23), (24) and (27), the formula

$$\begin{aligned}\mu_j^{(j)}x_j &= (1-\nu_n)^{-1}f_j(\theta,x_j,x_j)-\nu_n(1-\nu_n)^{-1}x_j\\ &=(1-\nu_n)^{-1}\big(f_j(\theta,x_j,x_j)-\nu_n(x_j,\theta)\big)=(1-\nu_n)^{-1}\nu_j(x_j,\theta) \quad (j=1,2,\ldots,n-1).\end{aligned}$$

Hence and from (26) it follows that the coefficients $\mu_k^{(j)}$ $(k,j=1,2,\ldots,n-1)$ do not depend on the choice of indices j. Consequently, taking into account formula (24), we may write equality (25) in the form

$$h(x_1,\ldots,x_{n-1},x_j)=\sum_{k=1}^{n-1}\mu_k x_k \quad (j=1,2,\ldots,n-1),$$

where $\sum_{k=1}^{n-1}\mu_k=1$. Hence it follows that the equality

$$h(x_1,\ldots,x_n)=\sum_{k=1}^{n-1}\mu_k x_k$$

holds for any $x_1,\ldots,x_{n-1}$, $x_n=x_1$, $x_n=x_2,\ldots,x_n=x_{n-1}$. Since by assumption $n\geqslant 3$, the last equality, in view of Lemma 1, holds for any $x_1,\ldots,x_n$. Hence and from (23) follows the representation

$$f(x_1,\ldots,x_n)=\sum_{k=1}^{n}\lambda_k x_k,$$

where $\lambda_k=(1-\nu_n)\mu_k$ $(1\leqslant k\leqslant n-1)$, $\lambda_n=\nu_n$ and consequently

$$\sum_{k=1}^{n}\lambda_k=(1-\nu_n)\sum_{k=1}^{n-1}\mu_k+\nu_n=1.$$

Now let us assume that, for every index k $(1\leqslant k\leqslant n)$, $\nu_k=1$. Moreover, let us suppose that the field $\mathcal{K}$ has a characteristic different from 2. Put

$$\begin{aligned}g_0(x_1,x_2)&=2^{-1}x_1+2^{-1}x_2,\\ f_0(x_1,\ldots,x_n)&=g_0\big(f(x_1,\ldots,x_n),x_n\big)=2^{-1}f(x_1,\ldots,x_n)+2^{-1}x_n.\end{aligned} \tag{28}$$

By Lemma 8, $g_0\in\widetilde{A}^{(2)}$, which implies $f_0\in\widetilde{A}^{(n)}$. Moreover,

$$f_0(x_1,x_2,\ldots,x_2)=2^{-1}\nu_1(x_1,x_2)+2^{-1}x_2=2^{-1}x_1+2^{-1}x_2\not\equiv x_1.$$

Consequently, applying the first part of the proof, we have the representation

$$f_0(x_1,\ldots,x_n)=\sum_{k=1}^{n}\mu_k x_k,$$

where $\mu_1,\ldots,\mu_n\in\mathcal{K}$ and $\sum_{k=1}^{n}\mu_k=1$. Hence and from (28) it follows that

$$f(x_1,\ldots,x_n)=\sum_{k=1}^{n}\lambda_k x_k,$$

where $\lambda_k=2\mu_k$ $(1\leqslant k\leqslant n-1)$, $\lambda_n=2\mu_n-1$ and consequently $\sum_{k=1}^{n}\lambda_n=1$.

Finally let us assume that $\nu_k=1$ $(k=1,2,\ldots,n)$ and that the field $\mathcal{K}$ has the characteristic 2. Put

$$f_1(x_2,x_3,\ldots,x_n)=f(x_2,x_2,x_3,\ldots,x_n), \tag{29}$$

$$f_2(x_2,x_3,\ldots,x_n)=f(x_3,x_2,x_3,\ldots,x_n). \tag{30}$$

Obviously, $f_1,f_2\in\widetilde{A}^{(n-1)}$ and, by the induction assumption, we have the representations

$$f_1(x_2,x_3,\ldots,x_n)=\sum_{k=2}^{n}\lambda_k^{(1)}x_k,\qquad f_2(x_2,x_3,\ldots,x_n)=\sum_{k=2}^{n}\lambda_k^{(2)}x_k, \tag{31}$$

where $\lambda_2^{(1)},\ldots,\lambda_n^{(1)},\lambda_2^{(2)},\ldots,\lambda_n^{(2)}\in\mathcal{K}$ and $\sum_{k=2}^{n}\lambda_k^{(1)}=\sum_{k=2}^{n}\lambda_k^{(2)}=1$. Setting $x_2=\ldots=x_{k-1}=x_{k+1}=\ldots=x_n=\theta$ $(3\leqslant k\leqslant n)$ in (31) and taking into account the definition of functions ν_k we infer that

$$\lambda_k^{(1)}x_k=\nu_k(x_k,\theta)=x_k \quad (3\leqslant k\leqslant n). \tag{32}$$

Replacing $x_3,\ldots,x_n$ by x_1, x_1 by x_3 in $f(x_1,\ldots,x_n)$ we obtain the expression $g(x_1,x_2,x_3)$, where $g\in\widetilde{A}^{(3)}$. By formula (12) of Lemma 3 we have the equality

$$g(x_1,x_2,x_3)=s\big(g(x_1,x_1,x_3),g(x_1,x_2,x_1),x_1\big).$$

Setting in the last equality $x_1=\theta$, $x_3=x_2$ and taking into account the definition of functions ν_k we infer that

$$g(\theta,x_2,x_2)=\nu_1(x_2,\theta)+\nu_2(x_2,\theta)=x_2+x_2=\theta. \tag{33}$$

Further putting $x_3=\ldots=x_n=\theta$ in equality (31) we obtain, according to (29) and (33), the formula

$$\lambda_2^{(1)}x_2=g(\theta,x_2,x_2)=\theta.$$

Hence and from (31) and (32) it follows that

(34) $$f_1(x_2, x_3, \ldots, x_n) = \sum_{k=3}^{n} x_k$$

and

(35) $$(n-2)\cdot 1 = 1.$$

Analogously we obtain the equality

(36) $$f_2(x_2, x_3, \ldots, x_n) = x_2 + \sum_{k=4}^{n} x_k.$$

Put

(37) $$h_0(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} x_k.$$

Since, in view of (35), $n\cdot 1 = 1$, we have, according to Lemma 8, $h_0 \in \widetilde{A}^{(n)}$. Moreover, in virtue of (29), (30), (34) and (36), we have the equalities

$$f(x_2, x_2, x_3, \ldots, x_n) = h_0(x_2, x_2, x_3, \ldots, x_n),$$
$$f(x_3, x_2, x_3, \ldots, x_n) = h_0(x_3, x_2, x_3, \ldots, x_n),$$

whence it follows that the equality

$$f(x_1, \ldots, x_n) = h_0(x_1, \ldots, x_n)$$

holds for all $x_2, \ldots, x_n$, $x_1 = x_2$ and $x_1 = x_3$. Consequently, by Lemma 1, $f = h_0$, which implies, in view of (37), the assertion of our Lemma. The Lemma is thus proved.

LEMMA 10. *If $A^{(3)} \neq A^{(3,1)}$, then the set*

(38) $$A_0 = \{f(\theta)\colon f \in A^{(1)}\}$$

is a linear subspace of A. Moreover, for every $f \in A^{(1)}$ there is an element $\lambda \in \mathcal{K}$ such that

(39) $$f(x) = \lambda x + f(\theta)$$

for any $x \in A$.

Proof. First we shall prove formula (39). By Lemma 8 the function g defined by the formula

$$g(x_1, x_2, x_3) = x_1 - x_2 + x_3$$

belongs to $\widetilde{A}^{(3)}$. Given $f \in A^{(1)}$ we put

(40) $$h(x_1, x_2, x_3) = g\big(f(x_1), f(x_2), x_3\big) = f(x_1) - f(x_2) + x_3.$$

Obviously, $\hat{h}(x) = x$ and, consequently, $h \in \widetilde{A}^{(3)}$. Thus, according to Lemma 9, there is a triplet $\lambda, \mu, \nu \in \mathcal{K}$ for which

$$h(x_1, x_2, x_3) = \lambda x_1 + \mu x_2 + \nu x_3.$$

Setting $x_2 = x_3 = \theta$, $x_1 = x$ in the last equality and taking into account formula (40), we infer that (39) is true.

For given $\lambda, \mu \in \mathcal{K}$ we put

(41) $$g_1(x_1, x_2, x_3) = \lambda x_1 - \lambda x_2 + x_3, \qquad g_2(x_1, x_2) = \mu x_1 + (1-\mu) x_2.$$

Evidently, in view of Lemma 8, $g_1 \in \widetilde{A}^{(3)}$, $g_2 \in \widetilde{A}^{(2)}$. Moreover, it is easy to see that for every pair $f_1, f_2 \in A^{(1)}$ the equality

(42) $$g_1\big(x_1, f_1(x_1), x_2\big) = g_2\big(f_2(x_1), x_1\big)$$

depends on x_2. Consequently, there is a function $f_3 \in A^{(1)}$ such that the last equality is equivalent to the equality $x_2 = f_3(x_1)$. Taking into account equalities (41) and (42) we infer that

$$f_3(\theta) = \lambda f_1(\theta) + \mu f_2(\theta).$$

Thus A_0 is a linear subspace of A.

LEMMA 11. *If $A^{(3)} = A^{(3,1)}$, then $A^{(n)} = A^{(n,1)}$ for every $n \geqslant 1$.*

Proof. To prove our assertion it suffices to show, in virtue of Lemma 2, that $\widetilde{A}^{(n)} = \widetilde{A}^{(n,1)}$ for every $n \geqslant 1$. We shall prove the last equality by induction. It is obvious for $n = 1, 2$ and 3. Let us suppose that $n \geqslant 4$ and

$$\widetilde{A}^{(k)} = \widetilde{A}^{(k,1)} \quad \text{for} \quad k = 1, 2, \ldots, n-1.$$

Let $f \in \widetilde{A}^{(n)}$. From the last equalities it follows that for every pair i, j ($i \neq j$, $1 \leqslant i, j \leqslant n$) there exists an integer $s(i,j)$ ($1 \leqslant s(i,j) \leqslant n$) such that

(43) $$f(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n) = x_{s(i,j)}.$$

Obviously, $s(i,j) \neq j$.

First we shall prove that there exists a triplet i_0, j_0, m_0 such that $s(j_0, i_0) = s(m_0, i_0)$ and $i_0 \neq j_0$, $i_0 \neq m_0$, $j_0 \neq m_0$. Contrary to this statement let us suppose that for every triplet i, j, m ($i \neq j$, $i \neq m$, $j \neq m$) we have the inequality

(44) $$s(j,i) \neq s(m,i).$$

We have, according to (43), the equalities

(45) $$f(x_2, x_2, x_3, x_4, \ldots, x_n) = x_{s(2,1)},$$

(46) $$f(x_3, x_2, x_3, x_4, \ldots, x_n) = x_{s(3,1)},$$

(47) $$f(x_4, x_2, x_3, x_4, \ldots, x_n) = x_{s(4,1)}.$$

Setting $x_2 = x_3$ in (45) and (46) we infer that the right sides of these equalities are equal. Taking into account formula (44) we have $x_{s(2,1)} = x_2$ or x_3 and $x_{s(3,1)} = x_2$ or x_3. Consequently,

$$s(2,1) = 2 \text{ or } 3, \qquad s(3,1) = 2 \text{ or } 3.$$

Similarly, setting $x_2 = x_4$ in (45) and (47), we obtain the equalities

$$s(2,1) = 2 \text{ or } 4, \qquad s(4,1) = 2 \text{ or } 4,$$

and, setting $x_3 = x_4$ in (46) and (47),

$$s(3,1) = 3 \text{ or } 4, \qquad s(4,1) = 3 \text{ or } 4.$$

Thus $s(2,1) = 2$, $s(3,1) = 3$ and $s(4,1) = 4$. Hence and from (45) and (46) it follows that

$$f(x_2, x_2, x_3, x_4, \ldots, x_n) = x_2, \qquad f(x_3, x_2, x_3, x_4, \ldots, x_n) = x_3,$$

which implies that the equality

$$f(x_1, \ldots, x_n) = x_1$$

holds for all $x_2, \ldots, x_n$, $x_1 = x_2$ and $x_1 = x_3$. Consequently, by Lemma 1, it holds for any $x_1, \ldots, x_n$. Hence we get the equalities $1 = s(2,3) = s(1,3)$, which contradicts inequality (44).

Let i_0, j_0, m_0 be a triplet satisfying the conditions $s(j_0, i_0) = s(m_0, i_0)$, $j_0 \neq i_0$, $m_0 \neq i_0$, $j_0 \neq m_0$. Setting for brevity $s_0 = s(j_0, i_0)$ we have the equalities

$$f(x_1, \ldots, x_{i_0-1}, x_{j_0}, x_{i_0+1}, \ldots, x_n) = x_{s_0},$$
$$f(x_1, \ldots, x_{i_0-1}, x_{m_0}, x_{i_0+1}, \ldots, x_n) = x_{s_0},$$

which imply that the equality

$$f(x_1, \ldots, x_n) = x_{s_0} \tag{48}$$

holds for $x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n$, $x_{i_0} = x_{j_0}$, $x_{i_0} = x_{m_0}$. Consequently, by Lemma 1, equality (48) holds for all $x_1, \ldots, x_n$, which implies $f \in \widetilde{A}^{(n,1)}$. The Lemma is thus proved.

Proof of the theorem. (i) If $A^{(0)} \neq 0$ and $A^{(3)} \neq A^{(3,1)}$, then, in virtue of Lemmas 5 and 6, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$. Moreover, taking into account the definition of addition and scalar-multiplication in A and the definition of θ, we infer that all functions f defined as

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \ldots, \lambda_n \in \mathcal{K}$, $a \in A_0$, A_0 is defined by formula (38) of Lemma 10, belong to A.

Now let $f \in A$. By Lemma 10 we have the equality

$$\hat{f}(x) = \lambda x + a,$$

where $\lambda \in \mathcal{K}$ and $a = f(\theta) \in A_0$. Put

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \lambda x_n - a + x_n, \tag{49}$$

if $f \in A^{(n)}$. Obviously, $\hat{g}(x) = x$, which implies $g \in \widetilde{A}^{(n)}$. Using Lemma 9 we have the equality

$$g(x_1, \ldots, x_n) = \sum_{k=1}^{n} \mu_k x_k,$$

where $\mu_1, \ldots, \mu_n \in \mathcal{K}$. Hence, according to (49), we get the representation

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \lambda_k x_k + a,$$

where $\lambda_1, \ldots, \lambda_n \in \mathcal{K}$ and $a \in A_0$. The assertion (i) of the Theorem is thus proved.

(ii) If $A^{(0)} = 0$ and $A^{(3)} \neq A^{(3,1)}$, then, in virtue of Lemmas 5 and 6, there is a field $\mathcal{K}$ such that A is a linear space over $\mathcal{K}$.

Now we shall prove that all functions f belonging to A are of the form

$$f(x_1, \ldots, x_n) = g\big(f_0(x_1, \ldots, x_n)\big), \tag{50}$$

where $g \in A^{(1)}$ and $f_0 \in \widetilde{A}^{(n)}$ if $f \in A^{(n)}$. Obviously, if $g \in A^{(1)}$ and $f_0 \in \widetilde{A}^{(n)}$, then the superposition $f \in A^{(n)}$. Conversely, let $f \in A^{(n)}$. Since $A^{(0)} = 0$, the function f is not constant, which implies that the equality

$$\hat{f}(x_1) = x_2$$

depends on x_1. Thus there is a function $h \in A^{(1)}$ such that the last equality is equivalent to the equality $x_1 = h(x_2)$. Hence, in particular, it follows that

$$\hat{f}\big(h(x)\big) = x = h\big(\hat{f}(x)\big).$$

Setting

$$f_0(x_1, \ldots, x_n) = h\big(f(x_1, \ldots, x_n)\big), \qquad g(x) = \hat{f}(x)$$

we have the equalities

$$\hat{f}_0(x) = h\big(\hat{f}(x)\big) = x,$$
$$g\big(f_0(x_1, \ldots, x_n)\big) = \hat{f}\Big(h\big(f(x_1, \ldots, x_n)\big)\Big) = f(x_1, \ldots, x_n),$$

which implies $f_0 \in \widetilde{A}^{(n)}$ and, consequently, formula (50).

Let $g \in A^{(1)}$. We have, by Lemma 10, the equality

$$g(x) = \lambda x + g(\theta),$$

where $\lambda \in \mathcal{K}$. We shall prove that $\lambda = 1$. Contrary to the last equality let us assume that $\lambda \neq 1$. Then the equality $g(x) = x$ depends on x. Thus there is a constant $g_0 \in A^{(0)}$ such that the last equality is equivalent to the equality $x = g_0$, which contradicts the assumption $A^{(0)} = 0$. Thus

$$g(x) = x + g(\theta)$$

for every $g \in A^{(1)}$. Hence and from (50) it follows that all functions f belonging to $A^{(n)}$ $(n = 1, 2, \ldots)$ are of the form

$$f(x_1, x_2, \ldots, x_n) = f_0(x_1, \ldots, x_n) + a\,,$$

where $a \in A_0$, A_0 is defined by formula (38) of Lemma 10 and $f_0 \in \tilde{A}^{(n)}$. The assertion (ii) of our Theorem is a direct consequence of the last equality and Lemmas 8 and 9.

(iii) If $A^{(3)} = A^{(3,1)}$, then, in view of Lemma 11, A is the class of all functions f:

$$f(x_1, \ldots, x_n) = h(x_j) \quad (h \in A^{(1)},\ 1 \leqslant j \leqslant n)\,. \tag{51}$$

First let us assume that $A^{(1)} = A^{(1,0)}$. This implies that A is one-point set: $A = \{a_0\}$ and, consequently,

$$f(x_1, \ldots, x_n) = a_0 \tag{52}$$

for every $f \in A$. Let $\mathcal{G}$ be the group containing the identity transformation of A only and let $A_0 = 0$. Evidently, A_0 is normal with respect to $\mathcal{G}$ and the assertion of the Theorem is a direct consequence of formula (52).

Now let us assume that $A^{(1)} \neq A^{(1,0)}$. Put $\mathcal{G} = A^{(1)} \setminus A^{(1,0)}$. We shall prove that $\mathcal{G}$ is a group with respect to the operation

$$(g_1 \cdot g_2)(x) = g_1(g_2(x))\,.$$

Obviously, if $g_1, g_2 \in \mathcal{G}$, then $g_1 \cdot g_2 \in \mathcal{G}$ and

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

for any $g_1, g_2, g_3 \in \mathcal{G}$.

Setting $e(x) = x$ we have $e \in \mathcal{G}$ and $e \cdot g = g \cdot e = g$ for every $g \in \mathcal{G}$. For given $g \in \mathcal{G}$ the equality $g(x_1) = x_2$ depends on x_1. Thus there is a function $g^{-1} \in A^{(1)}$ such that the last equality is equivalent to the equality $x_1 = g^{-1}(x_2)$. Evidently, $g^{-1} \in \mathcal{G}$ and $g \cdot g^{-1} = g^{-1} \cdot g = e$. Thus $\mathcal{G}$ is a group of transformations of A.

Let $g \in \mathcal{G}$ and $g \neq e$. If the equality $g(x) = x$ is independent of x, then $g(x) \neq x$ for every $x \in A$. If the last equality depends on x, then there is a constant $a \in A^{(0)}$ such that this equality is equivalent to the equality $x = a$. Thus every transformation which is not the identity has at most one fixed point in A. Moreover, setting $A_0 = A^{(0)}$, we infer that A_0 is a normal set with respect to $\mathcal{G}$. Finally, from (51) it follows that all functions f belonging to A are of the form

$$f(x_1, \ldots, x_n) = g(x_j) \quad (1 \leqslant j \leqslant n)\,,$$

or

$$f(x_1, \ldots, x_n) = a\,,$$

where $g \in \mathcal{G}$ and $a \in A_0$. The Theorem is thus proved.

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