On extending of models V

Embedding theorems for relational models

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If $\{h_t\}_{t\in T}$ is a family of homomorphisms of an algebra $\Gamma = \langle A, F_1, \ldots, F_m\rangle$, where $F_1, \ldots, F_m$ are functions defined in the set A, then the function h assigning to any $a \in A$ a function $x = h(a)$ such that $x(t) = h_t(a)$ for every $t \in T$ is a homomorphism of Γ into the product-algebra of all homomorphic images $h_t(\Gamma)$ (Birkhoff's theorem [1]). We are interested in the conditions under which an analogous theorem holds for any relational model $\mathfrak{M} = \langle A, R_1, \ldots, R_m\rangle$, where $R_1, \ldots, R_m$ are arbitrary relations defined in the set A. We formulate the conditions for the productability of families of homomorphisms and some general theorems concerning the embedding of models into products of models. As an application we prove a theorem about embedding models into decidable models.

We assume here Mostowski's notion of homomorphism [3] for relations. It has already been applied by some authors, e. g. [2], [5].

§ 1. Auxiliary notions and notation. Any subset of a Cartesian product $A_1 \times \ldots \times A_n$ is called an *n-ary relation*. For any n-ary relation R and any set A we denote by $R|A$ the n-ary relation for which $R|A\,(a_1, \ldots, a_n)$ holds if and only if $a_1, \ldots, a_n \in A$ and $R(a_1, \ldots, a_n)$. A relation R is said to be *defined in a set* A if $R|A = R$.

Every binary reflexive, symmetric and transitive relation $\sim$ defined in a set A is called an *equivalence in* A. The abstraction class of $\sim$ in A determined by $a \in A$, i. e. the set of all $c \in A$ such that $c \sim a$, is denoted by $a/\sim$. Similarly the set of all abstraction classes of $\sim$ in A is denoted by $A/\sim$.

If h is any mapping of a set A, then the equivalence $\sim$ such that $a_1 \sim a_2$ holds for $a_1, a_2 \in A$ if and only if $h(a_1) = h(a_2)$, is called the *equivalence induced in* A *by* h. If $h_0(a) = a/\sim$ for any $a \in A$, then $h_0^{-1}(a/\sim) = a/\sim$ for each $a \in A$.

If $\{A_t\}_{t\in T}$ is a family of sets, then its Cartesian product $\mathop{\boldsymbol{P}}_{t\in T} A_t$ is the set of all functions x such that $x(t) \in A$ for any $t \in T$.

§ 2. Homomorphims and congruences of relations. Let h be a function mapping the set A onto the set B. The mapping h is said to be a *homomorphism of an n-ary relation R defined in A onto an n-ary relation Q defined in B* if the following condition holds for any $b_1, \ldots, b_n \in B$:

(2.1) *$Q(b_1, \ldots, b_n)$ if and only if there are such elements $a_1, \ldots, a_n \in A$ that $b_1 = h(a_1), \ldots, b_n = h(a_n)$ and $R(a_1, \ldots, a_n)$*

or equivalently

(2.2) $$Q = \underset{\langle b_1, \ldots, b_n\rangle}{E} \left[\left(h^{-1}(b_1) \times \ldots \times h^{-1}(b_n)\right) \frown R \neq 0\right]^{(1)}.$$

The n-ary relations R, Q defined in A and B respectively are called *isomorphic* if there exists a homomorphism of R onto Q inducing an equivalence which is an identity in A.

If the mapping h of A into B is a homomorphism of an n-ary relation R defined in A onto the relation $Q|h(A)$ where Q is an n-ary relation defined in B, then h is said to be a *homomorphism of R into Q*.

Any equivalence in a set A induced by some homomorphism of a relation R defined in A is called a *congruence of R in A*. This definition, although analogous to corresponding definition of congruences in algebras, leads to the unexpected conclusion that

(2.3) *every equivalence in a set A is a congruence of any relation R defined in A.*

This results from the following observation.

For any equivalence $\sim$ in A and any relation R defined in A, one may define in $A/\sim$ a relation $R/\sim$ such that

(2.4) $$R/\sim\; = \underset{\langle a_1/\sim, \ldots, a_n/\sim\rangle}{E} [(a_1/\sim \times \ldots \times a_n/\sim) \frown R \neq 0].$$

Evidently, the mapping h_0 of A onto $A/\sim$ such that $h_0(a) = a/\sim$ for any $a \in A$ is a homomorphism of R onto $R/\sim$. It is easy to see that

(2.5) *the relation $R/\sim$ defined in $A/\sim$ is isomorphic to any relation Q defined in some set B such that there exists a homomorphism of R onto Q inducing the equivalence $\sim$ in A.*

§ 3. Products of relations and projections. Let $\{R_t\}_{t \in T}$ be a family of n-ary relations defined in the corresponding sets belonging to the family $\{A_t\}_{t \in T}$. The n-ary relation P defined in the product $\underset{t \in T}{\mathbf{P}} A_t$ in such a way that

(3.1) *$P(x_1, \ldots, x_n)$ if and only if $R_t(x_1(t), \ldots, x_n(t))$ for every $t \in T$,*

(1) The symbol $\frown$ denotes the set-theoretical meet-operation.

is called the *relational product of the family $\{R_t\}_{t \in T}$*. It is seen directly that if some relation R_t is empty, then the product P is also empty.

Consider a mapping p_t of $\underset{t \in T}{\mathbf{P}} A_t$ onto A_t such that

(3.2) $$p_t(x) = x(t)$$

for any $x \in \underset{t \in T}{\mathbf{P}} A_t$. This mapping is called the *projection of the product $\underset{t \in T}{\mathbf{P}} A_t$ onto t-axis A_t*. The projection p_t is a homomorphism of P onto R_t except the case $P = 0 \neq R_t$. I. e.

(3.3) *the projection p_t is a homomorphism of P onto R_t if and only if either R_t is the empty relation or every relation in $\{R_t\}_{t \in T}$ is non-empty.*

§ 4. Productability of homomorphisms and of congruences. Let h_t be, for any $t \in T$, a homomorphism of the relation R defined in A onto the relation R_t defined in A_t and let $\underset{t}{\sim}$ be the congruence induced in A by h_t. A mapping h^* of A into $\underset{t \in T}{\mathbf{P}} A$ such that for each $a \in A$:

(4.1) *$h^*(a) = x$ if and only if $x(t) = h_t(a)$ for every $t \in T$*

or equivalently

(4.2) $$p_t\left(h^*(a)\right) = h_t(a)$$

is called the *product of the family $\{h_t\}_{t \in T}$ of homomorphisms*. Let $\overset{*}{\sim}$ be the congruence in A induced by it. Of course, for any $a_1, a_2 \in A$,

(4.3) *$a_1 \overset{*}{\sim} a_2$ if and only if $a_1 \underset{t}{\sim} a_2$ for each $t \in T$.*

If the product h^* introduced above is a homomorphism of R into the relational product P of the family $\{R_t\}_{t \in T}$ defined in $\underset{t \in T}{\mathbf{P}} A_t$, then the family $\{h_t\}_{t \in T}$ of homomorphisms and the family $\{\underset{t}{\sim}\}_{t \in T}$ of congruences are said to be *productable*. One may see that the family $\{\underset{t}{\sim}\}_{t \in T}$ of congruences is *productable* if and only if the mapping h_0 of A into $\underset{t \in T}{\mathbf{P}} A_t$ such that for each $a \in A$

(4.4) *$h_0(a) = z$ if and only if $z(t) = a/\underset{t}{\sim}$ for each $t \in T$*

or equivalently

(4.5) $$p_t\left(h_0(a)\right) = a/\underset{t}{\sim}$$

is a homomorphism of R into the relational product P of the family $\{R/\underset{t}{\sim}\}_{t \in T}$ defined in $\underset{t \in T}{\mathbf{P}} A/\underset{t}{\sim}$.

We now want to formulate the necessary and sufficient conditions for the productability of homomorphisms and congruences.

From (2.1), (3.1), (3.2) and (4.2) we infer that the product h^* is a homomorphism of the relation R defined in A into the relational product P of the family $\{R_t\}_{t\in T}$ defined in $\underset{t\in T}{\boldsymbol{P}} A_t$ if and only if for any $a_1, \ldots, a_n \in A$:

(4.6) *$R_t(h_t(a_1), \ldots, h_t(a_n))$ holds for every $t \in T$ if and only if there exist such elements $c_1, \ldots, c_n \in A$ that $c_1 \overset{*}{\sim} a_1, \ldots, c_n \overset{*}{\sim} a_n$ and $R(c_1, \ldots, c_n)$.*

The implication in (4.6) from the right to the left follows directly from the assumption that h_t is a homomorphism for any $t \in T$. Therefore the following two theorems are true.

THEOREM 1. *For the productability of the family $\{h_t\}_{t\in T}$ of homomorphisms it is necessary and sufficient that the following condition holds for any $a_1, \ldots, a_n \in A$:*

(4.7) *if $R_t(h_t(a_1), \ldots, h_t(a_n))$ holds for every $t \in T$, then there exist such elements $c_1, \ldots, c_n \in A$ that $c_1 \overset{*}{\sim} a_1, \ldots, c_n \overset{*}{\sim} a_n$ and $R(c_1, \ldots, c_n)$.*

THEOREM 2. *For the productability of the family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences it is necessary and sufficient that the following condition holds for any $a_1, \ldots, a_n \in A$:*

(4.8) *if for every $t \in T$ there exist such elements $d_1, \ldots, d_n \in A$ that $d_1 \underset{t}{\sim} a_1, \ldots, d_n \underset{t}{\sim} a_n$ and $R(d_1, \ldots, d_n)$ then there exist such elements $c_1, \ldots, c_n \in A$ that $c_1 \overset{*}{\sim} a_1, \ldots, c_n \overset{*}{\sim} a_n$ and $R(c_1, \ldots, c_n)$.*

The conditions (4.7) and (4.8) may be considerably simplified in the case of homomorphisms and congruences of some special kind. Namely, the family $\{h_t\}_{t\in T}$ of homomorphisms and the family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences are said to be *separating* the set A if for any $a_1, a_2 \in A$

(4.9) *if $a_1 \neq a_2$ then there exists such a $t \in T$ that $h_t(a_1) \neq h_t(a_2)$*

or equivalently

(4.10) *if $a_1 \neq a_2$ then there exists such a $t \in T$ that non $a_1 \underset{t}{\sim} a_2$.*

In this special case we observe that the equivalence $\overset{*}{\sim}$ induced in A by the product h^* of homomorphisms is the identity relation in A and, therefore, we obtain the following two theorems, which are specialisation of theorems 1 and 2.

THEOREM 3. *For the productability of the family $\{h_t\}_{t\in T}$ of homomorphisms separating the set A it is necessary and sufficient that the following condition holds for any $a_1, \ldots, a_n \in A$:*

(4.11) *if $R_t(h_t(a_1), \ldots, h_t(a_n))$ holds for every $t \in T$, then $R(a_1, \ldots, a_n)$.*

THEOREM 4. *For the productability of the family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences separating the set A it is necessary and sufficient that the following condition holds for any $a_1, \ldots, a_n \in A$:*

(4.12) *if for every $t \in T$ there exist such elements $d_1, \ldots, d_n \in A$ that $d_1 \underset{t}{\sim} a_1, \ldots, d_n \underset{t}{\sim} a_n$ and $R(d_1, \ldots, d_n)$ then $R(a_1, \ldots, a_n)$.*

§ 5. Embedding theorems for relational models. The previous considerations concerning homomorphisms and congruences of relations may be generalized to the case of relational models.

Let $\mathfrak{M} = \langle A, R_1, \ldots, R_m\rangle$ be a relational model. If $R_1, \ldots, R_m$ are k_1-ary, ..., k_m-ary relations defined in the set A, then the model $\mathfrak{M}$ is said to be of the type $\langle k_1, \ldots, k_m\rangle$. Let $\mathfrak{N} = \langle B, Q_1, \ldots, Q_m\rangle$ be another model of the type $\langle k_1, \ldots, k_m\rangle$. Any function h mapping the set A into (onto) the set B such that h is a homomorphism for $i = 1, \ldots, m$ of the relation R_i defined in A into (onto) the relation Q_i defined in B is called a *homomorphism of the model $\mathfrak{M}$ into (onto) the model $\mathfrak{N}$*. A homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$ is called *isomorphism* if it induces in A an equivalence which is identity relation. The models $\mathfrak{M}$ and $\mathfrak{N}$ are then called isomorphic.

Any equivalence in A induced by a homomorphism of $\mathfrak{M}$ is called a *congruence of the model* $\mathfrak{M}$. It follows from (2.3) that any equivalence $\sim$ in A is a congruence of the model $\mathfrak{M} = \langle A, R_1, \ldots, R_m\rangle$. The mapping h_0 of A onto $A/\sim$ such that $h_0(a) = a/\sim$ for any $a \in A$, is a homomorphism of the model $\mathfrak{M}$ onto the model $\langle A/\sim, R_1/\sim, \ldots, R_m/\sim\rangle$ denoted in the following by $\mathfrak{M}/\sim$. The model $\mathfrak{M}/\sim$ is isomorphic to any model $\mathfrak{N}$ such that there exists a homomorphism h of $\mathfrak{M}$ onto $\mathfrak{N}$ inducing the equivalence $\sim$ in A.

Let $\{\mathfrak{M}_t\}_{t\in T}$ be a family of relational models of the same type and let $\mathfrak{M}_t = \langle A_t, R_t^{(1)}, \ldots, R_t^{(m)}\rangle$ for any $t \in T$. The *product* $\underset{t\in T}{\boldsymbol{P}} \mathfrak{M}_t$ of the family $\{\mathfrak{M}_t\}_{t\in T}$ of models is the relational model $\langle \underset{t\in T}{\boldsymbol{P}} A_t, P_1, \ldots, P_m\rangle$ where for $i = 1, \ldots, m$ the relation P_i is the relational product of the family $\{R_t^{(i)}\}_{t\in T}$.

Let h_t be, for any $t \in T$, a homomorphism of the model $\mathfrak{M}$ onto the model $\mathfrak{M}_t$. The family $\{h_t\}_{t\in T}$ of homomorphisms is said to be *productable* if the product of this family, i. e. the function h^* defined in (4.1) or (4.2) is a homomorphism of the model $\mathfrak{M}$ into the product $\underset{t\in T}{\boldsymbol{P}} \mathfrak{M}_t$.

The model $\mathfrak{M}$ is said to be *embedded in the model $\mathfrak{N}$ by h* if h is an isomorphism of $\mathfrak{M}$ into $\mathfrak{N}$.

We now want to formulate some theorems concerning the embedding of models into products of models. These theorems are analogous to the well-known product-theorem of Birkhoff concerning the embedding of abstract algebras [1]. We apply below the notation introduced above.

Theorem 5. *If h_t is, for any $t \in T$, a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}_t$ and if the family $\{h_t\}_{t\in T}$ is productable and separates the set A in $\mathfrak{M}$, then the model $\mathfrak{M}$ is embedded into the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}_t$ by the product h^* of family $\{h_t\}_{t\in T}$.*

Proof. It follows from the productability of $\{h_t\}_{t\in T}$ that h^* is a homomorphism of $\mathfrak{M}$ into $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}_t$. On the other hand h^* is an isomorphism as $\{h_t\}_{t\in T}$ separates the set A.

Theorem 6. *If h_t is, for any $t \in T$, a homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}_t$ and if the family $\{h_t\}_{t\in T}$ separates the set A in $\mathfrak{M}$ and fulfils condition* (4.11) *for any relation in the model $\mathfrak{M}$, then the model $\mathfrak{M}$ is embedded in the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}_t$ by the product h^* of the family $\{h_t\}_{t\in T}$.*

Proof. We observe that according to theorem 3 the product h^* is a homomorphism of any relation in $\mathfrak{M}$. Therefore it is a homomorphism of the whole model $\mathfrak{M}$ and the family $\{h_t\}_{t\in T}$ is productable. Now apply theorem 5.

Theorem 7. *If the family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences of the model $\mathfrak{M}$ separates the set A in $\mathfrak{M}$ and fulfils condition* (4.12) *for any relation in the model $\mathfrak{M}$, then the model $\mathfrak{M}$ is embedded in the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}/\underset{t}{\sim}$.*

Proof. Assume that $h_t(a) = a/\underset{t}{\sim}$ and $\mathfrak{M}_t = \mathfrak{M}/\underset{t}{\sim}$ and then apply theorems 4 and 6.

§ 6. Embedding in decidable models. Any relational model in the general sense is a system $\mathfrak{M} = \langle A, R_1, \ldots, R_\xi, \ldots\rangle_{\xi<\beta}$ of type $\langle a_1, \ldots, a_\xi, \ldots\rangle_{\xi<\beta}$. A is a set, β is an ordinal called the *order* of $\mathfrak{M}$, $\beta = \overline{\overline{\mathfrak{M}}}$ and $a_1, \ldots, a_\xi, \ldots$ $(\xi<\beta)$ are any ordinal numbers such that $R_1, \ldots, R_\xi, \ldots$ $(\xi<\beta)$ are relations defined in A a_1-ary, ..., a_ξ-ary, ... $(\xi<\beta)$ respectively. The least upper bound a of all ordinal numbers a_ξ $(\xi<\beta)$ is called the *rank* of $\mathfrak{M}$. If B is any subset of the set A then the model $\langle B, R_1|B, \ldots, R_\xi|B, \ldots\rangle_{\xi<\beta}$ is said to be a *submodel* of the given model $\mathfrak{M}$. If the set A in the model $\mathfrak{M}$ is finite, then the model $\mathfrak{M}$ is called *finite*.

We have previously considered only the *finitary* models, i. e. the models of finite order and finite rank. However, these considerations may be generalized to the general case and theorems 1-7 are valid for any relational models of any order and any rank.

Now we want to prove some embedding theorems holding for models having only finitary relations.

Theorem 8. *For any model $\mathfrak{M}$ having only finitary relations there exists a family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences of $\mathfrak{M}$ such that*

1° *the model $\mathfrak{M}$ is embedded in the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}/\underset{t}{\sim}$ and*

2° *any model $\mathfrak{M}/\underset{t}{\sim}$, for $t \in T$, is finite.*

Proof. Let the model $\mathfrak{M} = \langle A, R_1, R_2, \ldots\rangle$ be of type $\langle k_1, k_2, \ldots\rangle$ and let $t = \langle a_1, \ldots, a_n\rangle$. The element t, belonging to the Cartesian power A^n, determines a partition of the set A into at most $n+1$ classes,

$$(a_1), \ldots, (a_n), A-(a_1, \ldots, a_n).$$

Therefore the element t determines in a well-known way an equivalence $\underset{t}{\sim}$ in the set A, which is a congruence of the model $\mathfrak{M}$. The model $\mathfrak{M}/\underset{t}{\sim}$ is finite since the set $A/\underset{t}{\sim}$ contains at most $n+1$ elements. Consider now the set

$$T = \bigcup_{i=1,2,\ldots} A^{k_i}\ (^2)$$

and the corresponding family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences of $\mathfrak{M}$. This family separates the set A, since non $a\underset{t}{\sim}b$ if $a, b \in A$, $a \neq b$ and $t = \langle a, \ldots\rangle$. On the other hand, for any $a_1, \ldots, a_{k_i}$ if $t = \langle a_1, \ldots, a_{k_i}\rangle$ then the abstraction classes $a_1/\underset{t}{\sim}, \ldots, a_{k_i}/\underset{t}{\sim}$ are unit classes. It follows that the family $\{\underset{t}{\sim}\}_{t\in T}$ fulfils condition (4.12) for any relation in the model $\mathfrak{M}$. Therefore, following theorem 7, the model $\mathfrak{M}$ is embedded in the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}/\underset{t}{\sim}$ and theorem 8 is proved.

From the proof given above we see that if the order of the model $\mathfrak{M}$ is finite, i. e. if $\mathfrak{M} = \langle A, R_1, \ldots, R_m\rangle$ where m is a natural number, then the power of the model $\mathfrak{M}/\underset{t}{\sim}$, i. e. the power of the set $A/\underset{t}{\sim}$, does not exceed the natural number $k = 1+\max(k_1, k_2, \ldots, k_m)$. Therefore we have proved the following

Theorem 9. *For any finitary model $\mathfrak{M}$ there exists a family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences of $\mathfrak{M}$ such that*

1° *the model $\mathfrak{M}$ is embedded in the product $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}/\underset{t}{\sim}$ and*

2° *there exists a natural number k such that the power of any model $\mathfrak{M}/\underset{t}{\sim}$ is less than k for any $t \in T$.*

We pass to decidable models. Consider the class $\mathfrak{A}$ of all finitary models of a given type and an elementary logical language S corresponding to the class $\mathfrak{A}$. The construction of S for $\mathfrak{A}$ is well-known. A model $\mathfrak{M}$ is called *decidable* if the set $S(\mathfrak{M})$ of all sentences in S valid in $\mathfrak{M}$ is decidable. We present

Theorem 10. *Any finitary relational model $\mathfrak{M}$ is embedded in some decidable model.*

Proof. We apply theorem 9 and we consider such a family $\{\underset{t}{\sim}\}_{t\in T}$ of congruences of $\mathfrak{M}$ that the model $\mathfrak{M}$ is embedded in $\underset{t\in T}{\boldsymbol{P}}\mathfrak{M}/\underset{t}{\sim}$ and the

(2) The symbol $\bigcup$ means the set-theoretical union operation.

power of any model $\mathfrak{M}/\underset{t}{\sim}$ is less than some natural number k. There exists of course only a finite number of non-isomorphic models of power less than k. Therefore the set T may be decomposed in a finite union $T = T_1 \cup \ldots \cup T_r$ such that for any $i = 1, \ldots, r$ and all $t_1, t_2 \in T_i$ the models $\mathfrak{M}/\underset{t_1}{\sim}$ and $\mathfrak{M}/\underset{t_2}{\sim}$ are isomorphic. It follows that the model $\mathfrak{M}$ is embedded in the model $\mathop{\boldsymbol{P}}\limits_{i=1}^{r} \mathop{\boldsymbol{P}}\limits_{t \in T_i} \mathfrak{M}/\underset{t}{\sim}$ isomorphic to the model $\mathop{\boldsymbol{P}}\limits_{t \in T} \mathfrak{M}/\underset{t}{\sim}$. The model $\mathop{\boldsymbol{P}}\limits_{t \in T_i} \mathfrak{M}/\underset{t}{\sim}$ is a product of mutually isomorphic models, and therefore it is isomorphic to some Cartesian power of some model $\mathfrak{M}/\underset{t_i^*}{\sim}$ where $t_i^* \in T_i$. This model is finite and, subsequently, it is decidable, and — according to Mostowski's theorem [4] — its Cartesian power is decidable. Since S. Feferman has proved that the product of a finite number of decidable models is decidable, we infer that the model $\mathop{\boldsymbol{P}}\limits_{i=1}^{r} \mathop{\boldsymbol{P}}\limits_{t \in T_i} \mathfrak{M}/\underset{t}{\sim}$ is decidable, q. e. d.

§ 7. C. C. Chang's product theorem. We present here a simple proof of the following theorem of C. C. Chang [2]. Let $\{\mathfrak{M}_t\}_{t \in T}$ be a family of models of a fixed type.

If every model $\mathfrak{M}_t$ is of rank α and of order β, and does not contain the empty relation, then for any model $\mathfrak{M}$ which is a submodel of the product $\mathop{\boldsymbol{P}}\limits_{t \in T} \mathfrak{M}_t$ there exists a subset $T_0 \subset T$ such that

1° *$\mathfrak{M}$ is embedded in the product $\mathop{\boldsymbol{P}}\limits_{t \in T_0} \mathfrak{M}_t$,*

2° $\overline{\overline{T_0}} \leqslant \overline{\overline{\mathfrak{M}}}^{\alpha} \cdot \overline{\overline{\beta}} \cdot \aleph_0$.

Proof. Let A_t be a set in $\mathfrak{M}_t$ and let p_t be the projection of the product $\mathop{\boldsymbol{P}}\limits_{t \in T} A_t$ on the t-axis A_t. According to (3.3) the projection p_t is a homomorphism of $\mathop{\boldsymbol{P}}\limits_{t \in T} \mathfrak{M}_t$ into $\mathfrak{M}_t$. From the definitions of products and projections and from theorem 3 it follows that $\{p_t\}_{t \in T}$ is a separating and productable family of homomorphisms of $\mathop{\boldsymbol{P}}\limits_{t \in T} \mathfrak{M}_t$. Since $\mathfrak{M}$ is a submodel of $\mathop{\boldsymbol{P}}\limits_{t \in T} \mathfrak{M}_t$, the family $\{p_t\}_{t \in T}$ is also a separating and productable family of homomorphisms of model $\mathfrak{M}$. Let $\mathfrak{M} = \langle A, R_1, \ldots, R_\xi, \ldots \rangle_{\xi < \beta}$ and $\mathfrak{M}_t = \langle A_t, R_1^{(t)}, \ldots, R_\xi^{(t)}, \ldots \rangle_{\xi < \beta}$, $t \in T$. Let $\langle \alpha_1, \ldots, \alpha_\xi, \ldots \rangle_{\xi < \beta}$ be the type of the models considered and let $V^{(\xi)} = A^{\alpha_\xi} - R_\xi$, $\xi < \beta$. From the productability of $\{p_t\}_{t \in T}$ and (4.11) it follows that for every $v = \langle a_1, \ldots, a_\vartheta, \ldots \rangle_{\vartheta < \alpha_\xi} \in V^{(\xi)}$ there exists such a $t_v \in T$ that non $R_\xi^{(t_v)}(p_{t_v}(a_1), \ldots, p_{t_v}(a_\vartheta), \ldots)_{\vartheta < \alpha_\xi}$. Let $T^{(\xi)} = \bigcup\limits_{v \in V^{(\xi)}} (t_v)$. Since the family $\{p_t\}_{t \in T}$ separates the model $\mathfrak{M}$, for every $u = \langle a, b \rangle \in A^2$ with $a \neq b$ there exists such a $t_u \in T$ that $p_{t_u}(a) \neq p_{t_u}(b)$. T^* denotes the set of all t_u, where $u = \langle a, b \rangle \in A^2$ and $a \neq b$. We put

$$T_0 = T^* \cup \bigcup_{\xi < \beta} T^{(\xi)}.$$

Since $T^* \subset T_0$, the family $\{p_t\}_{t \in T_0}$ separates $\mathfrak{M}$. Since $\bigcup\limits_{\xi < \beta} T^{(\xi)} \subset T_0$, the family $\{p_t\}_{t \in T_0}$ is by (4.11) a productable family of homomorphisms of $\mathfrak{M}$. Hence $\mathfrak{M}$ is by theorem 6 embedded in $\mathop{\boldsymbol{P}}\limits_{t \in T_0} \mathfrak{M}_t$. By the definitions of $T^{(\xi)}$, $V^{(\xi)}$ and rank α we have

$$\overline{\overline{T^{(\xi)}}} \leqslant \overline{\overline{V^{(\xi)}}} \leqslant \overline{\overline{A^{\alpha_\xi}}} \leqslant \overline{\overline{A^{\alpha}}} = \overline{\overline{\mathfrak{M}}}^{\alpha}.$$

Moreover $\overline{\overline{T^*}} \leqslant \overline{\overline{A}}^2 = \overline{\overline{\mathfrak{M}}}^2$. Hence and from the definition of T_0 we obtain

$$\overline{\overline{T_0}} \leqslant \overline{\overline{T^*}} + \sum \overline{\overline{T^{(\xi)}}} \leqslant \overline{\overline{\mathfrak{M}}}^2 + \overline{\overline{\mathfrak{M}}}^{\alpha} \cdot \overline{\overline{\beta}} \leqslant \overline{\overline{\mathfrak{M}}}^{\alpha} \cdot \overline{\overline{\beta}} \cdot \aleph_0, \qquad \text{q. e. d.}$$

References

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