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On upper and lower limits in measure

by

C. Goffman and D. Waterman (U. S. A.)

1. In his memoir *On convergence in measure of trigonometric series* D. E. Menchoff introduces, and makes essential use of, the upper and lower limits in measure of a sequence of measurable functions.

The use of the definition given by Menchoff requires a complicated proof of existence and uniqueness, involving transfinite induction, and considerable labor is needed to establish the main properties.

The purpose of this note is to give a new definition which simplifies the entire matter. We show that our definition of the upper and lower limits in measure is equivalent to that of Menchoff, that existence and uniqueness follow immediately, and that it yields simple proofs of the main properties.

2. We first give the definition of Menchoff; he considers extended real valued, measurable functions defined on a closed interval $[a, b]$, i. e., infinite values may be assumed on sets of positive measure.

Let $\{f_n\}$ be a sequence of measurable functions. A function F is called an *upper limit in measure* of $\{f_n\}$ if it satisfies the following two conditions:

(i) For every measurable φ, $\lim\limits_{n\to\infty} m(S\cap E_n)=0$, where E_n is the set for which $f_n(x)>\varphi(x)$ and S is the set for which $\varphi(x)>F(x)$.

(ii) If $m(S)>0$, then $\limsup\limits_{n\to\infty} m(S\cap E_n)=0$, where ψ is a measurable function, S is the set for which $F(x)>\psi(x)$, and E_n is the set for which $f_n(x)>\psi(x)$.

Let

$$h(x)=\begin{cases}\dfrac{2}{\pi}\arctan x, & -\infty<x<+\infty,\\ 1, & x=+\infty,\\ -1, & x=-\infty.\end{cases}$$

Then h is an order preserving mapping of $[-\infty,+\infty]$ onto $[-1,1]$, with order preserving inverse. Since the above definition involves only the order relation between $f_n(x)$, $n=1,2,\ldots$, $\varphi(x)$, $\psi(x)$, and $F(x)$, it follows that hF is an upper limit in measure of $\{hf_n\}$ if and only if F is an upper limit in measure of $\{f_n\}$. There is, accordingly, no loss in generality in restricting the discussion to functions whose values are in the interval $[-1,1]$.

3. Let $\mathcal{M}$ be the set of equivalence classes of measurable functions on $[a,b]$, where, as usual, two functions are equivalent if they are equal almost everywhere. By the above remark, the range may be taken to be the interval $[-1,1]$.

For equivalence classes $[f]$ and $[g]$, by $[f]\leqslant[g]$ we shall mean that $f(x)\leqslant g(x)$ almost everywhere if $f\in[f]$ and $g\in[g]$. Clearly no ambiguity will result if we denote both $[f]$ and its elements by f, relying on the reader to make suitable distinction according to the context. For completeness, we prove the known

LEMMA 1. *$\mathcal{M}$ is a complete lattice.*

Proof. We need only show that every $\mathcal{S}\subset\mathcal{M}$ has a greatest lower bound. Let $\mathcal{L}$ be the set of lower bounds of $\mathcal{S}$. Let

$$k=\sup\Big[\int_a^b f(x)\,dx:\ f\in\mathcal{L}\Big].$$

For every $n=1,2,\ldots$ there is an $f_n\in\mathcal{L}$ for which

$$\int_a^b f_n(x)\,dx>k-\frac{1}{n}.$$

Now let

$$g_n=\sup[f_1,\ldots,f_n],\qquad n=1,2,\ldots$$

Then $g_n\in\mathcal{L}$, $n=1,2,\ldots$, and $g_1\leqslant g_2\leqslant\ldots\leqslant g_n\leqslant\ldots$ Let $g(x)=\lim_{n\to\infty}g_n(x)$

Now $g\in\mathcal{L}$ and $\int_a^b g(x)\,dx=k$. If there were $h\in\mathcal{L}$ with $\sup(g,h)>g$ then $\int_a^b\sup\big(g(x),h(x)\big)\,dx>k$ and $\sup(g,h)\in\mathcal{L}$. Since this is impossible, $g=\inf\mathcal{S}$.

Let $\{f_n\}$ be a sequence in $\mathcal{M}$. We call u an *upper function relative to* $\{f_n\}$ if $\lim_{n\to\infty}m(S_n)=0$, where S_n is the set for which $f_n(x)>u(x)$, $n=1,2,\ldots$

LEMMA 2. *If u and v are upper functions relative to $\{f_n\}$ then so is* $\inf(u,v)$.

Proof. Let S_n, S_n', S_n'' be the sets for which $f_n(x)>\inf\big(u(x),v(x)\big)$, $f_n(x)>u(x)$ and $f_n(x)>v(x)$, respectively. Then $S_n\subset S_n'\cup S_n''$, $n=1,2,\ldots$, and $\lim_{n\to\infty}m(S_n)\leqslant\lim_{n\to\infty}m(S_n')+\lim_{n\to\infty}m(S_n'')=0$.

We let U be the greatest lower bound of the set of all upper functions relative to $\{f_n\}$. The existence of U is assured by the fact that $\mathcal{M}$ is a complete lattice. Then U is the upper limit in measure of $\{f_n\}$ in our sense. We shall show that it is an upper limit in measure in the sense of Menchoff.

4. We point out that U need not itself be an upper function relative to $\{f_n\}$. For example, let $f_n(x)\equiv 1/n$, $n=1,2,\ldots$ Then $U=0$, so that $m(S_n)=b-a$, and $\lim_{n\to\infty}m(S_n)=b-a\neq 0$, where S_n is the set for which $f_n(x)>U(x)$. Suppose, however, that $f(x)>U(x)$ at almost every point where $U(x)<1$, and $f(x)=1$ almost everywhere else. We shall denote this by $f\gg U$. It is important that such a function, and in particular $U_\varepsilon=\inf[U+\varepsilon,1]$, is an upper function. We first prove

LEMMA 3. *There is a decreasing sequence $\{g_n\}$ of upper functions relative to $\{f_n\}$ which converges to U, i. e., $\lim_{n\to\infty}g_n(x)-U(x)=0$ almost everywhere.*

Proof. Let $\mathcal{U}$ be the set of upper functions relative to $\{f_n\}$ and let

$$k=\inf\Big[\int_a^b\big(u(x)-U(x)\big)\,dx:\ u\in\mathcal{U}\Big].$$

Let $h_n\in\mathcal{U}$ be such that

$$\int_a^b\big(h_n(x)-U(x)\big)\,dx<k+\frac{1}{n},\qquad n=1,2,\ldots$$

By lemma 2, $g_n=\inf(h_1,\ldots,h_n)\in\mathcal{U}$, $n=1,2,\ldots$ The sequence $\{g_n\}$ is decreasing. Let $g(x)=\lim_{n\to\infty}g_n(x)$. Suppose $g(x)>U(x)$ on a set of positive measure. There is then $u\in\mathcal{U}$ with $u(x)<g(x)$ on a set of positive measure. Now

$$u_n=\inf[u,g_n]\in\mathcal{U},\qquad n=1,2,\ldots,$$

and

$$\int_a^b\big(u_n(x)-U(x)\big)\,dx<k$$

for n sufficiently large. Hence $g(x)=U(x)$ almost everywhere.

COROLLARY 1. *If $u\gg U$ then u is an upper function relative to $\{f_n\}$.*

Proof. Let $\varepsilon>0$. There is a k such that $u(x)\geqslant g_k(x)$ on S where $m(CS)<\varepsilon$. Let E_n and S_n be sets for which $f_n(x)>g_k(x)$ and $f_n(x)>u(x)$ respectively. Since $\lim_{n\to\infty}m(E_n)=0$, and $S_n\subset E_n\cup CS$, it follows that $\limsup_{n\to\infty}m(S_n)\leqslant\varepsilon$.

5. Krickeberg has given a definition of upper limit in measure (for a directed system of functions) which is similar to ours. For a sequence $\{f_n\}$, Krickeberg's definition is as follows: Let $h \in \mathcal{H}$ if $h' \geqslant h$ implies $\lim_{n\to\infty} m(E_n) = 0$, where E_n is the set for which $f_n(x) > h'(x)$. Then $v = \inf \mathcal{H}$ is the upper limit in measure according to Krickeberg.

Now $h \in \mathcal{H}$ if and only if $h' \in \mathcal{U}$ for every $h' \geqslant h$. Thus $h \geqslant u$ for every $h \in \mathcal{H}$. Hence, $v \geqslant U$. Conversely, let $h \geqslant U$. By corollary 1, $h \in \mathcal{U}$ so that $h' \geqslant h$ implies $h' \in \mathcal{U}$ and thus $h \in \mathcal{H}$. But then $v \leqslant U$.

Krickeberg makes no reference to the work of Menchoff, but applies the limits in measure to the convergence of martingales. Theorems 2 and 3 below are given by Krickeberg as well as by Menchoff. Their proofs, which are an immediate consequence of our definition, are included here.

6. We now prove

THEOREM 1. *U is an upper limit in measure of $\{f_n\}$ in the sense of Menchoff.*

Proof. Let $\varphi \in \mathcal{M}$ and let S be the set for which $\varphi(x) > U(x)$. If we define

$$u(x) = \begin{cases} \varphi(x), & x \in S, \\ 1, & x \in CS, \end{cases}$$

then, by corollary 1, u is an upper function relative to $\{f_n\}$. Let E_n be the set for which $f_n(x) > \varphi(x)$ and E_n' the set for which $f_n(x) > u(x)$. Then $\lim_{n\to\infty} m(E_n') = 0$ and $E_n \cap S = E_n'$, $n = 1, 2, \ldots$, so that $\lim_{n\to\infty} m(E_n \cap S) = 0$, and U satisfies condition (i).

Let $\psi \in \mathcal{M}$ be such that $m(S) > 0$, where S is the set for which $U(x) > \psi(x)$. Let

$$v(x) = \begin{cases} \psi(x), & x \in S, \\ 1, & x \in CS. \end{cases}$$

Then v is not an upper function relative to $\{f_n\}$. Hence $\limsup_{n\to\infty} m(E_n) > 0$, where E_n is the set for which $f_n(x) > v(x)$. But $m(E_n \cap S) = m(E_n)$, so that U satisfies condition (ii).

We remark that this constitutes an alternate proof to the one given by Menchoff that upper limits in measure exist. In the next section we show that U is the only function satisfying the conditions of Menchoff. We emphasize, however, that our definition alone will be used in developing the properties of U.

7. Let F be an upper limit in measure in the sense of Menchoff. Then $F \leqslant U$. For, let u be any upper function relative to $\{f_n\}$. If S is

the set for which $F(x) > u(x)$, and E the set for which $f_n(x) > u(x)$, then $\limsup_{n\to\infty} m(S \cap E_n) \leqslant \lim_{n\to\infty} m(E_n) = 0$. Hence, by condition (ii), $m(S) = 0$. Hence, $F \leqslant U$. Suppose $F(x) < U(x)$ on a set S. Let

$$\varphi(x) = \begin{cases} \tfrac{1}{2}\big(F(x) + U(x)\big), & x \in S, \\ 1, & x \in CS. \end{cases}$$

Then, by condition (i), $\lim_{n\to\infty} m(S \cap E_n) = 0$, where E_n is the set for which $f_n(x) > \varphi(x)$. But $S \cap E_n \supset E_n$. Hence, φ is an upper function, $m(S) = 0$, and $F \geqslant U$. Accordingly, $F = U$.

8. We define the lower limit in measure of $\{f_n\}$ as the least upper bound L of the set $\mathcal{L}$ of lower functions relative to $\{f_n\}$, where $l \in \mathcal{L}$ if $\lim_{n\to\infty} m(S_n) = 0$, S_n being the set for which $f_n(x) < l(x)$. We may define $L_\varepsilon = \sup(L - \varepsilon, -1)$ for $\varepsilon > 0$. Clearly it has properties analogous to those of U_ε. Now, if $u \in \mathcal{U}$, $l \in \mathcal{L}$ then clearly $l \leqslant u$. It then follows that $L \leqslant U$.

9. We now prove

THEOREM 2. *The sequence $\{f_n\}$ converges in measure to f if and only if $U = L$, and then $f = U = L$.*

Proof. Suppose $U = L$. Let $\varepsilon > 0$. Then if S_n and T_n are the sets for which $f_n(x) > U_{\varepsilon/2}(x)$ and $f_n(x) < L_{\varepsilon/2}(x)$, respectively, then $\lim_{n\to\infty} m(S_n) = \lim_{n\to\infty} m(T_n) = 0$. But $|f_n(x) - U(x)| < \varepsilon$, except possibly on $S_n \cup T_n$ so that $\{f_n\}$ converges in measure to $f = U = L$.

Suppose, conversely, that $\{f_n\}$ converges in measure to f. Then $\inf\big(f(x) + \varepsilon, 1\big)$ is an upper function for every $\varepsilon > 0$. Hence $U \leqslant f$. Similarly, $L \geqslant f$. Thus, $f \leqslant L \leqslant U \leqslant f$, and $f = U = L$.

10. It is of interest to compare the upper and lower limits in measure with the customary limit superior and limit inferior.

THEOREM 3. *For every $\{f_n\}$, $\liminf_{n\to\infty} f(x) \leqslant L(x) \leqslant U(x) \leqslant \limsup_{n\to\infty} f_n(x)$ almost everywhere.*

Proof. Clearly, $\inf[\limsup_{n\to\infty} f_n(x) + \varepsilon, 1]$ is an upper function for every $\varepsilon > 0$. Hence $U(x) \leqslant \limsup_{n\to\infty} f_n(x)$ almost everywhere. Similarly, $L(x) \geqslant \liminf_{n\to\infty} f_n(x)$ almost everywhere.

11. We give one additional result which further elucidates the relation between upper and lower limits in measure and the usual limit superior and limit inferior. It should be noted that part of Theorem 2 is an immediate consequence of this theorem. The proof is of particular interest in view of its simplicity as compared with that of Menchoff.

We first prove

LEMMA 4. *If $\{\alpha_n\}$ converges in measure to zero, then $\{f_n\}$ and $\{f_n+\alpha_n\}$ have the same upper and lower limits in measure.*

Proof. Let U be the upper limit in measure of $\{f_n\}$. For every $\varepsilon>0$, U_ε is evidently an upper function relative to $\{f_n+\alpha_n\}$ so that, if U' is the upper limit in measure of $\{f_n+\alpha_n\}$, then $U'\leqslant U$. By the same token, $U\leqslant U'$.

THEOREM 4. *For every $\{f_n\}$, there are $\{h_n\}$ and $\{\alpha_n\}$ such that*

$$f_n=h_n+\alpha_n\,,\qquad n=1,2,\ldots,$$

$$U(x)=\limsup_{n\to\infty}h_n(x)\,,$$

$$L(x)=\liminf_{n\to\infty}h_n(x)\,,$$

and $\{\alpha_n\}$ converges in measure to zero.

Proof. Clearly, $U_\varepsilon\geqslant L_\varepsilon$, for every $\varepsilon>0$. Let

$$h_n(x)=\begin{cases}U_{1/n}(x)\,, & \text{if}\quad f_n(x)\geqslant U_{1/n}(x)\,,\\ f_n(x)\,, & \text{if}\quad L_{1/n}(x)<f_n(x)<U_{1/n}(x)\,,\\ L_{1/n}(x)\,, & \text{if}\quad f_n(x)\leqslant L_{1/n}(x)\,,\end{cases}$$

and let $\alpha_n(x)=f_n(x)-h_n(x)$, $n=1,2,\ldots$

Let $\varepsilon>0$. Let S_n be the set for which $|\alpha_n(x)|>\varepsilon$. Let T_n and T_n' be the sets for which $f_n(x)>U_{\varepsilon+1/n}(x)$ and $f_n(x)<L_{\varepsilon+1/n}(x)$, respectively. Clearly

$$S_n=T_n\cup T_n'\,.$$

If E_n and E_n' are the sets for which $f_n(x)>U_\varepsilon(x)$ and $f_n(x)<L_\varepsilon(x)$, respectively, then $T_n\subset E_n$ and $T_n'\subset E_n'$. But

$$\lim_{n\to\infty}m(E_n)=\lim_{n\to\infty}m(E_n')=0\,.$$

Hence, $\{\alpha_n\}$ converges in measure to zero. Now,

$$\limsup_{n\to\infty}h_n(x)\leqslant\limsup_{n\to\infty}U_{1/n}(x)=U(x)$$

and

$$\liminf_{n\to\infty}h_n(x)\geqslant\liminf_{n\to\infty}L_{1/n}(x)=L(x)\,.$$

But, by Lemma 4, U and L are the upper and lower limits in measure of $\{h_n\}$, respectively. Hence, by Theorem 3,

$$\limsup_{n\to\infty}h_n(x)\geqslant U(x)$$

and

$$\liminf_{n\to\infty}h_n(x)\leqslant L(x)\,.$$

12. Finally, we define an equivalence relation for sequences of measurable functions. Let $\{f_n\}$ be equivalent to $\{g_n\}$ if $\{f_n-g_n\}$ converges in measure to zero. This relation is evidently reflexive, symmetric, and transitive. By Lemma 4, sequences belonging to the same equivalence class have the same upper and lower limits in measure. Moreover, in proving Theorem 4, an equivalent sequence of functions was constructed whose limit superior and upper limit in measure are the same, and whose limit inferior and lower limit in measure are the same. By Theorem 3, we thus have

THEOREM 5. *For every sequence $\{f_n\}$, the upper and lower limits in measure are given by*

$$U(x)=\inf[\limsup_{n\to\infty}g_n(x):\ \{g_n\}\in\mathcal{F}]$$

and

$$L(x)=\sup[\liminf_{n\to\infty}h_n(x):\ \{h_n\}\in\mathcal{F}]$$

where $\mathcal{F}$ is the equivalence class to which $\{f_n\}$ belongs.

Indeed, this property could well be taken as the definition.

References

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PURDUE UNIVERSITY
LAFAYETTE, INDIANA

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