On cyclically ordered groups

by

S. Świerczkowski (Wrocław)

A relation $[x, y, z]$ which is defined on all ordered triplets of different elements x, y, z of a group G is called a *cyclic order* if it has the following properties:

I. *Either* $[x, y, z]$ *or* $[z, y, x]$,

II. $[x, y, z]$ *implies* $[y, z, x]$,

III. $[x, y, z]$ *and* $[y, u, z]$ *implies* $[x, u, z]$,

IV. $[x, y, z]$ *implies* $[uxv, uyv, uzv]$ *for* $u, v \in G$.

A group on which a cyclic order is defined will be called a *cyclically ordered group* (for references see [1]).

The natural order of points on a directed circle defines a cyclic order on the group of multiplication of complex numbers of absolute value one. We shall denote this group by K and the cyclic order on K by (x, y, z) (¹).

If Γ is a (linearly) ordered group, then a cyclic order $[x, y, z]$ is defined on Γ by

$$[x, y, z] \equiv x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < x < y .$$

We shall say that this cyclic order is *generated by the order* on Γ.

Cyclically ordered groups can be obtained by the following construction. Let Γ be an ordered group and let $[x, y, z]$ be the cyclic order generated by the order on Γ. We consider the direct product $\Gamma \times K$ (its elements are pairs $\langle x, a\rangle$, $x \in \Gamma$, $a \in K$) and we define a cyclic order on this group by

$$[\langle x, a\rangle, \langle y, b\rangle, \langle z, c\rangle] \equiv \begin{cases} (a, b, c) & \text{in } K \quad \text{if} \quad a \neq b \neq c \neq a, \\ x < y & \text{in } \Gamma \quad \text{if} \quad a = b \neq c, \\ y < z & \text{in } \Gamma \quad \text{if} \quad b = c \neq a, \\ z < x & \text{in } \Gamma \quad \text{if} \quad c = a \neq b, \\ [x, y, z] & \text{in } \Gamma \quad \text{if} \quad a = b = c. \end{cases}$$

This cyclic order on $\Gamma \times K$ will be called the *natural cyclic order*. Evidently every subgroup of $\Gamma \times K$ is also a cyclically ordered group. The aim of this paper is to prove that there exist no other cyclically ordered groups, i. e.

(¹) A more precise definition is given in the remark to Lemma 1.

THEOREM. *If G is a cyclically ordered group, then there exists an ordered group Γ such that*

$$G \subset \Gamma \times K$$

and the cyclic order on G is determined by the natural cyclic order on $\Gamma \times K$.

Let us call a cyclic ordered group *Archimedean* if it contains no elements x, y such that $[e, x^n, y]$ holds for every positive integer n ($e =$ the unity). From our theorem follows the

COROLLARY. *If G is an Archimedean cyclicly ordered group, then $G \subset K$ and the cyclic order on G is carried over from K.*

The proof of the theorem will be preceded by three lemmas. Let us consider first an arbitrary ordered group F. We suppose that F contains an element z such that

(*) z belongs to the centre of F, $z < e$, and for every $x \in F$ there exists an integer n for which $z^n > x$.

Let N be the cyclic group generated by z and let $G = F/N$. For each coset $a \in G$ we denote by r_a the unique representative of a in F such that $e \leqslant r_a < z$. Let $[x, y, z]$ be the cyclic order on F generated by the order.

LEMMA 1 (L. S. Rieger [1]). *If we define for $a, b, c \in G$*

$$[a, b, c] \equiv [r_a, r_b, r_c] \quad in \quad F,$$

then we obtain a cyclic order on G.

Remark. If F is the additive group of real numbers and N is the group of integers, then this cyclic order on G is the natural cyclic order on K.

LEMMA 2 (see Rieger [1]). *Given a cyclically ordered group G, there exists an ordered group F and an element $z \in F$ which satisfies (*) and generates a group N for which $G = F/N$. Moreover the cyclic order on G is then given by the definition in Lemma 1.*

LEMMA 3 ([2]). *If $z > e$ belongs to the centre of an ordered group F, then there exists an ordered group F' which contains F so that the ordering relation of F' passes over to F and is such that to every real number α corresponds an element $z^\alpha \in F'$ and these elements satisfy*

(**) $$z^\alpha z^\beta = z^{\alpha+\beta}, \quad z^\alpha > z^\beta \quad for \quad \alpha > \beta .$$

Moreover every element z^α belongs to the centre of F'.

Proof. In the sequel let z denote a fixed element belonging to the centre of F. For certain rational numbers $r = m/n$ the group F contains an element x for which $x^n = z^m$. We shall denote such element x by z^r. It is easily seen that the set of all these $z^r \in F$ forms a subgroup D of the centre of F. Also (**) holds if α, β are rationals for which $z^\alpha, z^\beta \in F$.

We consider now the group R of real numbers and a group Δ which is isomorphic with R. We denote by ζ^α the element of Δ which corresponds to $\alpha \in R$ so that $\zeta^\alpha \zeta^\beta = \zeta^{\alpha+\beta}$. Let

$$H = F \times \Delta$$

be the direct product of the groups F, Δ. So the elements of H are pairs $\langle x, \zeta^\alpha \rangle$ where $x \in F$, $\zeta^\alpha \in \Delta$. Let F' be the group which we obtain from H if we extend the family of all group-relations in H by adding

$$z^r = \zeta^r \quad \text{for every} \quad z^r \in D .$$

In other words, F' is the factor group of H by its normal subgroup

$$U = \{\langle z^r, \zeta^{-r} \rangle \in H : z^r \in D\} .$$

The elements of F' are U-cosets in H. In the sequel a coset which contains a representative $\langle x, e \rangle$, $x \in F$, will be denoted simply by x. This notation is unique since to $\langle x, e \rangle \neq \langle y, e \rangle$ correspond different cosets. It follows that F is contained in F'. A coset corresponding to $\langle e, \zeta^\alpha \rangle$ will be denoted by z^α. This is justified by the fact that if $z^r \in F$ then the coset with the representative $\langle z^r, e \rangle$, which we have already decided to denote by z^r, is the same as that one which has the representative $\langle e, \zeta^r \rangle$. Let us observe that all elements $z^\alpha \in F'$ belong to the centre of F' and the law of their multiplication is given by the first part of (**).

In our new notation every element of F' has a factorization xz^α where $x \in F$. This factorization is not unique and all others are $xz^r z^{\alpha-r}$ where $z^r \in D$. We consider the elements of F' which have a factorization xz^ϱ where ϱ is rational and $x \in F$. It is evident that these elements form a subgroup of F'. We shall denote this subgroup by Ψ. We extend now the ordering relation on $F \subset \Psi$ to an order on Ψ by defining there the positive elements (i. e. $> e$). We define for $a = xz^\varrho \in \Psi$, $\varrho = m/n$

$$a > e \quad \text{in} \quad \Psi \quad \text{if and only if} \quad x^n z^m > e \quad \text{in} \quad F .$$

It is easily verified that this definition does not depend on the factorization of a. We define an order on Ψ by $a > b \equiv ab^{-1} > e$. This is indeed an order if

I. $a, b > e$ implies $ab > e$.

II. $a > e$ implies $bab^{-1} > e$ for every b.

III. If $a > e$ is not true for some $a \neq e$, then $a^{-1} > e$.

These postulates are equivalent to statements valid in F and they are easily verified. For example we shall prove here III. Let $a = xz^\varrho$. If

([2]) I am much indebted to the reviewer, M. Król, for suggesting an important simplification to my original proof of this lemma.

$x^nz^m > e$ is not true and $xz^\varrho \neq e$, then $x^nz^m \neq e$ and thus $x^nz^m < e$. Consequently $z^{-m}x^{-n} = x^{-n}z^{-m} > e$ and this implies $a^{-1} = x^{-1}z^{-\varrho} > e$.

We shall prove now that there exists a subgroup Φ of F' such that

$$F' = \Psi \times \Phi ,$$

i. e. that Ψ is a direct factor of F'. Let us define Φ. Let $M \subset R$ be a maximal set of irrational numbers which are rationally independent. All rational combinations of elements of M form a subgroup of R. We denote this subgroup by Ω. It is clear that no rational number belongs to Ω except 0. We now define Φ as the subgroup of all those elements z^α for which $\alpha \in \Omega$. Evidently $\Psi \cap \Phi = \{e\}$. Every element $a \in F'$ has a factorization $a = xz^\beta$ where $x \in F$. Since for every β there exists a rational number ϱ such that $\alpha = \beta - \varrho \in \Omega$ we have $a = xz^\varrho z^\alpha$ where $xz^\varrho \in \Psi$, $z^\alpha \in \Phi$. Thus we have proved $F' \subset \Psi\Phi$.

We now define the positive elements of F'. Let $a = xz^\alpha \in F'$ where $x \in \Psi$ and $\alpha \in \Omega$. We set

$$xz^\alpha > e$$

if and only if one of the following conditions holds:

A. $\alpha = 0$ and $x > e$ in Ψ.

B. $xz^\varrho > e$ in Ψ for some rational $\varrho < \alpha$ and $\alpha \neq 0$.

C. $xz^\varrho > e$ for every rational $\varrho > \alpha$ and $xz^\varrho < e$ for every rational $\varrho < \alpha$ in Ψ and $\alpha > 0$.

We define the order in F' by $a > b = ab^{-1} > e$. It follows that the second part of (**) holds. We have to verify that conditions I, II, and III hold. We assume that $a = xz^\alpha$, $b = yz^\beta$; $x, y \in \Psi$; $\alpha, \beta \in \Omega$. Let us verify I. Since Ψ is an ordered group, it easily follows that if $a > e$, $b > e$ both hold by A, then $ab > e$ also by A. The same is true for B and C. Suppose $a > e$ by A but $b > e$ not by A. If $b > e$ by B, then also $ab > e$ by B. If $b > e$ by C, then we evidently have $xyz^\varrho > e$ for every rational $\varrho > \beta$ and it can also be $xyz^\varrho > e$ for some $\varrho < \beta$ and then $ab > e$ by B or $xyz^\varrho < e$ for all $\varrho < \beta$, and then $ab > e$ by C. If $a > e$ holds by B and $b > e$ by C, then we can find rationals ϱ_1, ϱ_2 such that $\varrho_1 + \varrho_2 < \alpha + \beta$, $xz^{\varrho_1} > e$, $yz^{\varrho_2} > e$ and thus $xyz^{\varrho_1+\varrho_2} > e$, i. e. $ab > e$ by B. We obtain the remaining possible assumptions on $a > e$, $b > e$ if we transpose a and b in those considered above. The proofs will be similar.

Now let us verify II. We have $bab^{-1} = yxy^{-1}z^\alpha$. It is sufficient to observe that if one of the conditions A, B, C holds for a, then the same condition holds after substituting yxy^{-1} for x.

III has already been verified for $a \in \Psi$. Thus we may assume $a \notin \Psi$ and consequently a non $> e$ implies non B and non C. From non B follows

$$xz^\varrho < e \quad \text{for every rational} \quad \varrho < \alpha$$

and this in conjunction with non C implies that

(i) $xz^\varrho < e$ for some rational $\varrho > \alpha$

or

(ii) $xz^\varrho > e$ for every rational $\varrho > \alpha$ and $\alpha < 0$.

If (i) holds, then $x^{-1}z^\varrho > e$ for some $\varrho < -\alpha$ and thus $a^{-1} > e$ by B. If (ii) holds, then we have

$$x^{-1}z^\varrho > e \text{ for every rational } \varrho > -\alpha \text{ and}$$
$$x^{-1}z^\varrho < e \text{ for every rational } \varrho < -\alpha \text{ and } -\alpha > 0$$

Thus $a^{-1} > e$ by C. We have proved the lemma.

Proof of the Theorem. Let G be a cyclically ordered group. We consider F, z, N, F' as they are defined by Lemmas 2 and 3. Let Λ be the ordered subgroup of F' consisting of all elements z^α. Let Γ be the ordered subgroup of all those $x \in F'$ which satisfy

$$z^{-\beta} < x < z^\beta$$

for every $\beta > 0$. We shall prove that

$$F' = \Gamma \times \Lambda$$

and the order on F' is defined lexicographically by the order on Γ and on Λ. Since $\Gamma \cap \Lambda = \{e\}$, it is sufficient to prove that if $y \in F'$, then $y = xz^\alpha$ for some $x \in \Gamma$. Let us observe that there exists a number α such that

$$z^{\alpha-\beta} < y < z^{\alpha+\beta} \quad \text{for every} \quad \beta > 0 .$$

Consequently $z^{-\beta} < yz^{-\alpha} < z^\beta$ and this proves $yz^{-\alpha} \in \Gamma$. Thus $y = xz^\alpha$ with $x \in \Gamma$. It is obvious that F' is lexicographically ordered.

We have $G = F/N \subset F'/N$ and thus from $F' = \Gamma \times \Lambda$ and $N \subset \Lambda$ follows

$$G \subset F'/N = \Gamma \times \Lambda/N = \Gamma \times K .$$

For every $\langle x, a\rangle \in \Gamma \times K$, let us denote by $\langle x, r_a\rangle \in \Gamma \times \Lambda$ that element which is mapped in $\langle x, a\rangle$ by the natural homomorphism of $\Gamma \times \Lambda$ on $\Gamma \times K$ and for which $e \leqslant r_a < z$ holds. We now define a cyclic order on $\Gamma \times K$ by (see Lemma 1)

$$[\langle x, a\rangle, \langle y, b\rangle, \langle z, c\rangle] \equiv [\langle x, r_a\rangle, \langle y, r_b\rangle, \langle z, r_c\rangle] \quad \text{in } F'$$

where the cyclic order on the right is generated by the order on F'. Since the ordering of F' is lexicographical, it follows that the cyclic order on

$\Gamma \times K$ defined above is the natural cyclic order. If in this definition we restrict ourselves to the subgroup G of $\Gamma \times K$ then by Lemma 2 we obtain the cyclic order on G which was initially given. Thus the cyclic order on G is carried over from $\Gamma \times K$.

Reference

[1] L. S. Rieger, *O uspořádaných a cyklicky uspořádaných grupách,* (English summary), Mémoires Soc. Roy. Sc. de Bohème (1946).

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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On cyclically ordered intervals of integers

by

S. Świerczkowski (Wrocław)

A relation $[x, y, z]$ defined on all ordered triplets of different integers x, y, z from the interval $\{0, 1, \ldots, N\}$ is called a *cyclically ordering relation* in this interval if it satisfies for $0 \leqslant x, y, z, x+v, y+v, z+v, u \leqslant N$ the following postulates

I. *Either* $[x, y, z]$ *or* $[z, y, x]$,
II. $[x, y, z]$ *implies* $[y, z, x]$,
III. $[x, y, z]$ *and* $[y, u, z]$ *implies* $[x, u, z]$,
IV. $[x, y, z]$ *implies* $[x+v, y+v, z+v]$.

An example. Let η be a real number such that $p_x = \exp(2\pi i x \eta)$, where $x = 0, 1, \ldots, N$, are different points on the circle $|z| = 1$. We establish a sense on this circle, say the counter-clock-wise sense, and denote the open arc with the initial point p_x and the endpoint p_y by (p_x, p_y). Thus (p_x, p_y) is empty if and only if $x = y$. Defining

$$[x, y, z]_\eta \equiv p_y \in (p_x, p_z)$$

we obtain a cyclically ordering relation on $\{0, 1, \ldots, N\}$.

The purpose of this paper is to prove the following (announced in [1])

Theorem. *For every cyclically ordering relation* $[x, y, z]$ *on* $\{0, 1, \ldots, N\}$ *there exists an interval* I *of real numbers* η *for which*

$$[x, y, z] \equiv [x, y, z]_\eta .$$

If η is irrational, then $[x, y, z]_\eta$ is a cyclically odering relation on the set of all integers and thus

Corollary 1. *Every relation* $[x, y, z]$ *on* $\{0, 1, \ldots, N\}$ *can be extended to a cyclically ordering relation on the set of all integers.*

Let us say that y *follows immediately after* x if $[x, z, y]$ is always false $(0 \leqslant x, y, z \leqslant N)$. If η satisfies the assertion of our theorem, then a number y follows immediately after another one, say x, if and only if the arc (p_x, p_y) contains no points p_z with $z \leqslant N$. Thus there exists for every x strictly one y which follows immediately after x. Let a be the number which follows immediately after 0 and b the one which is