On cyclically ordered intervals of integers

by

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A relation $[x, y, z]$ defined on all ordered triplets of different integers x, y, z from the interval $\{0, 1, \ldots, N\}$ is called a *cyclically ordering relation* in this interval if it satisfies for $0 \leqslant x, y, z, x+v, y+v, z+v, u \leqslant N$ the following postulates

I. *Either* $[x, y, z]$ *or* $[z, y, x]$,

II. $[x, y, z]$ *implies* $[y, z, x]$,

III. $[x, y, z]$ *and* $[y, u, z]$ *implies* $[x, u, z]$,

IV. $[x, y, z]$ *implies* $[x+v, y+v, z+v]$.

An example. Let η be a real number such that $p_x = \exp(2\pi i x\eta)$, where $x = 0, 1, \ldots, N$, are different points on the circle $|z| = 1$. We establish a sense on this circle, say the counter-clock-wise sense, and denote the open arc with the initial point p_x and the endpoint p_y by (p_x, p_y). Thus (p_x, p_y) is empty if and only if $x = y$. Defining

$$[x, y, z]_\eta \equiv p_y \in (p_x, p_z)$$

we obtain a cyclically ordering relation on $\{0, 1, \ldots, N\}$.

The purpose of this paper is to prove the following (announced in [1])

Theorem. *For every cyclically ordering relation* $[x, y, z]$ *on* $\{0, 1, \ldots, N\}$ *there exists an interval* I *of real numbers* η *for which*

$$[x, y, z] \equiv [x, y, z]_\eta .$$

If η is irrational, then $[x, y, z]_\eta$ is a cyclically odering relation on the set of all integers and thus

Corollary 1. *Every relation* $[x, y, z]$ *on* $\{0, 1, \ldots, N\}$ *can be extended to a cyclically ordering relation on the set of all integers.*

Let us say that y *follows immediately after* x if $[x, z, y]$ is always false $(0 \leqslant x, y, z \leqslant N)$. If η satisfies the assertion of our theorem, then a number y follows immediately after another one, say x, if and only if the arc (p_x, p_y) contains no points p_z with $z \leqslant N$. Thus there exists for every x strictly one y which follows immediately after x. Let a be the number which follows immediately after 0 and b the one which is immediately followed by 0. From a result concerning the distribution of the points p_x on the circle $|z| = 1$ (see [2]) and from our theorem it follows easily that

COROLLARY 2. *The differences $y-x$, where y follows immediately after x, take at most three values. They are $a, -b$ and $a-b$ (the last value occurs only in the case $N < a+b-1$).*

It is not true that every relation $[x, y, z]$ defined on the set of all integers is of the form $[x, y, z]_\eta$ (see [3]).

Outline of the proof of the theorem. We show first that it is sufficient to prove the theorem for those relations for which $[0, 1, 2]$ is true. Then we prove that two relations $_1[x, y, z]$, $_2[x, y, z]$ which coincide on all triplets $\{0, x, x+1\}$, where $0 \leqslant x < N$, coincide on $\{0, \ldots, N\}$ (Lemma 1). Then we consider an arbitrary fixed relation $[x, y, z]$ and prove that for the sequence $m_1 < m_2 < \ldots < m_r$ of those numbers which satisfy $[0, m_i+1, m_i]$ we have (Lemma 2)

$$m_k + m_l \leqslant m_{k+l} \leqslant m_k + m_l + 1 .$$

Finally (Lemma 3) we show that the above inequalities enable us to construct an interval I such that for $\eta \in I$ and $0 \leqslant x < N$ the relation $[0, x+1, x]_\eta$ holds only with $x = m_1, \ldots, m_r$. Thus, by Lemma 1 and postulate I, $[x, y, z]$ and $[x, y, z]_\eta$ coincide on $\{0, 1, \ldots, N\}$.

We suppose first that our theorem is already proved for those relations which hold on $\langle 0, 1, 2\rangle$. If $[x, y, z]$ is a relation which does not hold on $\langle 0, 1, 2\rangle$, then let us define the relation $[x, y, z]^*$ by

$$[x, y, z]^* \equiv [z, y, x] \quad \text{for} \quad 0 \leqslant x, y, z \leqslant N .$$

Since evidently $[0, 1, 2]^*$ holds, we have $[x, y, z]^* \equiv [x, y, z]_\eta$ for some interval I and $\eta \in I$. Thus $[x, y, z] \equiv [x, y, z]_{-\eta}$ for $\eta \in I$ and it follows that our theorem is true for $[x, y, z]$.

In the sequel we shall consider only relations which hold on the triplet $\langle 0, 1, 2\rangle$.

LEMMA 1. *If $_1[x, y, z]$ and $_2[x, y, z]$ are such relations that*

$$_1[0, x, x+1] \equiv {_2[0, x, x+1]} ,$$

then these relations are equal.

Proof. **1.1.** *If $x, y, z, u \in \{0, 1, \ldots, N\}$ are different and $p = [x, y, z]$, $q = [y, z, u]$, $s = [x, u, z]$, $t = [u, x, y]$, then*

$$p \equiv s \cdot q \vee q \cdot t \vee t \cdot s$$

($\cdot$, $\vee$ are the symbols of conjunction and alternation of sentences).

Indeed, suppose that $s \cdot q \vee q \cdot t \vee t \cdot s = 1$ ($1 =$ true, $0 =$ false). Let be $s \cdot q = 1$. Then $[x, u, z]$ and $[y, z, u]$, what by II and III implies $[x, y, z]$, i. e. $p = 1$. Similarly we obtain $p = 1$ from $q \cdot t = 1$ or $t \cdot s = 1$.

Now suppose that $s \cdot q \vee q \cdot t \vee t \cdot s = 0$. We verify $(s \cdot q \vee q \cdot t \vee t \cdot s)' \equiv s' \cdot q' \vee q' \cdot t' \vee t' \cdot s'$ ($'$ denotes negation). As before we find that $s' \cdot q' \vee q' \cdot t' \vee t' \cdot s' = 1$ implies $[z, y, x]$, i. e. $p' = 1$.

As a consequence of 1.1 we infer that

1.2. *If two relations $_1[x, y, z]$ and $_2[x, y, z]$ coincide on three triplets out of four arguments, then they coincide also on the remaining triplet.*

1.3. Bearing in mind the postulates I, ..., IV we observe that the conclusion of Lemma 1 follows if we prove the equivalences $E_{x,y}$:

$$_1[0, x, y] \equiv {_2[0, x, y]} \quad \text{for} \quad 0 < x < y \leqslant N .$$

We shall prove them by induction. $E_{1,2}$ is true since we consider only relations which hold on $\langle 0, 1, 2\rangle$. Suppose that we have proved $E_{1,y}$ for some $y \geqslant 2$. Then both relations considered coincide on the triplets $\{0, 1, y\}$, $\{0, y, y+1\}$ and $\{1, y, y+1\}$ (by IV and $E_{y-1,y}$). Thus 1.2 implies their coincidence on $\{0, 1, y+1\}$, i. e. $E_{1,y+1}$.

Now suppose that $E_{x,y}$ for some x. Proving $E_{x+1,y}$ we may assume $x+1 < y$. Thus both relations coincide on $\{0, x, y\}$, $\{0, x, x+1\}$, $\{x, x+1, y\}$ (by $E_{1,y-x}$). Consequently they coincide also on the remaining triplet $\{0, x+1, y\}$.

DEFINITION 1. Let $[x, y, z]$ be a cyclically ordering relation on $\{0, 1, \ldots, N\}$. We shall denote by $m_1, m_2, \ldots, m_r$ the *increasing sequence of all numbers m_k for which $[0, m_k+1, m_k]$ holds.*

Remark. For a relation $[x, y, z]_\eta$, where $0 < \eta < 1$, the numbers $m_1, \ldots, m_r$ are exactly those which satisfy $m_k\eta < k < (m_k+1)\eta$, $k = 1, 2, \ldots, r$ and $m_k < N$.

LEMMA 2. $m_k + m_l \leqslant m_{k+l} \leqslant m_k + m_l + 1$.

Proof. **2.1.** *If the numbers $m_k + m_l$, $m_k + m_l + 1$ are smaller than N, then one of them belongs to the sequence $m_1, \ldots, m_r$.*

Assume $m_k + m_l \neq m_i$, $i = 1, 2, \ldots, r$. *Then*

(1) $[m_k, m_k+m_l+1, m_k+m_l]$ by $[0, m_l+1, m_l]$;

(2) $[m_k+1, m_k+m_l+2, m_k+m_l+1]$ by (1);

(3) $[0, m_k, m_k+m_l+1]$ by $[0, m_k+m_l, m_k+m_l+1]$ and (1);

(4) $[0, m_k+1, m_k+m_l+1]$ by $[0, m_k+1, m_k]$ and (3);

(5) $[0, m_k+m_l+2, m_k+m_l+1]$ by (2) and (4);

This proves $m_k + m_l + 1 = m_i$ for some i.

2.2. $m_k+m_1 \leqslant m_{k+1} \leqslant m_k+m_1+1$.

We prove first $m_{k+1} \leqslant m_k+m_1+1$. This inequality evidently holds if $m+m_1+1 \geqslant N$. In the other case it follows easily from 2.1. By 2.1 we also see that $m_k+m_1 \leqslant m_{k+1}$ will be proved if we show that $[0, m_k+x-1, m_k+x]$ for $x=2, \ldots, m_1$. We have

(1) $$[0,1,2], [0,2,3], \ldots, [0, m_1-1, m_1].$$

(2) $$[0,1,x] \quad \text{for} \quad x=2, \ldots, m_1.$$

Indeed this is true for $x=2$ and if $[0,1,x]$ holds for some $x<m_1$, then also $[0,x,x+1]$ is true by (1), which implies $[0,x,x+1]$. Thus for $x=2, \ldots, m_1$

(3) $$[m_k, m_k+1, m_k+x] \quad \text{by} \quad (2);$$

(4) $$[0, m_k+1, m_k+x] \quad \text{by} \quad [0, m_k+1, m_k] \text{ and } (3).$$

Substituting $x=2$ in (4) we obtain the first relation which we wish to prove. Now let $x>2$. Then

(5) $$[m_k+1, m_k+x-1, m_k+x] \quad \text{by} \quad (1);$$

(6) $$[0, m_k+x-1, m_k+x] \quad \text{by} \quad (4) \text{ and } (5).$$

Thus 2.2. is proved.

2.3. We shall prove

$$m_k+m_l \leqslant m_{k+l} \leqslant m_k+m_l+1$$

by induction on l. The first step is done in 2.2. Now suppose that both the above equalities hold for some k, l.

Let us show that $m_k+m_{l+1} \leqslant m_{k+l+1}$. If $m_k+m_l<m_{k+l}$ then the inequality proved easily follows. Indeed it is sufficient to observe that (by 2.2) m_k+m_{l+1} exceeds m_k+m_l by m_1 or m_1+1 and m_{k+l+1} exceeds m_{k+l} also by m_1 or m_1+1. Now if $m_k+m_l=m_{k+l}$, then the above argument is insufficient only in the case where

$$m_{l+1}=m_l+m_1+1 \quad \text{and} \quad m_{k+l+1}=m_{k+l}+m_1.$$

Let us prove that this case is impossible. Indeed, observe that $[0,1,2]$ implies $m_1 \geqslant 2$ and thus

(1) $$[0, m_k-1, m_k] \quad \text{by} \quad m_{k-1} \leqslant m_k-m_1<m_k-1;$$

(2) $$[m_l+m_1, m_k+m_l+m_1-1, m_k+m_l+m_1] \quad \text{by} \quad (1);$$

(3) $$[m_l+m_1-1, m_k+m_l+m_1, m_k+m_l+m_1-1] \quad \text{by} \quad [0, m_k+1, m_k];$$

(4) $$[m_l+m_1-1, m_k+m_l+m_1, m_l+m_1] \quad \text{by (2) and (3)};$$

(5) $$[m_l+m_1, m_k+m_l+m_1+1, m_l+m_1+1] \quad \text{by} \quad (4);$$

(6) $$[0, m_k+m_l+m_1, m_l+m_1] \quad \text{by} \quad [0, m_l+m_1-1, m_l+m_1] \text{ and } (4);$$

(7) $$[0, m_l+m_1, m_k+m_l+m_1+1] \quad \text{by} \quad [0, m_l+m_1, m_l+m_1+1] \text{ and } (5);$$

(8) $$[0, m_k+m_l+m_1, m_k+m_l+m_1+1] \quad \text{by} \quad (6) \text{ and } (7).$$

Thus we have obtained $[0, m_{k+l+1}, m_{k+l+1}+1]$, which is a contradiction.

It remains to prove $m_{k+l+1} \leqslant m_k+m_{l+1}+1$. This inequality easily follows if $m_{k+l}<m_k+m_l+1$ (it is sufficient to apply 2.2). If $m_{k+l}=m_k+m_l+1$ then 2.2 is inapplicable only in the case of

$$m_{l+1}=m_l+m_1, \quad m_{k+l+1}=m_{k+l}+m_1+1.$$

This case however is impossible, since by 2.1 one of the numbers m_k+m_{l+1}, $m_k+m_{l+1}+1$ must belong to the sequence $m_1, \ldots, m_r$ and this contradicts

$$m_{k+l}<m_k+m_{l+1}<m_k+m_{l+1}+1=m_k+m_l+m_1+1<m_{k+l+1}.$$

Lemma 3. *There exists an interval I such that for $\eta \in I$ the relation $[x,y,z]_\eta$ is defined on $\{0,1,\ldots,N\}$ and $[0,x+1,x]_\eta$ holds strictly for $x=m_1, m_2, \ldots, m_r$.*

Proof. We shall find a number $m_{r+1} \geqslant N$ such that there will exist an interval I of numbers η satisfying $m_k\eta<k<(m_k+1)\eta$ for $k=1,2,\ldots, r+1$. Then Lemma 3 holds by our remark on Definition 1.

3.1. There exists a number $m_{r+1} \geqslant N$ such that

(*) $$m_k+m_l \leqslant m_{k+l} \leqslant m_k+m_l+1 \quad \text{for} \quad k+l \leqslant r+1.$$

Consider the numbers $q_k=m_k+m_{r-k+1}$ where $k \leqslant r$. If they are all equal, say, to some q, then (*) evidently holds if $m_{r+1}=q+1$ (cf. Lemma 2). In the other case define $m_{r+1}=\max q_k$. Let $m_{r+1}=q_s$. The first inequality in (*) obviously holds. In order to prove the second one we ought to show that $m_s+m_{r-s+1} \leqslant m_k+m_{r-k+1}+1$. This follows easily from Lemma 2. Indeed, if $k<s$ then it is sufficient to add the inequalities $m_s \leqslant m_k+m_{s-k}+1$, $m_{r-s+1}+m_{s-k} \leqslant m_{r-k+1}$. If $k>s$, then we repeat the above proof with $r-k+1$ and $r-s+1$ interchanged with k, s.

It remains to verify that $N \leqslant m_{r+1}$. Assume the contrary. We have $m_{r+1}=q_l+1$ for some l. Indeed, this is evident when the numbers q_k are equal. In the other case we have $m_{r+1}=q_s \leqslant q_k+1$ by $m_s+m_{r-s+1} \leqslant m_k+m_{r-k+1}+1$ and $q_s=\max q_k$, which proves $q_s=q_l+1$ for some l. From $N>m_{r+1}=q_l+1=m_l+m_{r-l+1}+1$ it follows by 2.1 (Lemma 2) that one of the numbers $m_{r+1}-1$, m_{r+1} belongs to the sequence $m_1, \ldots, m_r$. But this is impossible since $m_{r+1}-1 \geqslant m_r+m_1-1>m_r$ by (*).

3.2. *If (*) holds, then the numbers η which satisfy simultaneously all inequalities $m_k\eta<k<(m_k+1)\eta$, $k=1,2,\ldots,r+1$ fill up a non empty interval I.*

Evidently

$$I = \bigcap_{1\leqslant k\leqslant r+1} \left(\frac{k}{m_k+1}, \frac{k}{m_k}\right)$$

and it remains to prove that

$$\frac{k}{m_k+1} < \frac{l}{m_l} \quad \text{for} \quad k, l \leqslant r+1 .$$

If $k = l = 1$ then this inequality is obvious. Let us suppose that it holds for $k, l \leqslant h$. We shall prove it for $k, l \leqslant h+1$. If $k = h+1$ then a proof is necessary only if $l \leqslant h$. Then $k-l, l \leqslant h$ and thus by our supposition

$$\frac{k-l}{m_{k-l}+1} < \frac{l}{m_l},$$

i. e. $km_l < (m_{k-l}+m_l+1)l$. Since $m_{k-l}+m_l \leqslant m_k$ by Lemma 2, we obtain $km_l < (m_k+1)l$, which we wished to prove.

Now if $l = h+1$, and $k \leqslant h$ then from $k, l-k \leqslant h$ follows

$$\frac{k}{m_k+1} < \frac{l-k}{m_{l-k}}$$

and $k(m_{l-k}+m_k+1) < (m_k+1)l$. It remains to apply $m_l \leqslant m_{l-k}+m_k+1$.

It is evident that Lemma 3 implies

$$[0, x+1, x] \equiv [0, x+1, x]_\eta \quad \text{for} \quad \eta \in I, \quad x = 1, 2, \ldots, N-1 .$$

Thus our theorem follows from Lemma 1.

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Dense families of continuous selections *

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1. Introduction. Let X be a metric space, Y a Banach space, $\mathcal{C}(Y)$ the family of non-empty, closed, convex subsets of Y, and let $\varphi\colon X \to \mathcal{C}(Y)$ be lower semi-continuous (i. e. $\{x \in X \mid \varphi(x) \cap U \neq \varphi\}$ is open in X for every open $U \subset Y$). Under these circumstances, it was proved in [4], Theorem 3.2″ (see also Theorem 1 of the expository paper [3]) that there exists a *selection* f for φ, that is, a continuous $f\colon X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$. In the present paper, this result is applied to prove Theorem 1.1 below and some of its consequences. A special case of Theorem 1.1 will be used by V. L. Klee [2].

THEOREM 1.1. *For every infinite cardinal a, there exists a family Φ of selections for φ, with card $\Phi \leqslant a$, such that, whenever $x \in X$ and $\varphi(x)$ has a dense subset of cardinality $\leqslant a$, then $\{f(x)\}_{f\in\Phi}$ is dense in $\varphi(x)$* (1).

Our first corollary generalizes the well-known result that the Banach space of continuous, real-valued functions on a compact metric space is separable.

COROLLARY 1.2. *If X is compact and if, for some infinite cardinal a, $\varphi(x)$ has a dense subset of ordinality $\leqslant a$ for all $x \in X$, then the space of selection for φ has a uniformly dense subset of cardinality $\leqslant a$.*

If $C \subset \mathcal{C}(Y)$, then a *face* of C is a closed, convex subset F of C such that any line segment in C, which has an interior point in F, must be entirely in F; the *inside* of C, denoted by $I(C)$, is the set of points in C which lie in no face of C. It is known that every separable $C \subset \mathcal{C}(Y)$ has a non-empty inside ([4], Lemma 5.1). As another application of Theorem 1.1, we have the following result, which was obtained in [4], Theorem 3.1‴, for separable Y.

COROLLARY 1.3. *There exists a selection f for φ such that $f(x) \in I(\varphi(x))$ whenever $\varphi(x)$ is separable.*

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(1) For separable Y, with $a = \aleph_0$, this result was already obtained in [4], Lemma 5.2.