On the countable sum of zero-dimensional metric spaces *

by

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1. Introduction. As is well known, not every infinite-dimensional metric space is the countable sum of zero-dimensional spaces; in fact the Hilbert-cube I_ω is not the countable sum of zero-dimensional spaces. It is known that by the generalized decomposition-theorem due to M. Katětov [3] and to K. Morita [4] a metric space is the countable sum of zero-dimensional spaces if and only if it is the countable sum of finite-dimensional spaces. We call such a space a *countable-dimensional space*. It seems, however, that our knowledge of countable dimensional spaces, owing to peculiar difficulties in the infinite-dimensional case, is very little if compared to that of finite-dimensional spaces.

The purpose of this paper is to extend the theory of finite-dimensional spaces to the countable-dimensional case. First we shall characterize countable-dimensional spaces by extending Eilenberg-Otto's theorem [1]. Then we shall characterize such spaces by closed coverings and show that every countable-dimensional space is an image of a generalized Baire zero-dimensional space $N(\Omega)$ (1) by a closed continuous mapping such that the inverse image of each point consists of finitely many points. Furthermore, it will be shown that a countable-dimensional space with a σ-star-finite basis can be imbedded in $N(\Omega)\times R_\omega$, where R_ω is the set of points in I_ω at most a finite number of whose coordinates are rational. Finally we shall discuss on a metric space which is the countable sum of finite-dimensional closed sets.

All spaces considered in the present paper will be assumed to be metric spaces unless the contrary is explicitly stated. Dim R denotes the strong inductive dimension of R, i. e. $\dim \emptyset = -1$, $\dim R \leqslant n$ if and only if for any pair of a closed set F and an open set G

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(1) $N(\Omega) = \{(\alpha_1, \alpha_2, \ldots) |\ \alpha_i \in \Omega,\ i = 1, 2, \ldots\}$. We define the metric of $N(\Omega)$ as follows: if $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$, $\alpha_i = \beta_i$ for $i < n$, $\alpha_n \neq \beta_n$, then $\varrho(\alpha, \beta) = 1/n$. As is well known, $N(\Omega)$ is a 0-dimensional metric space. This notion is due to [4].

with $F \subset G \subset R$ there exists an open set U such that $F \subset U \subset G$, $\dim(\overline{U}-U) \leqslant n-1$ (2).

2. An extension of Eilenberg-Otto's theorem. For a point p and for a covering $\mathfrak{U}$ of a space R we denote by $\operatorname{order}_p \mathfrak{U}$ the largest integer n such that there are n members of $\mathfrak{U}$ which contain p. We also use the notation $B(\mathfrak{U}) = \{B(U) \mid U \in \mathfrak{U}\}$, where $B(U)$ means the boundary of U.

LEMMA 2.1. *Let A_n, $n=1,2,\ldots$, be a countable number of zero-dimensional sets of a space R. Let $\{U_\alpha \mid \alpha < \tau\}$ (3) be a collection of open sets and $\{F_\alpha \mid \alpha < \tau\}$ a collection of closed sets such that $F_\alpha \subset U_\alpha$, $\alpha < \tau$, and such that $\{U_\beta \mid \beta < \alpha\}$ is locally finite for every $\alpha < \tau$. Then there exists a collection of open sets V_α, $\alpha < \tau$, such that*

(1) $F_\alpha \subset V_\alpha \subset U_\alpha$, $\alpha < \tau$,

(2) $\operatorname{order}_p B(\mathfrak{V}) \leqslant n-1$ *for every* $p \in A_n$,

where $\mathfrak{V} = \{V_\alpha \mid \alpha < \tau\}$.

Proof. We shall define, by induction on α, V_α satisfying (1) and

$(2)_\alpha$ $\operatorname{order}_p B(\mathfrak{V}_\alpha) \leqslant n-1$ *for every* $p \in A_n$, *where* $\mathfrak{V}_\alpha = \{V_\beta \mid \beta \leqslant \alpha\}$.

We take open sets G_1, W_1 such that

$$G_1 \supset F_1, \quad W_1 \supset U_1^c \ (4), \quad \overline{G}_1 \cap \overline{W}_1 = \emptyset .$$

Since A_1 is zero-dimensional, there exists an open closed set N_1 of A_1 satisfying

$$\overline{G}_1 \cap A_1 \subset N_1 \subset (\overline{W}_1)^c \cap A_1 .$$

If we put $B_1 = N_1 \cup F_1$, $C_1 = (A_1 - N_1) \cup U_1^c$, then $(\overline{B}_1 \cap C_1) \cup (B_1 \cap \overline{C}_1) = \emptyset$. Hence there exists an open set V_1 such that $B_1 \subset V_1 \subset \overline{V}_1 \subset C_1^c$. Since $B(V_1) \cap A_1 = \emptyset$ is clear, V_1 satisfies (1) and $(2)_\alpha$ for $\alpha = 1$.

Suppose that V_β has been constructed for every $\beta < \alpha$ $(< \tau)$. Then we put

$$H_1 = A_1, \quad H_n = \bigcup \{B(V_{\beta_1}) \cap \ldots \cap B(V_{\beta_{n-1}}) \cap A_n \mid \beta_1, \ldots, \beta_{n-1} < \alpha\}, \quad n = 2, 3, \ldots,$$

$$K_\alpha = \bigcup_{n=1}^{\infty} H_n .$$

It follows from $\dim A_n = 0$, $n = 1, 2, \ldots$, that $\dim H_n \leqslant 0$, $n = 1, 2, \ldots$

(2) The equivalence of this dimension with the Lebesgue dimension in every metric space was proved by [3] and [4].

(3) We denote by $\alpha, \beta, \gamma, \tau$ ordinal numbers.

(4) We denote by U_1^c the complement set of U_1.

We easily see that, for every n, $\bigcup_{i=1}^{n} H_i$ is open in K_α. For let $p \in \bigcup_{i=1}^{n} H_i$, then $p \in \bigcup_{i=1}^{n} A_i$; hence we have $\operatorname{order}_p \{B(V_\beta) \mid \beta < \alpha\} \leqslant n-1$ by the assumption of induction. Therefore we can find, from every collection $\{B(V_{\beta_1}), \ldots, B(V_{\beta_j})\}$ with $\beta_1, \ldots, \beta_j < \alpha$, $B(V_{\beta_i})$ with $B(V_{\beta_i}) \not\ni p$ if $j \geqslant n$. On the other hand, $U(p) = \bigcap \{R - B(V_\beta) \mid \beta < \alpha, p \notin B(V_\beta)\}$ is an nbd (= neighborhood) of p by the local finiteness of $\{U_\beta \mid \beta < \alpha\}$. Hence we conclude $U(P) \cap \big(\bigcup_{i=n+1}^{\infty} H_i\big) = \emptyset$, which implies the openess of $\bigcup_{i=1}^{n} H_i$ in K_α. Thus, for every n, $H_n - \bigcup_{i=1}^{n-1} H_i$ is a 0-dimensional F_σ-set. Hence we have, by the generalized sum-theorem [4], $\dim K_\alpha \leqslant 0$. Consequently we can define, in the same way as for the case of $\alpha = 1$, an open set V_α such that

$$F_\alpha \subset V_\alpha \subset U_\alpha, \quad B(V_\alpha) \cap K_\alpha = \emptyset ,$$

which implies $(2)_\alpha$. This completes the proof.

THEOREM 2.2. *A space R is countable-dimensional if and only if there exists a countable collection of locally finite open coverings $\mathfrak{V}_i$ such that $\mathfrak{V} = \{V \mid V \in \mathfrak{V}_i, i = 1, 2, \ldots\}$ is a basis of open sets of R and $\operatorname{order}_p B(\mathfrak{V}) < +\infty$ for every point p of R.*

Proof. Let R be countable-dimensional, i. e. $R = \bigcup_{n=1}^{\infty} A_n$ for 0-dimensional A_n; then by a theorem of A. H. Stone [8] there exists a countable collection of locally finite open coverings $\mathfrak{U}_i$ of R such that $\{S(p, \mathfrak{U}_i) \mid i = 1, 2, \ldots\}$ (5) is an nbd basis of each point p. Let $\mathfrak{U}_i = \{U_\alpha \mid \tau_{i-1} \leqslant \alpha < \tau_i\}$, $\tau_0 = 1$. Then there exists, by the local finiteness of $\mathfrak{U}_i$, a closed covering $\{F_\alpha \mid \tau_{i-1} \leqslant \alpha < \tau_i\}$ such that $F_\alpha \subset U_\alpha$. Putting $\tau = \sup \{\tau_i \mid i = 1, 2, \ldots\}$ we have the collection $\{U_\alpha \mid \alpha < \tau\}$ of open sets satisfying the condition of Lemma 2.1. Hence there exists a collection $\mathfrak{V} = \{V_\alpha \mid \alpha < \tau\}$ of open sets satisfying (1), (2) of Lemma 2.1. $\mathfrak{V}_i = \{V_\alpha \mid \tau_{i-1} \leqslant \alpha < \tau_i\}$, $i = 1, 2, \ldots$, are clearly open coverings satisfying the condition of this proposition.

Conversely, if there exists such a collection $\{\mathfrak{V}_i \mid i = 1, 2, \ldots\}$ of open coverings of R, then we let

$$\{p \mid \operatorname{order}_p B(\mathfrak{V}) = n-1\} = A_n, \quad n = 1, 2, \ldots$$

Since $\mathfrak{V} = \{V \cap A_n \mid V \in \mathfrak{V}_i, i = 1, 2, \ldots\}$ is an open basis of A_n such that $\operatorname{order}_p B(\mathfrak{V}) = n-1$, we have, by [4], $\dim A_n \leqslant n-1$. Thus it follows from the generalized decomposition-theorem that $R = \bigcup_{n=1}^{\infty} A_n$ is countable-dimensional.

(5) We denote by $S(p, \mathfrak{U}_i)$ the union of all the sets of $\mathfrak{U}_i$ containing p.

THEOREM 2.3. *A space R is countable-dimensional if and only if for every collections $\{U_\alpha \mid \alpha < \tau\}$ of open sets and $\{F_\alpha \mid \alpha < \tau\}$ of closed sets such that $F_\alpha \subset U_\alpha$, $\alpha < \tau$, and such that $\{U_\beta \mid \beta < \alpha\}$ is locally finite for every $\alpha < \tau$, there exists a collection of open sets V_α, $\alpha < \tau$, satisfying*

(1) $F_\alpha \subset V_\alpha \subset U_\alpha$, $\alpha < \tau$,

(2) $\mathrm{order}_p B(\mathfrak{V}) < +\infty$ *for every* $p \in R$,

where $\mathfrak{V} = \{V_\alpha \mid \alpha < \tau\}$.

Proof. The "only if" part is a direct consequence of Lemma 2.1. The "if" part is a direct consequence of Theorem 2.2.

3. Characterizations by closed coverings and by $N(\Omega)$.

LEMMA 3.1. *Let A_n, $n = 1, 2, \ldots$, be a countable number of 0-dimensional sets of a space R. Let $\mathfrak{U} = \{U_\alpha \mid \alpha < \tau\}$ be a locally finite open covering. Then there exists a closed covering $\mathfrak{F} = \{F_\alpha \mid \alpha < \tau\}$ such that $\mathfrak{F} < \mathfrak{U}$ and $\mathrm{order}_p \mathfrak{F} \leqslant n$ for every $p \in A_n$.*

Proof. We obtain by Lemma 2.1 an open covering $\mathfrak{V} = \{V_\alpha \mid \alpha < \tau\}$ such that $\overline{V}_\alpha \subset U_\alpha$, $\mathrm{order}_p B(\mathfrak{V}) \leqslant n-1$ for $p \in A_n$, where we notice that we can easily choose $V_\alpha^\cdot$ satisfying $\overline{V}_\alpha \subset U_\alpha^\cdot$ instead of (1) of Lemma 2.1. Let

$$F_1 = \overline{V}_1, \quad F_\alpha = \overline{V_\alpha - \bigcup_{\beta<\alpha} \overline{V}_\beta}, \quad \alpha < \tau .$$

Then $\mathfrak{F} = \{F_\alpha \mid \alpha < \tau\}$ is a closed covering satisfying the condition of this lemma. In fact, let $p \in A_n$, $p \in V_\alpha$, $p \notin V_\beta$ for every $\beta < \alpha$. Then it is clear that $p \notin F_\gamma$ for every $\gamma > \alpha$, and $p \in F_\beta$ for some $\beta < \alpha$ implies $p \in \overline{V}_\beta - V_\beta = B(V_\beta)$. Thus it follows from $\mathrm{order}_p B(\mathfrak{V}) < n-1$ that $\mathrm{order}_p \mathfrak{F} \leqslant n$.

THEOREM 3.2. *A space R is countable-dimensional if and only if there exists a countable collection $\{\mathfrak{F}_i \mid i = 1, 2, \ldots\}$ of locally finite closed coverings of R satisfying*

(1) *for every nbd $U(p)$ of every point p of R there exists some i with $S(p, \mathfrak{F}_i) \subset U(p)$,*

(2) $\mathfrak{F}_i = \{F(\alpha_1, \ldots, \alpha_i) \mid \alpha_k \in \Omega,\ k = 1, 2, \ldots, i\}$, *where $F(\alpha_1, \ldots, \alpha_i)$ may be empty,*

(3) $F(\alpha_1, \ldots, \alpha_{i-1}) = \bigcup \{F(\alpha_1, \ldots, \alpha_{i-1}, \beta) \mid \beta \in \Omega\}$,

(4) $\sup \{\mathrm{order}_p \mathfrak{F}_i \mid i = 1, 2, \ldots\} < +\infty$ *for each point p of R.*

Proof. Let R be a countable-dimensional space with $R = \bigcup_{n=1}^{\infty} A_n$ for 0-dimensional A_n and $\mathfrak{S}_1 > \mathfrak{S}_2 > \ldots$ a uniformity of R. Then we shall define $\mathfrak{F}_i$ satisfying (2), (3), $\mathfrak{F}_i < \mathfrak{S}_i$ and $\mathrm{order}_p \mathfrak{F}_i \leqslant n$ for each point $p \in A_n$. We can define $\mathfrak{F}_1$ by Lemma 3.1. Assume that we have defined $\mathfrak{F}_k$ for every $k < i$; then we put $\mathfrak{F}_{i-1} = \{F_\alpha \mid \alpha < \tau\}$ for brevity. To obtain $\mathfrak{F}_i$ we shall define closed sets $F_{\alpha\beta}$, $\alpha < \tau$, $\beta \in \Omega$, such that

(i) $F_\alpha = \bigcup \{F_{\alpha\beta} \mid \beta \in \Omega\}$, $\{F_{\alpha\beta} \mid \beta \in \Omega\} < \mathfrak{S}_i$,

(ii) $\mathfrak{G}_\alpha^\cdot = \{F_{\alpha'\beta} \mid \alpha' \leqslant \alpha,\ \beta \in \Omega\} \cup \{F_{\alpha'} \mid \alpha' > \alpha\}$ is locally finite for every $\alpha < \tau$,

(iii) $\mathrm{order}_p \mathfrak{G}_\alpha \leqslant n$ for every $\alpha < \tau$ and for each point $p \in A_n$.

First we define $F_{1\beta}$, $\beta \in \Omega$, as follows. We let

$$H_{10} = F_1 \cap A_1 ,$$
$$H_{21} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = 1,\ p \in F_1 \cap A_2\} ,$$
$$H_{32} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = 2,\ p \in F_1 \cap A_3\} ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$K_1 = H_{10} \cup H_{21} \cup H_{32} \cup \ldots$$

and

$$H_{20} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = 0,\ p \in F_1 \cap A_2\} ,$$
$$H_{31} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = 1,\ p \in F_1 \cap A_3\} ,$$
$$H_{42} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = 2,\ p \in F_1 \cap A_4\} ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$K_2 = H_{20} \cup H_{31} \cup H_{42} \cup \ldots$$

and generally

$$H_{r+s\,s} = \{p \mid \mathrm{order}_p \{F_\alpha \mid 1 < \alpha < \tau\} = s,\ p \in F_1 \cap A_{r+s}\} ,$$
$$K_r = \bigcup_{s=0}^{\infty} H_{r+s\,s} .$$

If $p \in H_{r+s\,s}$, then $p \in F_{\alpha_1} \cap \ldots \cap F_{\alpha_s}$ for some $\alpha_1, \ldots, \alpha_s$ and $p \notin F_{\alpha_1} \cap \ldots \cap F_{\alpha_{s+1}}$ for every $\alpha_1, \ldots, \alpha_{s+1}$. Hence the nbd $U(p) = \bigcap \{F_\alpha^c \mid p \notin F_\alpha,\ 1 < \alpha < \tau\}$ of p satisfies $U(p) \cap H_{r+t\,t} = \emptyset$ for every $t > s$, which implies the openness of $\bigcup_{s'=1}^{s} H_{r+s'\,s'}$ in K_r. Since evidently $\dim H_{r+s\,s} \leqslant 0$ for every $s \geqslant 0$, we have $\dim K_r \leqslant 0$, $r = 1, 2, \ldots$ Therefore we can define by Lemma 3.1 a locally finite closed covering $\mathfrak{G}_1' = \{F_{1\beta} \mid \beta \in \Omega\}$ of F_1 such that $\mathfrak{G}_1' < \mathfrak{S}_i$, $\mathrm{order}_p \mathfrak{G}_1' \leqslant n$ for every $p \in K_n$.

To show $\mathrm{order}_p \mathfrak{G}_1 \leqslant n$ for $p \in A_n$ and for $\mathfrak{G}_1 = \mathfrak{G}_1' \cup \{F_{\alpha'} \mid \alpha' > 1\}$, we assume $\mathrm{order}_p \mathfrak{F}_{i-1} = s+1$ and $p \in F_1 \cap F_{\alpha_1} \cap \ldots \cap F_{\alpha_s}$, where $\alpha_1, \ldots, \alpha_s > 1$ and $0 \leqslant s \leqslant n-1$ because of $\mathrm{order}_p \mathfrak{F}_{i-1} \leqslant n$. Then $p \in H_{ns} \subset K_{n-s}$, and hence $\mathrm{order}_p \mathfrak{G}_1' \leqslant n-s$, proving $\mathrm{order}_p \mathfrak{G}_1 \leqslant n$. Assume that we have defined $F_{\alpha'\beta}$, $\beta \in \Omega$, for every $\alpha' < \alpha$; then we can define $F_{\alpha\beta}$, $\beta \in \Omega$, satisfying (i)-(iii). The method of defining $F_{\alpha\beta}$ is parallel to that of $F_{1\beta}$ except that we use F_α and $\{F_{\alpha'\beta} \mid \alpha' < \alpha,\ \beta \in \Omega\} \cup \{F_{\alpha'} \mid \alpha' > \alpha\}$ instead of F_1 and $\{F_\alpha \mid \alpha > 1\}$ respectively; hence its proof is left to the reader. Thus we can define the required covering $\mathfrak{F}_i = \{F_{\alpha\beta} \mid \alpha < \tau,\ \beta \in \Omega\}$.

Conversely, if there exists a countable collection $\{\mathfrak{F}_i \mid i = 1, 2, \ldots\}$ satisfying (1)-(4), then we set $\{p \mid \sup \{\mathrm{order}_p \mathfrak{F}_i \mid i = 1, 2, \ldots\} = n\} = A_n$.

Since $\operatorname{order}_p \mathfrak{F}_i \leqslant n$, $i = 1, 2, \ldots$, for every $p \in A_n$, we have, by [5], $\dim A_n \leqslant n-1$. Hence $R = \bigcup_{n=1}^{\infty} A_n$ is countable-dimensional.

THEOREM 3.3. *A space R is countable-dimensional if and only if there exist a subset S of $N(\Omega)$ for suitable Ω and a closed continuous mapping f of S onto R such that for each point p of R the inverse image $f^{-1}(p)$ consists of finitely many points.*

Proof. The "only if" part is a direct consequence of Theorem 3.2. In fact, let R be countable-dimensional and $\mathfrak{F}_i$ closed coverings satisfying (1)-(4) of Theorem 3.2. Then we define a subset S of $N(\Omega)$ by $S = \{(\alpha_1, \alpha_2, \ldots) \mid \bigcap_{i=1}^{\infty} F(\alpha_1, \ldots, \alpha_i) \neq \emptyset\}$ and a mapping f of S onto R by $f(a) = p = \bigcap_{i=1}^{\infty} F(\alpha_1, \ldots, \alpha_i)$ for $a = (\alpha_1, \alpha_2, \ldots)$. To show the closedness of f we put $N(\alpha_1, \ldots, \alpha_i) = \{(\alpha_1', \alpha_2', \ldots) \mid \alpha_k' = \alpha_k, k = 1, \ldots, i\}$, $\mathfrak{N}_i = \{N(\alpha_1, \ldots, \alpha_i) \mid \alpha_k \in \Omega, k = 1, \ldots, i\}$. Let K be a closed subset of S and let $f(K) \not\ni p$. Then it follows from the finiteness of $f^{-1}(p)$ that $S(K, \mathfrak{N}_i) \cap f^{-1}(p) = \emptyset$ [6] for some i. Hence $S(f(K), \mathfrak{F}_i) \not\ni p$, i. e. we obtain an nbd $U(p) = [S(f(K), \mathfrak{F}_i)]^c$ of p satisfying $U(p) \cap f(K) = \emptyset$, which shows that $f(K)$ is closed. The other conditions of f are clearly satisfied.

Conversely, if there exist such $N(\Omega)$, S and f, then putting $A_n = \{p \mid f^{-1}(p)$ consists of n points$\}$ we obtain, by [5], an at most n-dimensional subset A_n of R. Thus $R = \bigcup_{n=1}^{\infty} A_n$ is countable-dimensional.

4. Imbedding.

LEMMA 4.1. *Let R be a countable-dimensional space with $R = \bigcup_{n=1}^{\infty} A_n$, $\dim A_n = 0$. Let $\{U_m \mid m = 1, 2, \ldots\}$ be a collection of open sets and $\{F_m \mid m = 1, 2, \ldots\}$ a collection of closed sets such that $F_m \subset U_m$, $m = 1, 2, \ldots$ Then there exists a collection of open sets U_{mr}, $m = 1, 2, \ldots$, $|r| < \sqrt{2}/2m$, r rational, such that*

(1) $F_m \subset U_{mr} \subset \overline{U}_{mr} \subset U_{mr'} \subset \overline{U}_{mr'} \subset U_m$ *for* $r < r'$,

(2) $\overline{U}_{mr} = \bigcap \{U_{mr'} \mid r' > r\}$, $U_{mr} = \bigcup \{\overline{U}_{mr'} \mid r' < r\}$,

(3) $\operatorname{order}_p \{B(U_{mr}) \mid m = 1, 2, \ldots, |r| < \sqrt{2}/2m,\ r \text{ rational}\} \leqslant n-1$ *for each point $p \in A_n$.*

Proof. First we number all rational numbers with $|r| < \sqrt{2}/2m$ so that

$$r_{m1},\quad r_{m2} < r_{m1} < r_{m3},\quad r_{m4} < r_{m2} < r_{m5} < r_{m1} < r_{m6} < r_{m3} < r_{m7},\quad \ldots$$

Then we put

$$N_{m1} = \{r_{m1}\},\quad N_{m2} = \{r_{m2}, r_{m3}\},\quad N_{m3} = \{r_{m4}, r_{m5}, r_{m6}, r_{m7}\},\quad \ldots$$

[6] $S(K, \mathfrak{N}_i)$ denotes the union of all the sets of $\mathfrak{N}_i$ intersecting K.

We shall define U_{mr} satisfying (1), (3) and

(2′) *if r_{mi} and r_{mk} are adjoining numbers contained in $\bigcup_{h=1}^{s-1} N_{mh}$, $r_{mj} \in N_{ms}$ and $r_{mi} < r_{mj} < r_{mk}$, then*

$$U_{r_{mj}} \subset S_{1/s}(\overline{U}_{r_{mi}})\ [7] \quad \text{if } s \text{ is odd},$$

$$(\overline{U}_{r_{mj}})^c \subset S_{1/s}((\overline{U}^c_{r_{mk}})) \quad \text{if } s \text{ is even},$$

where we denote, for brevity, $U_{mr_{mi}}$ by $U_{r_{mi}}$. Then it will easily be seen that $\{U_{mr}\}$ also satisfies (2). In fact, let $U_{r_{mi}}$ satisfy (2′). Let $p \notin \overline{U}_{r_{mi}}$, $r_{mi} \in N_{ms-1}$; then we take an odd t with $t \geqslant \max[s, \varrho(p, \overline{U}_{r_{mi}})]$ and $r_{mj} \in N_{mt}$ which is next to r_{mi} in $\bigcup_{k=1}^{t} N_{mk}$. It follows from (2′) that $U_{r_{mj}} \subset S_{1/t}(\overline{U}_{r_{mi}}) \not\ni p$. Let $p \in U_{r_{mk}}$, $r_{mk} \in N_{ms-1}$; then we take an even t with $t \geqslant \max[s, \varrho(p, (\overline{U}^c_{r_{mk}}))]$ and $r_{mj} \in N_{mt}$ to which r_{mk} is next in $\bigcup_{k=1}^{t} N_{mk}$. It follows from (2′) that $p \in [S_{1/t}((\overline{U}^c_{r_{mk}}))]^c \subset \overline{U}_{r_{mj}}$, proving (2).

We define, by induction, all U_{mr} in such order that $U_{r_{11}}, U_{r_{12}}, U_{r_{21}}, U_{r_{13}}, U_{r_{22}}, U_{r_{31}}, U_{r_{14}}, \ldots$ First we define, in the same way as in the proof of Lemma 2.1, an open set $U_{r_{11}}$ such that

$$F_1 \subset U_{r_{11}} \subset \overline{U}_{r_{11}} \subset U_1, \quad B(U_{r_{11}}) \cap A_1 = \emptyset.$$

Assume that we have defined all $U_{r_{nh}}$ before $U_{r_{mj}}$ and that $r_{mj} \in N_{ms}$. Then we define $U_{r_{mj}}$ as follows: we take $r_{mi}, r_{mk} \in \bigcup_{k=1}^{s-1} N_{mk}$ such that $r_{mi} < r_{mj} < r_{mk}$ and such that r_{mi} and r_{mk} are adjoining in $\bigcup_{k=1}^{s-1} N_{mk}$. We can define $U_{r_{mj}}$ such that

$$\overline{U}_{r_{mi}} \subset U_{r_{mj}} \subset \overline{U}_{r_{mj}} \subset U_{r_{mk}} \cap S_{1/s}(\overline{U}_{r_{mi}}) \quad \text{if } s \text{ is odd},$$

$$[S_{1/s}((\overline{U}^c_{r_{mk}}))]^c \cup \overline{U}_{r_{mi}} \subset U_{r_{mj}} \subset \overline{U}_{r_{mj}} \subset U_{r_{mk}} \quad \text{if } s \text{ is even}$$

and such that

$$\operatorname{order}_p B(\mathfrak{U}_{mj}) \leqslant n-1 \quad \text{for each point} \quad p \in A_n,$$

where $\mathfrak{U}_{mj} = \{U_{r_{11}}, U_{r_{12}}, U_{r_{21}}, \ldots, U_{r_{mj}}\}$. The method of defining $U_{r_{mj}}$ is parallel to that of V_α in the proof of Lemma 2.1, and hence it is left to the reader. Thus the proof of this lemma is complete.

[7] $S_\varepsilon(U) = \{y \mid \inf\{\varrho(x, y) \mid x \in U\} < \varepsilon\}$, where $\varrho(x, y)$ denotes the distance between x and y.

DEFINITION 4.2. Let $\{\mathfrak{N}_i \mid i=1,2,\ldots\}$ be a collection of star-finite open coverings [8] of a space R. If $\mathfrak{N}=\bigcup_{i=1}^{\infty}\mathfrak{N}_i$ is a basis of open sets, then we call $\{\mathfrak{N}_i \mid i=1,2,\ldots\}$ a *σ-star-finite basis*.

THEOREM 4.3. *Let R be a space with a σ-star-finite basis. Denote by R_ω the set of points in I_ω at most finitely many of whose coordinates are rational. Then R is countable-dimensional if and only if R is homeomorphic to a subset of $N(\Omega)\times R_\omega$ for suitable Ω.*

Proof. Let R be a space with a σ-star-finite basis. Then, since R is homeomorphic to a subset of $N(\Omega)\times I_\omega$ by a theorem of K. Morita [9], we obtain a sequence $\mathfrak{N}_1>\mathfrak{N}_2>\ldots$ of star-finite open coverings such that $\{S(p,\mathfrak{N}_i)\mid i=1,2,\ldots\}$ is an nbd basis of each point p of R. We let

$$\mathfrak{S}_i=\{S^\infty(N,\mathfrak{N}_i)\mid N\in\mathfrak{N}_i\},\quad \text{where}\quad S^\infty(N,\mathfrak{N}_i)=\bigcup_{n=1}^{\infty}S^n(N,\mathfrak{N}_i)\ \text{[10]}.$$

Then we can put $\mathfrak{S}_i=\{S_\alpha \mid \alpha\in\Omega_i\}$, $S_\alpha\cap S_\beta=\emptyset$ for $\alpha\neq\beta$. Since $\mathfrak{N}_i$ is star-finite, for $\alpha\in\Omega_i$ we can put $S_\alpha=\bigcup\{N_{\alpha j}\mid j=1,2,\ldots\}$, $N_{\alpha j}\in\mathfrak{N}_i$. We take an open covering $\mathfrak{P}_i$ of R such that

$$\mathfrak{P}_i=\{P_{\alpha j}\mid \alpha\in\Omega_i,\ j=1,2,\ldots\},\qquad \overline{P}_{\alpha j}\subset N_{\alpha j},\qquad \alpha\in\Omega_i,\qquad j=1,2,\ldots$$

Letting

$$U_{ij}=\bigcup\{N_{\alpha j}\mid \alpha\in\Omega_i\},\qquad F_{ij}=\bigcup\{\overline{P}_{\alpha j}\mid \alpha\in\Omega_i\},$$

we obtain an open set U_{ij} and a closed set F_{ij} with $F_{ij}\subset U_{ij}$, $i,j=1,2,\ldots$ Then we put

$$\{F_{ij}\mid i,j=1,2,\ldots\}=\{F_m\mid m=1,2,\ldots\},$$
$$\{U_{ij}\mid i,j=1,2,\ldots\}=\{U_m\mid m=1,2,\ldots\}.$$

Now, if R is countable-dimensional, then for these F_m and U_m we define U_{mr} by Lemma 4.1. Next we define a real-valued continuous function f_m of R by

$$f_m(p)=\inf\{r\mid p\in U_{mr}\}.$$

Then it is obvious that

$$f_m(F_m)=-\sqrt{2}/2m,\qquad f_m(U_m^c)=\sqrt{2}/2m,\qquad |f_m|\leqslant\sqrt{2}/2m.$$

To show that $f_m(p)$ has an irrational value at every point $p\notin\bigcup\{B(U_{mr})\mid |r|<\sqrt{2}/2m,\ r \text{ rational}\}$ we take any point $p\notin\bigcup_r B(U_{mr})$ and any rational number r with $|r|<\sqrt{2}/2m$. If $p\in U_{mr}$, then there exists, by (2) of Lemma 4.1, r' with $r'<r$, $p\in U_{mr'}$; hence $f_m(p)\leqslant r'<r$. If $p\notin\overline{U}_{mr}$, then there exists, by (2), r' with $r'>r$, $p\notin\overline{U}_{mr'}$; hence $f_m(p)\geqslant r'>r$. Therefore $f_m(p)\neq r$ in either case. Hence at most finitely many of $\{f_1(p),f_2(p),\ldots\}$ are rational by (3) of Lemma 4.1.

[8] A covering $\mathfrak{U}$ is called *star-finite* if each member of $\mathfrak{U}$ intersects finitely many members of $\mathfrak{U}$.

[9] The proof of the theorem is unpublished.

[10] $S^1(N,\mathfrak{N})=S(N,\mathfrak{N})$, $S^n(N,\mathfrak{N})=S(S^{n-1}(N,\mathfrak{N}),\mathfrak{N})$.

Putting $\Omega=\bigcup_{i=1}^{\infty}\Omega_i$, we define a continuous mapping c of R into $N(\Omega)$ by

$$c(p)=a=(a_1,a_2,\ldots)\quad\text{for}\quad p\in S_{a_i},\ a_i\in\Omega_i,\quad i=1,2,\ldots$$

Finally we define a continuous mapping φ of R into $N(\Omega)\times R_\omega$ by

$$\varphi(p)=\big(c(p),f_1(p),f_2(p),\ldots\big)\in N(\Omega)\times R_\omega\quad\text{for}\quad p\in R.$$

To see that φ is homeomorphic, let $U(p)$ be an nbd of a point p of R. Let $S(p,\mathfrak{N}_i)\subset U(p)$, $p\in F_{ij}=F_m$, $p\in\bigcap_{k=1}^{i}S_{a_k}$. Then we define an nbd $V(f(p))$ of $\varphi(p)$ by

$$V(\varphi(p))=N(a_1,\ldots,a\ )\times N(\varphi(p)),$$

where

$$N(a_1,\ldots,a_i)=\{(a_1',a_2',\ldots)\mid a_k'=a_k,\ k=1,\ldots,i\}\subset N(\Omega),$$
$$N(\varphi(p))=\{(a_1,a_2,\ldots)\mid a_m>0\}\subset R_\omega.$$

It is clear that $\varphi^{-1}[V(\varphi(p))]\subset U(p)$, proving this assertion. Thus the "only if" part of this theorem is valid.

Conversely, since

$$N(\Omega)\times R_\omega=\bigcup_{n=1}^{\infty}[N(\Omega)\times R_n]$$

for $R_n=\{(a_1,a_2,\ldots)\mid a_j \text{ are irrational for } j>n,\ |a_i|\leqslant 1/i \text{ for } i=1,2,\ldots\}$ and since $\dim R_n=n$, $N(\Omega)\times R_\omega$ is a countable-dimensional space. This proves the "if" part of the theorem.

The following corollary is a direct consequence of Theorem 4.3:

COROLLARY 4.4. *Let R be a separable space. Then R is countable-dimensional if and only if R is homeomorphic to a subset of R_ω.*

5. The countable sum of zero-dimensional closed sets.

DEFINITION 5.1. If a space R is the countable sum of finite-dimensional closed sets, then we call R a *strong countable-dimensional space*.

A strong countable-dimensional space is countable-dimensional, but the converse is not true.

EXAMPLE 5.2. *The countable-dimensional space R_ω of Theorem 4.3 is not a strong countable-dimensional space.*

Proof. Assume the contrary, i. e. $R_\omega = \bigcup_{n=1}^{\infty} F_n$ for finite-dimensional closed sets F_n, $n = 1, 2, \ldots$ First we notice that for F_j and for every number a_k with $a_k \in L_k = \{x \mid |x| \leqslant 1/k\}$ and every open interval I_{j+p} with $I_{j+p} \subset L_{j+p}$, $p = 1, 2, \ldots$, there exist open intervals $J_{j+p} \subset I_{j+p}$, $p = 1, 2, \ldots$, such that

$$\{a_1\} \times \ldots \times \{a_j\} \times J_{j+1} \times J_{j+2} \times \ldots \subset F_j^c,$$

For, if the assertion is false, then there exist $a_1, \ldots, a_j$ and $I_{j+1}, I_{j+2}, \ldots$ such that for every $J_{j+p} \subset I_{j+p}$, $p = 1, 2, \ldots$,

$$[\{a_1\} \times \ldots \times \{a_j\} \times J_{j+1} \times J_{j+2} \times \ldots] \cap F_j \neq \emptyset .$$

Hence $x \in [\{a_1\} \times \ldots \times \{a_j\} \times I_{j+1} \times I_{j+2} \times \ldots] \cap R_\omega$ implies $x \in \overline{F}_j = F_j$, which means

$$[\{a_1\} \times \ldots \times \{a_j\} \times I_{j+1} \times \ldots] \cap R_\omega \subset F_j .$$

Since $\dim[\{a_1\} \times \ldots \times \{a_j\} \times I_{j+1} \times \ldots] \cap R_{j+p} = p$, $\dim[\{a_1\} \times \ldots \times \{a_j\} \times I_{j+1} \times \ldots] \cap R_\omega = \infty$, which contradicts $\dim F_j < \infty$.

Now we take an irrational a_1 with $a_1 \in L_1$. Then there exist, by the above notice, open intervals $J_{1k} \subset L_k$, $k = 2, 3, \ldots$, such that

$$\{a_1\} \times J_{12} \times J_{13} \times \ldots \subset F_1^c .$$

Let a_2 be an arbitrary irrational with $a_2 \in J_{12}$; there exist open intervals $J_{2k} \subset J_{1k}$, $k = 3, 4, \ldots$, such that

$$\{a_1\} \times \{a_2\} \times J_{23} \times J_{24} \times \ldots \subset F_2^c .$$

Let a_3 be an arbitrary irrational with $a_3 \in J_{23}$; then there exist open intervals $J_{3k} \subset J_{2k}$, $k = 4, 5, \ldots$, such that

$$\{a_1\} \times \{a_2\} \times \{a_3\} \times J_{34} \times J_{35} \times \ldots \subset F_3^c .$$

By repeating such processes we have a sequence $a_1, a_2, \ldots$ of irrational numbers satisfying

$$(a_1, a_2, \ldots) \in R_\omega - \bigcup_{n=1}^{\infty} F_n ,$$

which contradicts $R_\omega = \bigcup_{n=1}^{\infty} F_n$. Therefore R_ω cannot be a countable sum of finite-dimensional closed sets.

THEOREM 5.3. *In order that R be a strong countable-dimensional space it is necessary and sufficient that there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \ldots$ (11) of open coverings $\mathfrak{U}_i$, $i = 1, 2, \ldots$, of R such that*

(1) $\{S(p, \mathfrak{U}_i) \mid i = 1, 2, \ldots\}$ *is an nbd basis of each point p of R,*

(2) $\sup\{\mathrm{order}_p\, \mathfrak{U}_i \mid i = 1, 2, \ldots\} < +\infty$ *for each point p of R.*

(11) $\mathfrak{U}^* = \{S(U, \mathfrak{U}) \mid U \in \mathfrak{U}\}$.

Proof. Necessity. Let $R = \bigcup_{k=1}^{\infty} F_k$ for closed sets F_k, $k = 1, 2, \ldots$, with $\dim F_k = n_k$. In order to prove the necessity it is enough to show that for every open covering $\mathfrak{S}$ there exists an open covering $\mathfrak{U}$ such that $\mathfrak{U} < \mathfrak{S}$, $\mathrm{order}_p\, \mathfrak{U} \leqslant m_k = n_1 + \ldots + n_k + k$ for each point p of F_k. Let $\mathfrak{U}_k$ be an open covering of F_k satisfying $\mathfrak{U}_k < \mathfrak{S}$, order $\mathfrak{U}_k \leqslant n_k + 1$. Let

$$\mathfrak{U}_k = \{U_\alpha \mid \alpha \in \Omega\}, \quad U_\alpha \subset S_\alpha \in \mathfrak{S}, \quad \alpha \in \Omega .$$

Then for every point $p \in U_\alpha$ we define $\varepsilon(p) > 0$ such that

$$S_{\varepsilon(p)}(p) \subset S_\alpha , \quad [S_{\varepsilon(p)}(p)] \cap F_k \subset U'_\alpha .$$

Letting

$$U'_\alpha = \bigcup \{S_{\varepsilon(p)/2}(p) \mid p \in U_\alpha\},$$

we obtain a collection $\mathfrak{U}'_k = \{U'_\alpha \mid \alpha \in \Omega\} < \mathfrak{S}$ of open sets with order $\mathfrak{U}'_k \leqslant n_k + 1$. Then

$$\mathfrak{V}_k = \{[\bigcup_{i=1}^{k-1} F_i]^c \cap U'_\alpha \mid \alpha \in \Omega\}$$

is a collection of open sets covering $F_k - \bigcup_{i=1}^{k-1} F_i$ such that

$$\mathfrak{V}_k < \mathfrak{S},$$

$$\mathrm{order}_p\, \mathfrak{V}_k \leqslant n_k + 1 \quad \text{for} \quad p \in \bigcup_{i=k}^{\infty} F_i ,$$

$$\mathrm{order}_p\, \mathfrak{V}_k = 0 \quad \text{for} \quad p \in \bigcup_{i=1}^{k-1} F_i .$$

Therefore $\mathfrak{U} = \bigcup_{k=1}^{\infty} \mathfrak{V}_k$ is the required open covering.

Sufficiency. We let

$$F_k = \{p \mid \sup\{\mathrm{order}_p\, \mathfrak{U}_i \mid i = 1, 2, \ldots\} \leqslant k\}, \quad k = 1, 2, \ldots$$

Then F_k is cleary a closed set. Since $\mathfrak{U}_i$, restricted to F_k, is of order $\leqslant k$, it follows from [6], [7] that $\dim F_k \leqslant k$. In consequence $R = \bigcup_{k=1}^{\infty} F_k$ is a strong countable-dimensional space.

THEOREM 5.4. *In order that a space R with a σ-star-finite basis be a strong countable-dimensional space it is necessary and sufficient that R be homeomorphic with a subset of $N(\Omega) \times K_\omega$ for suitable Ω, where $K_\omega = \bigcup_{k=1}^{\infty} K_k$, $K_k = \{(a_1, a_2, \ldots) \mid |a_i| \leqslant 1/i$ for $i = 1, \ldots, k$, $a_i = 0$ for $i > k\}$.*

Proof. Since the sufficiency is obvious, we prove only the necessity. Let $R = \bigcup_{k=1}^{\infty} F_k$ for closed sets F_k with $\dim F_k = n_k$, $k = 1, 2, \ldots$ For convenience we rewrite K_{m_k} for $m_k = 2(n_1 + \ldots + n_k + k) - 1$ with K_k; then $\dim K_k = m_k$. If R has a σ-star-finite basis, then we obtain the open sets U_{ij} and closed sets F_{ij} in the proof of Theorem 4.3. Since $F_{ij} \subset U_{ij}$, $\{F_{ij}^c, U_{ij}\}$ is an open covering of R. We denote by $\{\mathfrak{U}_n \mid n = 1, 2, \ldots\}$ the totality of such coverings. Furthermore, we define the following notation:

$$C(R) = \{f \mid f \text{ is a continuous mapping of } R \text{ into } K_\omega \text{ and maps } F_k \text{ into } K_k \text{ for } k = 1, 2, \ldots\},$$

$$C_n(R) = \{f \mid f \in C(R), \text{ for every point } x \text{ of } I_\omega \text{ there exists an nbd } U(x) \text{ of } x \text{ such that } f^{-1}(U(x)) \in \mathfrak{U}_n\},$$

where we denote by $f^{-1}(U(x)) \in \mathfrak{U}_n$ the fact that $f^{-1}(U(x)) \subset U$ for some $U \in \mathfrak{U}_n$.

Now let us show that $\bigcap_{n=1}^{\infty} C_n(R) \neq \emptyset$. To see this we prove first that $C_n(R)$ is open in the functional space $C(R)$, which is a complete space with strong topology. The method of proof is analogous to that of the finite-dimensional case [2]. Assume that $f \in C_n(R)$. This means that every point x of I_ω has an nbd $U(x)$ with $f^{-1}(U(x)) \in \mathfrak{U}_n$. Since I_ω is compact, there exists a finite sub-collection of these nbds which covers I_ω, i. e. $I_\omega = \bigcup_{j=1}^{s} U(x_j)$. We take a positive number δ such that for every $x \in I_\omega$ and for some x_j, $S_\delta(x) \subset U(x_j)$ holds. Now let g be any mapping satisfying $\varrho'(f, g) < \delta/6$, where we denote by ϱ' the metric of $C(R)$. Let $x \in I_\omega$, $g^{-1}(S_{\delta/6}(x)) = N$. Then it is easy to see that $f(N) \subset S_\delta(f(y)) \subset U(x_j)$ for some x_j. Hence $N \subset f^{-1}(U(x_j)) \in \mathfrak{U}_n$, proving $g \in C_n(R)$.

Next we shall show that $C_n(R)$ is dense in $C(R)$. Let f be an arbitrary element of $C(R)$ and ε a positive number. We shall construct g such that $\varrho'(f, g) < \varepsilon$, $g \in C_n(R)$. Let

$$I_\omega = \bigcup_{j=1}^{s} S_{\varepsilon/4}(x_j), \qquad \mathfrak{V} = \{f^{-1}(S_{\varepsilon/4}(x_j)) \mid j = 1, \ldots, s\}.$$

Then we define a finite open covering $\mathfrak{N}$ of R such that

(1) $\mathfrak{N}^\Delta < \mathfrak{V} \wedge \mathfrak{U}_n$ [12].

[12] $\mathfrak{N}^\Delta = \{S(p, \mathfrak{N}) \mid p \in R\}$.

By the proof of Theorem 5.3 we can select an open covering $\mathfrak{W}$ so that

(2) $\mathfrak{N} > \mathfrak{W} = \bigcup_{k=1}^{r} \mathfrak{W}_k$,

(3) $\operatorname{order}_p \mathfrak{W}_k \leqslant n_k + 1$ for $p \in F_i$, $i \geqslant k$,

(4) $\operatorname{order}_p \mathfrak{W}_k = 0$ for $p \in F_i$, $i < k$.

We notice that we can assume without loss of generality, that $n_1 < n_2 < \ldots$ and hence we can assume that r is a finite number not greater than the number of elements of $\mathfrak{N}$. To see this, let $\mathfrak{N} = \{N_j \mid j = 1, \ldots, l\}$, and assume that $\bigcup_{k=1}^{l-1} \mathfrak{W}_k$ does not yet cover R. Then, since $l \leqslant n_l + 1$, putting $\mathfrak{W}_l = \{[\bigcup_{i=1}^{l-1} F_i]^c \cap N_j \mid j = 1, \ldots, l\}$, we get a covering $\mathfrak{W} = \bigcup_{k=1}^{l} \mathfrak{W}_k$ satisfying (2)-(4) for $r = l$. Now let $\mathfrak{W}_k = \{W_{ki} \mid i = 1, \ldots, t_k\}$. Then we can select vertices $x(W_{ki})$ in $K_k - K_{k-1}$ such that

(5) $\varrho(x(W_{ki}), f(W_{ki})) < \varepsilon/4$, $k = 1, \ldots, r$, $i = 1, \ldots, t_k$,

(6) any $2n_1 + 2$ of the vertices $x(W_{1i})$ and any $2n_2 + 2$ of the vertices $x(W_{2i})$ and ... and any $2n_r + 2$ of the vertices $x(W)_{ir}$ are linearly independent,

because K_k can be regarded as the $2(n_1 + \ldots + n_k + k) - 1 = m_k$-cube containing K_{k-1}.

Then we define a Kuratowski mapping g by

$$g(p) = \frac{\sum_{k,i} \varrho(p, W_{ki}^c) x(W_{ki})}{\sum_{k,i} \varrho(p, W_{ki}^c)};$$

in this formula we regard $x(W_{ki})$ as a point-vector. Let p be an arbitrary point of R and assume $p \in W_{ki}$. Then from (1), (2), (5) we get $\varrho(f(p), x(W_{ki})) < 3\varepsilon/4$ for every W_{ki} with $p \in W_{ki}$. Hence the centre of gravity $g(p)$ of the $x(W_{ki})$ satisfies

$$\varrho(f(p), g(p)) < 3\varepsilon/4, \quad \text{i. e.} \quad \varrho'(f, g) < \varepsilon.$$

Finally we shall prove $g \in C_n(R)$. Suppose that $W_{k_1 i_1}, \ldots, W_{k_s i_s}$ are all the members of $\mathfrak{W}$ containing a given point p of R. Then we denote by $L(p)$ the linear $(s-1)$-space spanned by $x(W_{k_1 i_1}), \ldots, x(W_{k_s i_s})$. Since there is only a finite number of linear subspaces $L(p)$, there exists a number $\delta > 0$ such that any two of those linear subspaces, $L(p)$ and $L(p')$, either meet or else are at a distance $\geqslant \delta$ from each other. If $p, p' \in g^{-1}(S_{\delta/2}(y))$ for some $y \in I_\omega$, then $\varrho(g(p), g(p')) < \varepsilon$. Since $g(p) \in L(p)$ and $g(p') \in L(p')$ are clearly verified, we have $L(p) \cap L(p') \neq \emptyset$. Let $L(p)$ be spanned by $x(W_{k_1 i_1}), \ldots, x(W_{k_s i_s})$ and $L(p')$ by $x(W_{h_1 j_1}), \ldots, x(W_{h_t j_t})$. Then

$x(W_{k_1i_1}), \ldots, x(W_{k_si_s}), x(W_{h_1j_1}), \ldots, x(W_{h_tj_t})$ are linearly dependent. On the other hand, by (3), (4) at most n_k+1 of the $x(W_{k_1i_1}), \ldots, x(W_{k_si_s})$ spanning $L(p)$ are vertices corresponding to members of $\mathfrak{W}_k$. This combined with (6) implies that at least one of the $x(W_{k_1i_1}), \ldots, x(W_{k_si_s})$ is also one of the $x(W_{h_1j_1}), \ldots, x(W_{h_tj_t})$. Hence p and p' are contained in a common member of $\mathfrak{W}$. It follows from (1), (2) that $g^{-1}(S_{\delta/2}(y)) \in \mathfrak{U}_n$, meaning $g \in C_n(R)$. Thus we have concluded that $C_n(R)$ is open and dense in $C(R)$. In consequence, by Baire's theorem, $\bigcap_{n=1}^{\infty} C_n(R)$ is dense in $C(R)$, and especially $\bigcap_{n=1}^{\infty} C_n(R) \neq \emptyset$. Since, for any element f of $\bigcap_{n=1}^{\infty} C_n(R)$ and for the same mapping $c(p)$ with the one in the proof of theorem 4.3, $\varphi(p) = (c(p), f(p))$ is clearly a homeomorphic mapping of R onto a subset of $N(\Omega) \times K_\omega$, the assertion is established.

COROLLARY 5.5. *Let R be a separable space. Then R is strongly countable-dimensional if and only if R is homeomorphic to a subset of K_ω.*

References

[1] S. Eilenberg et E. Otto, *Quelques propriétés charactéristiques de la dimension*, Fund. Math. 31 (1938), p. 149-153.

[2] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1941.

[3] M. Katětov, *On the dimension of non-separable spaces I*, Czechoslovak Mathematical Journal 2 (77) (1952), p. 333-368.

[4] K. Morita, *Normal families and dimension theory for metric spaces*, Math. Annalen 128 (1954), p. 350-362.

[5] — *A condition for the metrizability of topological spaces and for n-dimensionality*, Sciences Reports of the Tokyo Kyoiku Daigaku, Sec. A, 5 (1955), p. 33-36.

[6] J. Nagata, *Note on dimension theory*, Proc. Japan Acad. 32 (1956), p. 568-573.

[7] — *Note on dimension theory for metric spaces*, Fund. Math. 45 (1957), p. 143-181.

[8] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), p. 977-982.

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Concerning dense metric subspaces of certain non-metric spaces

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In this paper it is shown that if Σ is a space satisfying R. L. Moore's Axioms 0 and 1, [1], then Σ contains a complete metric subspace Σ' such that the set of all points of Σ' forms a dense subset of the set of all points of Σ. A sufficient condition is given for a point set M in order that it be the set of all points of some such Σ'. The terminology used in the paper is largely that of R. L. Moore.

AXIOM 0. Every region is a point set.

AXIOM 1. There exists a sequence $G_1, G_2, G_3, \ldots$ such that

(1) for each positive integer n, G_n is a collection of regions covering the set of all points,

(2) for each positive integer n, G_{n+1} is a subcollection of G_n,

(3) if R is a region and A is a point of R and B is a point of R, there is a positive integer n such that if g is a region of G_n containing A, then $\bar{g}$ is a subset of R and, unless B is A, $\bar{g}$ does not contain B,

(4) if $M_1, M_2, M_3, \ldots$ is a sequence of closed point sets and for each positive integer n there is a region g_n of G_n such that M_n is a subset of $\bar{g}$ and for each positive integer n, M_{n+1} is a subset of M_n, then there is a point common to all the sets of this sequence.

It has been shown that every space satisfying Axiom 0 and the following Axiom C is metric [2]:

AXIOM C. There exists a sequence $G_1, G_2, G_3, \ldots$ satisfying conditions (1), (2) and (4) of Axiom 1 together with the following condition

(3) if A is a point of a region R and B is a point of R, there is a positive integer n such that if x is a region of G_n containing A, and y is a region of G_n intersecting x, then $x+y$ is a subset of R and, unless B is A, $x+y$ does not contain B.

PROPERTY Q. A point set M is said to *have Property* Q provided it is true that if G is a collection of domains covering S, the set of all