Axiomatic characterization of the family of all clusters in a proximity space

by

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In paper [1], S. Leader introduced the notion of cluster in proximity spaces. According to his definition a cluster $\mathfrak{c}$ in a proximity space X is a class of subsets of X which satisfies the following conditions:

(a) *If A and B both belong to $\mathfrak{c}$, then A is close to B*;

(b) *if A is close to each B in $\mathfrak{c}$, then A belongs to $\mathfrak{c}$*;

(c) *if $A \cup B$ belongs to $\mathfrak{c}$, then either A or B belongs to $\mathfrak{c}$.*

The paper quoted above contains an interesting applications of the notion of clusters.

In this paper necessary and sufficient conditions will be given in order that a family $\mathfrak{C}$ of class of subsets of a set X be the family of all clusters for some proximity relation in X. (¹)

We note that each cluster $\mathfrak{c}$ satisfies the following conditions:

(d) *if A belongs to $\mathfrak{c}$ and A is contained in B, then B also belongs to $\mathfrak{c}$*;

(e) *let p and q both belong to X. If $\{p\}$ and $\{q\}$ belong simultaneously to the cluster $\mathfrak{c}$, then $p = q$*;

(f) *if A is a subset of X and A meets each B from $\mathfrak{c}$, then A also belongs to $\mathfrak{c}$.*

A class of subsets of X satisfying conditions (c), (d), (e), (f) will be called a *semi-ultrafilter* of X.

(¹) We recall here the axioms of proximity relation:

(P_1) *if A is close to B, then B is close to A*;

(P_2) *A is close to $B \cup C$ if and only if either B or C is close to A*;

(P_3) *A is close to 0 if and only if $A = 0$*;

(P_4) *p is close to q if and only if $p = q$ (p, q — points)*;

(P_5) *if for each E either A is close to E or B is close to $X \setminus E$, then A is close to B.*

We write "$A\,\delta\,B$" instead of "A is close to B" and "$A\,\bar{\delta}\,B$" instead of "A is not close to B". We shall also write "$A \Subset B$" instead of "$A\,\bar{\delta}\,X \setminus B$", where X is the space. Using this notation, the axiom (P_5) may be rewritten as follows:

(P_5') *if $A \Subset B$, then there is C with $A \Subset C \Subset B$.*

Now, we shall show some lemmas on clusters.

LEMMA 1. *If* $\mathfrak{c}$, $\mathfrak{c}'$ *are clusters and* $\mathfrak{c} \subset \mathfrak{c}'$, *then* $\mathfrak{c} = \mathfrak{c}'$.

Proof. If A belongs to $\mathfrak{c}'$, then A is close to each B from $\mathfrak{c}$ and hence, by condition (b), A belongs to $\mathfrak{c}$. Therefore $\mathfrak{c}' \subset \mathfrak{c}$ and the Lemma holds.

LEMMA 2. *If* $\mathfrak{s}$ *is a maximal centred family, then the class*

$$\mathfrak{c} = \{A \subset X\colon\ A\,\delta\,B \text{ for each } B \text{ in } \mathfrak{s}\}$$

is a cluster. Moreover, $\mathfrak{c}$ *is the only cluster which contains* $\mathfrak{s}$.

Proof. We prove first all the conditions for clusters step-by-step.

Condition (a). If $A\,\overline{\delta}\,B$, then there exists such an $E \subset X$ that $E\,\overline{\delta}\,A$ and $X \setminus E\,\overline{\delta}\,B$. $\mathfrak{s}$ being a maximal centred family implies either $E \in \mathfrak{s}$ or $X \setminus E \in \mathfrak{s}$. But $E \in \mathfrak{s}$ implies $A \notin \mathfrak{c}$ and $X \setminus E \in \mathfrak{s}$ implies $B \notin \mathfrak{c}$. Therefore A and B cannot simultaneously belong to $\mathfrak{c}$ and condition (a) is proved.

Condition (b) is obvious.

Condition (c). Supposing that A and B do not belong to $\mathfrak{c}$, there are E, E' in $\mathfrak{s}$ such that $A\,\overline{\delta}\,E$, $B\,\delta\,E'$. Since $E \cap E' \in \mathfrak{s}$ and $A\,\overline{\delta}\,(E \cap E')$ and $B\,\overline{\delta}\,(E \cap E')$, by the axiom (P_2), $(A \cup B)\,\overline{\delta}\,(E \cap E')$ and thus $A \cup B \notin \mathfrak{c}$.

Therefore $\mathfrak{c}$ is a cluster.

We shall show that $\mathfrak{c}$ is the only cluster containing $\mathfrak{s}$. Let us suppose that $\mathfrak{c}'$ is a cluster and $\mathfrak{s} \subset \mathfrak{c}'$. If $A \in \mathfrak{c}'$, then $A\,\delta\,B$ for each B in $\mathfrak{c}'$. Hence $A\,\delta\,B$ for each B in $\mathfrak{s}$ and it follows that $A \in \mathfrak{c}$. Consequently, $\mathfrak{c}' \subset \mathfrak{c}$, and by Lemma 1, $\mathfrak{c}' = \mathfrak{c}$.

LEMMA 3. *If* A *is close to* B, *then there exists a cluster* $\mathfrak{c}$ *to which both* A *and* B *belong.*

This lemma and its proof is presented in paper [1].

Now we prove the theorems which are crucial for the purpose of this paper.

THEOREM 1. *Let* $\mathfrak{C}$ *be the family of all clusters in a proximity space* X. *Then* $\mathfrak{C}$ *is a family of semi-ultrafilters which satisfies the following conditions:*

1. *Let* A *and* B *be arbitrary subsets of* X. *If for each* $E \subset X$ *there is a member* $\mathfrak{c} \in \mathfrak{C}$ *such that either* $A, E \in \mathfrak{c}$ *or* $B, X \setminus E \in \mathfrak{c}$, *then there is* $\mathfrak{c}_0$ *in* $\mathfrak{C}$ *such that* $A, B \in \mathfrak{c}_0$.

2. *Let* $\mathfrak{c}_0$ *be an arbitrary member from* $\mathfrak{C}$. *If for each member* B *from* $\mathfrak{c}_0$ *there is a* $\mathfrak{c}$ *in* $\mathfrak{C}$ *such that* $A, B \in \mathfrak{c}$, *then* $A \in \mathfrak{c}_0$.

3. *Each centred family of subsets of* X *is contained in some member of* $\mathfrak{C}$.

THEOREM 2. *If* $\mathfrak{C}$ *is a family of semi-ultrafilters of* X *which satisfies conditions* 1, 2, 3, *then* $\mathfrak{C}$ *is the family of all clusters for some proximity relation in* X.

Proof of Theorem 1. Of course, each cluster is a semi-ultrafilter. Condition 1 follows from axiom (P_5) and from Lemma 3. Condition 2 directly follows from Lemma 3 and from the definition of cluster, condition (b). Finally, condition 3 follows from Lemma 2 and thus Theorem 1 holds.

Proof of Theorem 2. Let δ be the relation between subsets of X defined by the condition

$A\,\delta\,B$ *if and only if there is a* $\mathfrak{c}$ *in* $\mathfrak{C}$ *to which both* A *and* B *belong.*

It is not hard to verify that δ is really a proximity relation. We shall show that that each member of $\mathfrak{C}$ is a cluster (with respect to δ). In fact, condition (a) directly follows from the definition of the relation δ. Condition (b) is simply condition 3, written in terms of the relation δ. Finally, condition (c) immediately follows from Lemma 2.

There remains the most essential part of the proof, namely to show that each cluster with respect to δ is a member of $\mathfrak{C}$. Let $\mathfrak{c}_0$ be a cluster. Let $\mathfrak{s}_0$ be the family of all members A of $\mathfrak{c}_0$ which satisfies the following condition:

there is a B *in* $\mathfrak{c}_0$ *such that* $B \Subset A$.

We shall show that $\mathfrak{s}_0$ is a centred family. Indeed, suppose that A_1, A_2 belong to $\mathfrak{s}_0$ and $A_1 \cap A_2 = 0$. Then there are B_1 and B_2 from $\mathfrak{c}_0$ with $B_1 \Subset A_1$ and $B_2 \Subset A_2$. Since $A_1 \cap A_2 = 0$, $A_1 \subset X \setminus A_2$, whence $B_1 \Subset X \setminus A_2$ and thus $B_1\,\overline{\delta}\,A_2$. Therefore $B_1\,\overline{\delta}\,B_2$ which contradicts the definition of clusters.

Now, let $\mathfrak{s}$ be a maximal centred family which contains $\mathfrak{s}_0$. We shall show that $\mathfrak{s}$ is contained in $\mathfrak{c}_0$. Indeed, let $A \in \mathfrak{s}$ and B be any member of $\mathfrak{c}_0$ and let $A\,\overline{\delta}\,B$. Thus $A \Subset X \setminus B$. By axiom (P_5'), we can find $C \subset X$ with $A \Subset C \Subset X \setminus B$. Then $X \setminus C \Subset B$, whence, by the definition of $\mathfrak{s}_0$, $X \setminus C \in \mathfrak{s}_0$, and moreover $A\,\overline{\delta}\,(X \setminus C)$. Consequently $A \cap (X \setminus C) = 0$, which leads to a contradiction.

Therefore we have shown that $A\,\delta\,B$ for each B in $\mathfrak{c}_0$ and it follows, by condition (b) on clusters, that $A \in \mathfrak{c}_0$ and finally $\mathfrak{s} \subset \mathfrak{c}_0$.

By condition 3 there is a member $\mathfrak{c} \in \mathfrak{C}$ which contains $\mathfrak{s}$. But we have actually shown that each member of $\mathfrak{C}$ is a cluster and it follows that $\mathfrak{c}_0$ and $\mathfrak{c}$ are clusters containing the maximal centred family $\mathfrak{s}$. Therefore, by Lemma 2, we obtain $\mathfrak{c}_0 = \mathfrak{c}$ and thus $\mathfrak{c}_0$ is a member of $\mathfrak{C}$. This completes the proof of Theorem 2.

According to Theorems 1 and 2 the notion of cluster may be adopted as the primitive notion of the theory of proximity spaces. This point of view leads to the following definition of proximity spaces:

A proximity space is an abstract set X with a distinguished family $\mathfrak{C}$ of semi-ultrafilters of X, named ex definitione clusters, which satisfies conditions 1, 2, 3.

The reader may easily verify that, starting with the above definition, many theorems on proximity spaces, in particular, the compactification theorem, may be proved without the help of the axiom of choice.

Reference

[1] S. Leader, *On clusters in proximity spaces*, Fund. Math. 47 (1959), p. 205-213.

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On upper and lower limits in measure

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1. In his memoir *On convergence in measure of trigonometric series* D. E. Menchoff introduces, and makes essential use of, the upper and lower limits in measure of a sequence of measurable functions.

The use of the definition given by Menchoff requires a complicated proof of existence and uniqueness, involving transfinite induction, and considerable labor is needed to establish the main properties.

The purpose of this note is to give a new definition which simplifies the entire matter. We show that our definition of the upper and lower limits in measure is equivalent to that of Menchoff, that existence and uniqueness follow immediately, and that it yields simple proofs of the main properties.

2. We first give the definition of Menchoff; he considers extended real valued, measurable functions defined on a closed interval $[a, b]$, i. e., infinite values may be assumed on sets of positive measure.

Let $\{f_n\}$ be a sequence of measurable functions. A function F is called an *upper limit in measure* of $\{f_n\}$ if it satisfies the following two conditions:

(i) For every measurable φ, $\lim\limits_{n\to\infty} m(S\cap E_n)=0$, where E_n is the set for which $f_n(x)>\varphi(x)$ and S is the set for which $\varphi(x)>F(x)$.

(ii) If $m(S)>0$, then $\limsup\limits_{n\to\infty} m(S\cap E_n)=0$, where ψ is a measurable function, S is the set for which $F(x)>\psi(x)$, and E_n is the set for which $f_n(x)>\psi(x)$.

Let

$$h(x)=\begin{cases}\dfrac{2}{\pi}\arctan x, & -\infty<x<+\infty,\\ 1, & x=+\infty,\\ -1, & x=-\infty.\end{cases}$$

Then h is an order preserving mapping of $[-\infty, +\infty]$ onto $[-1, 1]$, with order preserving inverse. Since the above definition involves only the order relation between $f_n(x)$, $n=1,2,\ldots$, $\varphi(x)$, $\psi(x)$, and $F(x)$, it