Conditions under which a surface in E^3 is tame *

by

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1. Introduction. A surface (closed set that is a 2-manifold) M in E^3 is *tame* if there is a homeomorphism of E^3 onto itself that takes M onto a polyhedron (finite or infinite). If there is no such homeomorphism, M is called *wild*. The main purpose of this paper is to give a condition under which surfaces are tame. This paper deals with surfaces in E^3, so if a surface is mentioned, it is to be understood that this surface lies in E^3.

If A, B are two homeomorphic sets,

$$H(A, B) \leqslant \varepsilon$$

is used to denote the fact that there is a homeomorphism of A onto B that moves no point by more than ε.

If S is a 2-sphere, we use $\operatorname{Int} S$ and $\operatorname{Ext} S$ to denote the bounded and unbounded components respectively of $E^3 - S$. Theorem 2.2 of the next section states that a 2-sphere S is tame if for each positive number ε there are 2-spheres S', S'' in $\operatorname{Int} S$, $\operatorname{Ext} S$ respectively such that

$$H(S, S') \leqslant \varepsilon, \quad H(S, S'') \leqslant \varepsilon .$$

This theorem (Theorem 2.2) is not proved directly but is reduced by way of one theorem (Theorem 2.1) to a simpler one (Theorem 3.1) which in turn is proved on the basis of a still easier one (Theorem 3.2). The proof given in Section 4 of this last theorem is based on an assortment of results developed in Sections 5-9.

While this treatment is not the logical one where simpler results are proved first and the big ones are proved in terms of those already proved, it does reveal the chronological order of a logical approach in trying to prove a big result where the big result is reduced to a simpler result, the simpler result is reduced, ..., and finally a result is proved on which the sequence of theorems leading to the main result depends.

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Such an approach shows the purpose of the subsidiary results which might seem unexciting in themselves.

The person preferring the logical approach may start by studying Sections 5, 6, 7, 8, 9, 10 which are independent except for 6 which depends on 5, then 4 which depends on 6-9, then 3 which depends on 4, and then 2 which depends on 3 and 10.

Some of the results developed in Sections 5-10 are pursued for their own interest rather than for their contributions alone to the study of tame embedding.

In Section 11 we discuss the possibility of extending the result (Theorem 2.2) about tame 2-spheres to apply to tame surfaces. We give conditions under which a surface is locally tame at a point p. A surface S is locally tame at a point p if there is a neighborhood N of p in E^3 and a homeomorphism of $\bar{N}$ into E^3 that takes $\bar{N}\cdot S$ onto a polyhedron.

Previous papers have given conditions under which surfaces are tame. Bing [4] and Moise [18] showed that a surface is tame if it is locally tame. Griffiths [11] and Harrold [14] have given other sets of conditions under which a surface is tame.

In another paper [7] we shall apply the results of the present paper to show that E^3 does not contain uncountably many mutually exclusive wild surfaces. This result will be used to give still another proof that each 3-manifold can be triangulated. Moise gave a proof of the triangulation theorem in [17] and Bing gave an alternate proof in [6].

Another future paper [8] will extend the results of the present paper to show that a surface in E^3 is tame if its complement is uniformly locally simply connected. A set X is *uniformly locally simply connected* if for each positive number ε there is a positive number δ such that each closed curve in X of diameter less than δ can be shrunk to a point on a subset of X of diameter less than ε.

The above results do not all extend to surfaces with boundaries. Stallings has given an example of an uncountable collection of mutually exclusive wild disks in E^3. In another paper [9] we shall show that a disk in E^3 is tame if its complement is uniformly locally simply connected.

In showing in [3] that each surface M in E^3 could be approximated by a polyhedral surface, we used the 2-skeleton of a triangulation to chop up E^3, and hence M, into small pieces. We could use such an approach for the present paper but decided instead to try a slightly different approach, — namely, adjust M so that it contains no vertical interval and then use a fence to chop up the adjusted M. If a person chose to redo the work in [3], he might choose to use fences as defined below rather than triangulations.

A *fence* is defined to be the sum of all vertical lines intersecting the 1-skeleton of a rectilinear triangulation of a horizontal plane. The mesh of the triangulation is called the *mesh* of the fence. Each vertical line intersecting a vertex of the triangulation is called a *corner* of the fence and the sum of all vertical lines intersecting an edge of the triangulation is called a *section* of the fence. The term "fence" was used to define a somewhat similar set in [5] but there the fences were not infinite and extended in only one direction.

Closely related to a fence is a vertical triangular cylinder. A *vertical triangular cylinder* C is the sum of a collection of vertical lines such that a horizontal cross section of C is a triangle. The sum of the lines intersecting the interior of the triangle is denoted by $\operatorname{Int} C$ and $\operatorname{Ext} C = E^3-(C+\operatorname{Int} C)$. If L is a vertical line in $\operatorname{Int} C$, we call C a vertical triangular cylinder about L. We use vertical triangular cylinders about the corners of a fence to study a 2-sphere S near these corners.

If M is a manifold with boundary, we use $\operatorname{Bd} M$ to denote this boundary and $\operatorname{Int} M$ to denote $M-\operatorname{Bd} M$. There is an inconsistency here in the way that we defined $\operatorname{Int} S$ for a 2-sphere S in E^3 and $\operatorname{Int} C$ for a cylinder C, but it seems unlikely that this double meaning of the symbol Int will lead to confusion.

A finite graph is used to subdivide a 2-sphere. A *finite graph* is the sum of a finite collection of arcs such that if two of these arcs have a point in common, this point is an end point of each. A finite graph is called *planar* if it can be embedded in a 2-sphere. A connected finite graph is called *stable* if it is planar and if each homeomorphism between two of its images in a 2-sphere S can be extended to a homeomorphism of S onto itself.

The distance function is denoted by ϱ.

An isotopy $H_t(0\leqslant t\leqslant 1)$ on X is a one parameter (denoted by t) family of homeomorphisms of X onto itself. All of the isotopies we use start at the identity map — that is $H_0(x)=x$. Hence, it is to be understood without the condition being imposed further that if we use an isotopy $H_t(0\leqslant t\leqslant 1)$, then

$$H_0=I \text{ (the identity)}.$$

A map f defined on E^3 is called piecewise linear if there is a rectilinear triangulation of E^3 such that f is linear on each tetrahedron of the triangulation. An isotopy H_t $(0\leqslant t\leqslant 1)$ is called piecewise linear if each of the homeomorphisms H_t in the one parameter family is piecewise linear.

We say that H_t moves no point by more than ε if $\varrho(H_t(x),x)\leqslant\varepsilon$ for each point x and write

$$\varrho(H_t, I)\leqslant\varepsilon.$$

In general, if f and g are two maps defined on X

$$\varrho(f,g) \leqslant \varepsilon \quad \text{means} \quad \varrho\big(f(x), g(x)\big) \leqslant \varepsilon \quad \text{for each} \quad x \in X.$$

2. A condition under which 2-spheres in E^3 are tame. In this section we give the main results of the paper — a condition under which a 2-sphere in E^3 is tame. The possibility of extending this result is discussed in Section 11. The intervening sections are used to verify some of the facts used in the present section.

THEOREM 2.1. *Suppose S is a 2-sphere in E^3 such that for each positive number ε there is a 2-sphere S' on* Int S *such that $H(S, S') \leqslant \varepsilon$. Then $S + \operatorname{Int} S$ is a topological 3-cell.*

Proof. We suppose that S contains no vertical interval. That there is no loss of generality in supposing this follows from Theorem 10.1.

Let $S_1, S_2, \ldots$ be a sequence of 2-spheres on $\operatorname{Int} S$ such that $H(S, S_i) \leqslant 1/i$. Since each 2-sphere can be approximated by a polyhedral 2-sphere [3], we suppose with no loss of generality that each S_i is polyhedral. We also suppose that $S_i \subset \operatorname{Int} S_{i+1}$.

Let $T(x)$ $(0 < x \leqslant 1)$ be the 2-sphere in E^3 with center at the origin and radius x. We shall prove Theorem 2.1 by showing that there is a homeomorphism h taking $S + \operatorname{Int} S$ onto $T(1) + \operatorname{Int} T(1)$.

Let $X(i,j)$ denote $S_j + \operatorname{Int} S_j - \operatorname{Int} S_i$ and $Y(i,j)$ denote $T\big(j/(j+1)\big) + \operatorname{Int} T\big(j/(j+1)\big) - \operatorname{Int} T\big(i/(i+1)\big)$. See Figure 1.

We prove Theorem 2.1 by sewing together homeomorphisms between certain of the $X(i,j)$'s and the corresponding $Y(i,j)$'s so as to get a homeomorphism of $S + \operatorname{Int} S$ onto $T(1) + \operatorname{Int} T(1)$. We must be able to control the homeomorphisms of the $X(i,j)$'s onto the $Y(i,j)$'s so that the homeomorphisms on the individual parts combine to give a homeomorphism on the sum.

Let h be a homeomorphism of S onto $T(1)$ and g_i a homeomorphism of S_i onto S such that g_i moves no point by more than $1/i$. Let h_i be the homeomorphism of S_i onto $T\big(i/(i+1)\big)$ defined so that the interval from the origin to $hg_i(x)$ goes through $h_i(x)$. See Figure 1. The h_i's give an approximation of the homeomorphism h of $S + \operatorname{Int} S$ onto $T(1) + \operatorname{Int} T(1)$ but rather than agreeing with h on each S_i, they only agree on a sequence of S_i's converging to S.

It follows from Theorem 3.1 that for each positive number ε there is an integer k so large that if $k < i < j$, then there is a homeomorphism $f(i,j;x)$ of $X(i,j)$ onto $Y(i,j)$ such that

$f(i,j) = h_i$ on S_i, and h_j on S_j while diameter $f^{-1}(i,j;A) < \varepsilon$ for each straight line interval A in $Y(i,j)$ on a ray through the origin.

Let $k_1, k_2, \ldots$ be a monotone increasing sequence of positive integers such that if $k_n \leqslant i < j$, then there is such a homeomorphism $f(i,j)$ for $\varepsilon < 1/n$. Hence we suppose that $f(k_n, k_{n+1})$ is a homeomorphism of $X(k_n, k_{n+1})$ onto $Y(k_n, k_{n+1})$ such that

$f(k_n, k_{n+1}) = h_{k_n}$ on S_{k_n} and $h_{k_{n+1}}$ on $S_{k_{n+1}}$, while diameter $f^{-1}(k_n, k_{n+1}; A) < 1/n$ for each straight line interval A in $Y(k_n, k_{n+1})$ on a line through the origin.

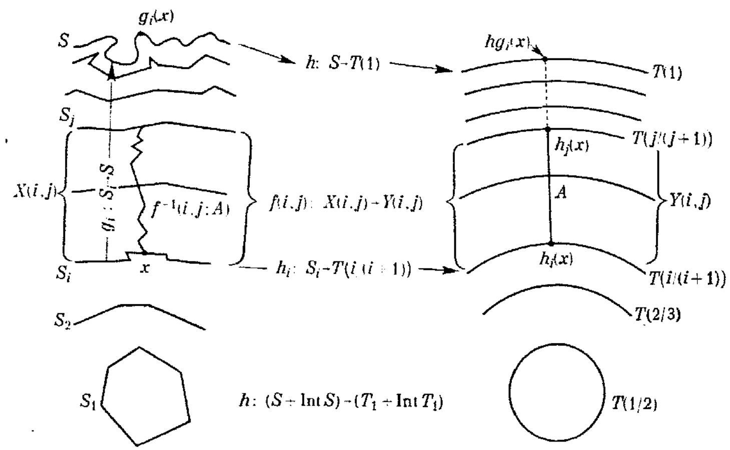

Fig. 1

It follows from a theorem by Alexander [1] that the homeomorphism h_{k_1} on S_{k_1} can be extended to take $S_{k_1} + \operatorname{Int} S_{k_1}$ homeomorphically onto $T_{k_1} + \operatorname{Int} T_{k_1}$. Then the homeomorphism h of $S + \operatorname{Int} S$ on $T(1) + \operatorname{Int} T(1)$ is defined as follows:

$$\begin{aligned} h &= h_{k_1} && \text{on} \quad S_{k_1} + \operatorname{Int} S_{k_1}, \\ h &= f(k_n, k_{n+1}) && \text{on} \quad X(k_n, k_{n+1}), \\ h &= h && \text{on} \quad S. \end{aligned}$$

By applying Theorem 2.1 to both the interior and exterior of a 2-sphere in E^3, one can obtain the following result.

THEOREM 2.2. *A 2-sphere S in E^3 is tame if for each positive number ε there are 2-spheres S', S'' in* Int S, Ext S *respectively such that $H(S, S') < \varepsilon$, $H(S, S'') < \varepsilon$.*

3. A set bounded by two polyhedral 2-spheres. As in the proof of Theorem 2.1 we keep the following notation which will be used throughout Sections 2-9.

S is a 2-sphere in E^3 that contains no vertical interval.

$S_1, S_2, \dots$ is a sequence of polyhedral 2-spheres in $\operatorname{Int} S$ such that $S_i \subset \operatorname{Int} S_{i+1}$.

g_i is a homeomorphism of S_i onto S that moves no point by more than $1/i$.

$T(r)$ $(0 < r \leqslant 1)$ is the 2-sphere with center at the origin and radius r.

h is a homeomorphism of S onto $T(1)$.

h_i is a homeomorphism of S_i onto $T\big(i/(i+1)\big)$ such that $h_i(x)$ is between $hg_i(x)$ and the origin.

$X(i, j) = S_j + \operatorname{Int} S_j - \operatorname{Int} S_i$.

$Y(i, j) = T\big(j/(j+1)\big) + \operatorname{Int} T\big(j/(j+1)\big) - \operatorname{Int} T\big(i/(i+1)\big)$.

We use fences to chop up images of certain $X(i, j)$'s so as to get a homeomorphism $f(i, j)$ of such an $X(i, j)$ onto the corresponding $Y(i, j)$ satisfying the conditions of Theorem 3.1.

THEOREM 3.1. *For each positive number ε there is an integer k such that if $k < i < j$, there is a homeomorphism $f(i, j; x)$ of $X(i, j)$ onto $Y(i, j)$ such that*

1. *$f(i, j; x)$ agrees with $h_i(x)$ on S_i and $h_j(x)$ on S_j and*
2. *diameter $f^{-1}(i, j; A) < \varepsilon$ for each straight line interval A in $Y(i, j)$ on a line through the origin.*

We now state Theorem 3.2 on which the proof of Theorem 3.1 is based. The proof of Theorem 3.2 is outlined in Section 4. Much tedious pushing and pulling, discussed in Section 7, is a basis for the proof of Theorem 3.2. It is in the proof of Theorem 3.2 that we start the use of fences to chop up the images of certain $X(i, j)$'s.

THEOREM 3.2. *For each positive number ε there are an integer k and a positive number δ such that if F is a fence of mesh less than δ and $k < i < j$, then there is a homeomorphism H_3 of E^3 onto itself, a stable finite graph G on S, and a homeomorphism g of $G \times [0, 1]$ into F such that*

1. $\varrho(H_3, I) < \varepsilon$,
2. $H_3(S_i + S_j)$ *is a polyhedron*,
3. *each component of $S - G$ is of diameter less than ε*,
4. $g(G \times 0) \subset H_3(S_i)$ *with* $g_i H_3^{-1} g(a \times 0) = a$,
5. $g(G \times 1) \subset H_3(S_j)$ *with* $g_j H_3^{-1} g(a \times 1) = a$,
6. *each $g(a \times [0, 1])$ is of diameter less than ε*,
7. $H_3(S_i + S_j) \cdot g(G \times [0, 1]) = g(G \times 0) + g(G \times 1)$.

See Figure 2. We note that H_3 removes certain feelers from S_i and S_j so that these feelers will not run in and out of the fence F and prevent Condition 7 from being satisfied.

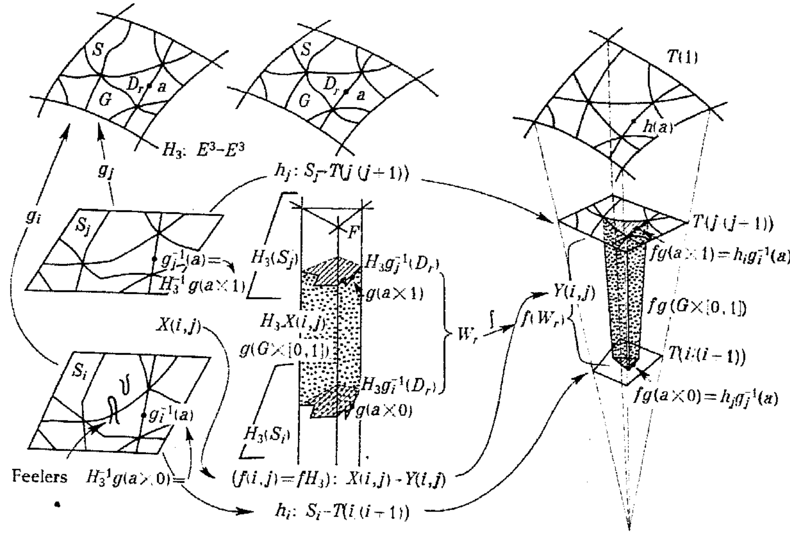

Fig. 2

Proof of Theorem 3.1. Let $\varepsilon_1 < \frac{1}{5}\varepsilon$ and k, F, G, H_3, and g be as promised in Theorem 3.2 for the positive number ε_1. We suppose $1/k < \varepsilon_1 < (\text{diameter } S)/11$.

We describe the homeomorphism $f(i, j)$ in terms of a homeomorphism f of $H_3\big(X(i, j)\big)$ onto $Y(i, j)$ where

$$f(i, j) = fH_3.$$

Since the conclusion of Theorem 3.1 requires that

$$f(i, j) = h_i \text{ on } S_i \text{ and } h_j \text{ on } S_j,$$

it follows from Conditions 4 and 5 of the statement of Theorem 3.2 that for each point a of G

$$fg(a \times 0) = h_i g_i^{-1}(a) = fH_3 g_i^{-1}(a)$$

and

$$fg(a \times 1) = h_j g_j^{-1}(a) = fH_3 g_j^{-1}(a).$$

See Figure 2. The homeomorphism f is extended to the rest of $g(G\times[0,1])$ so that

$fg(a\times t)$ is the point of $T\big((1-t)i/(i+1)+jt/(j+1)\big)$ between the origin and $h(a)$.

Let $D_1, D_2, \ldots, D_n$ be the disks which are the closures of the components of $S-G$ and

$$W_r = H_3g_i^{-1}(D_r)+H_3g_j^{-1}(D_r)+g(\mathrm{Bd}\,D_r\times[0,1])\,.$$

The 2-sphere W_r is shown in Figure 2.

Then W_m is a polyhedral 2-sphere of diameter less than $5\varepsilon_1$ since $H_3g_i^{-1}(D_m)+H_3g_j^{-1}(D_m)$ lies in a $2\varepsilon_1$ neighborhood of D_m and W_m lies in the convex hull of this neighborhood.

It follows from the fact that $G-\mathrm{Bd}\,D_r$ is a connected set of diameter more than $9\varepsilon_1$ that

$$\text{diameter } H_3g_i^{-1}(G-\mathrm{Bd}\,D_r) > 5\varepsilon_1\,.$$

Then W_r does not intersect $\mathrm{Int}\,W_m$ since $\mathrm{Int}\,W_m$ is too small to contain $H_3g_i^{-1}(G-\mathrm{Bd}\,D_r)$. Hence no two of the W's have interiors that intersect and $H_3\big(X(i,j)\big)$ is the sum of polyhedral cubes bounded by the W's.

We suppose that f is extended to the interiors of the W's so that

$$f(W_r+\mathrm{Int}\,W_r) = f(W_r)+\mathrm{Int}\,f(W_r)\,.$$

Then f takes $H_3\big(X(i,j)\big)$ homeomorphically onto $Y(i,j)$. Since $f(i,j) = fH_3$ and $H_3^{-1}(W_r)$ is of diameter less than $5\varepsilon_1$, for each arc A in $Y(i,j)$ on a line through the origin

$$\text{diameter } f^{-1}(i,j;A) < 5\varepsilon_1 < \varepsilon\,.$$

4. An isotopy that simplifies 2-spheres near fences. In this section we give the proof of Theorem 3.2. The proof depends on some somewhat unrelated theorems developed in Sections 5-9. As the proof of Theorem 3.2 is somewhat long, we break it into 8 steps.

Step 1. *Building cylinders about fence corners.* Consider any fence F. Let $L_1, L_2, \ldots, L_n$ be the corners of the fence that intersect S. It follows from the fact that S contains no vertical interval that for each positive number ε_1, we can find about each L_r a vertical triangular cylinder C_r with small horizontal cross section and such that

1. no corner of C_r lies on F,
2. each C_r is in general position with respect to each S_j,
3. for i sufficiently large, $S_i-\sum C_r$ has a component U_i such that each component of S_i-U_i is of diameter less than ε_1.

Let $K(r,j)$ denote the collection of components of $\mathrm{Bd}\,U_j$ on C_r that separate C_r into two unbounded pieces. It follows from Theorem 6.3 that we can find such C's and an integer k so that for i, j larger than k, there is a 1-1 correspondence between the elements of $K(r,i)$ and $K(r,j)$ such that

1. the distance between corresponding elements is less than ε_1,
2. no element of $K(r,i)+K(r,j)$ separates two corresponding elements from each other on C_r.

We suppose that k is so large that S_i+S_j misses each corner of F that S misses and that $1/k<\varepsilon_1$. If we take ε_1 to be very small, k must be large. As a substitute for saying "for k sufficiently large" we can say "for ε_1 sufficiently small".

This ε_1 we have introduced is the first of a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7$ that we shall use. In general we suppose that each ε_r is very small but this can only be accomplished by making the preceding ones very very small. We are interested in knowing that such ε_r's exist rather than in knowing how big they are. If one wants a workout in epsilontics, he can start with ε_7 and work back to ε_1 but this is not recommended.

Step 2. *An isotopy that simplifies near fence corners.* Since a solid cylinder can be chopped up into topological cubes, it follows from Theorems 8.3 and 7.5 that there is a piecewise linear isotopy H_t $(0\leqslant t\leqslant 1)$ of E^3 onto itself such that

1. H_1 is fixed on U_i+U_j,
2. $\varrho(H_1, I)<9\varepsilon_1$,
3. $H_1(S_i+S_j)$ is in general position with respect to F,
4. each component of $H_1(S_i+S_j)-(U_i+U_j)$ is a disk E of diameter less than ε_1 such that $\mathrm{Int}\,E\subset\sum\mathrm{Int}\,C_r$, $E\cdot L_r$ contains only one point if $\mathrm{Bd}\,E$ separates C_r into two unbounded pieces and E misses L_r otherwise.

Figure 3 shows that H_1 removes bulges (shown in S_i-U_i) having extraneous intersections with C_r and also removes feelers (shown in S_j-U_j) winding through C_r.

The correspondence between the elements of $K(r,i)$ and the elements of $K(r,j)$ sets up a 1-1 correspondence between points of $L_r\cdot H_1(S_i)$ and $L_r\cdot H_1(S_j)$. If π takes a point of $L_r\cdot H_1(S_i)$ onto the corresponding point of $L_r\cdot H_1(S_j)$, then

$$\varrho(\pi, I)<3\varepsilon_1 \quad\text{on}\quad L_r\cdot H_1(S_i)\,.$$

There is no point of $H_1(S_i+S_j)$ between a point and its image under π.

It follows from Theorem 8.4 that if k is sufficiently large and the disks E are properly chosen, then for each section V of F and arc pq in

$V \cdot H_1(S_i)$ joining two corner points of F, there is an arc in $V \cdot H_1(S_j)$ joining the points corresponding to p and q. See Figure 4.

Hence for each section V of F, we have a 1-1 correspondence between the components of $V \cdot H_1(S_i)$ that are arcs and the components of $V \cdot H_1(S_j)$ that are arcs.

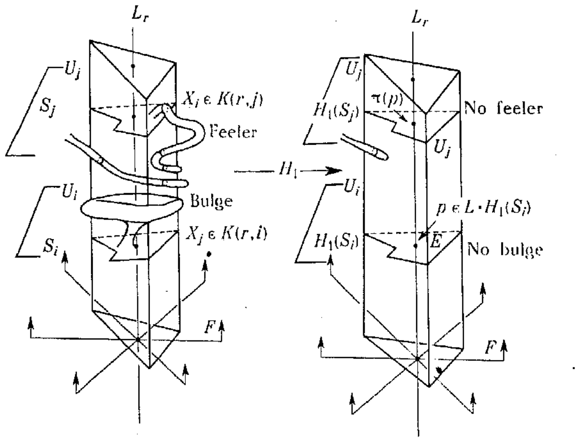

Fig. 3

Let G_i be the sum of all such arcs in $F \cdot H_1(S_i)$ and G_j be the sum of all such arcs in $F \cdot H_1(S_j)$. The aforementioned map π can be extended to a homeomorphism π of G_i onto G_j such that an arc in $V \cdot H_1(S_i)$ goes into the corresponding arc in $V \cdot H_1(S_j)$. Then $F \cdot H_1(S_i)$ is the sum of G_i and various simple closed curves in the sections of F. Also, $F \cdot H_1(S_j)$ is the sum of $G_j = \pi(G_i)$ and various simple closed curves in the sections of F.

Step 3. *The finite graph G.* The aforementioned G_i is a finite graph on $H_1(S_i)$ and $g_i H_1^{-1}(G_i)$ is a finite graph on S. We use these to get G.

Let ε_2 be the mesh of F.

Let ε_3 be a positive number such that each component of $H_1(S_i+S_j)-(G_i+G_j)$ is of diameter less than ε_3. Since S contains no vertical interval we can make ε_3 small by making ε_1 and ε_2 sufficiently small.

For each section V of F and each component A of $V \cdot (G_i+G_j)$

$$\text{diameter}\, A < \varepsilon_3 .$$

Hence

$$\varrho(\pi, I) < 3\varepsilon_1 + 2\varepsilon_3 \text{ on } G_i .$$

Since $\varrho(g_i, I) < \varepsilon_1$ and $\varrho(H_1, I) < 9\varepsilon_1$, each component of $S - g_i H_1^{-1}(G_i)$ is of diameter less than $\varepsilon_3 + 20\varepsilon_1$. It follows from Theorems 9.2 and 9.6 that for each positive number ε_4 and $\varepsilon_3 + 20\varepsilon_1$ sufficiently small one can cause $g_i H_1^{-1}(G_i)$ to contain a stable graph G such that

3. each component of $S - G$ is of diameter less than ε_4.

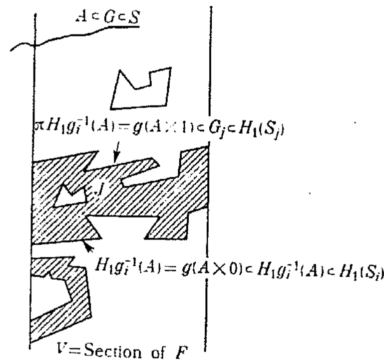

Fig. 4

For $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, sufficiently small, this finite graph G is the one mentioned in the statement of Theorem 3.2. We have just shown that Condition 3 in the statement of Theorem 3.2 can be realized.

Step 4. *The homeomorphism g.* We recall that G_i is the sum of arcs in the sections of F and that G_j is the sum of corresponding arcs under π. Also $H_1 g_i^{-1}(G)$ is the sum of a subcollection of the arcs in G_i.

For each point a of G we define

$$g(a \times 0) = H_1 g_i^{-1}(a) \quad \text{and} \quad g(a \times 1) = \pi H_1 g_i^{-1}(a) .$$

Since H_3 will be defined to be H_1 on $g_i^{-1}(G)$, we find that Condition 4 of the statement of Theorem 3.2 can be realized.

If V is a section of F, A is an arc in G such that $H_1 g_i^{-1}(A)$ is a component of $V \cdot H_1 g_i^{-1}(G)$, then g takes $A \times [0, 1]$ onto the disk in V bounded by the arcs $H_1 g_i^{-1}(A)$, $\pi H_1 g_i^{-1}(A)$, and two vertical intervals. See Figure 4. Since the diameter of this disk is less than $3\varepsilon_1 + 2\varepsilon_3$, it follows that

$$\text{diameter}\, g(a \times [0, 1]) < 3\varepsilon_1 + 2\varepsilon_3 .$$

This suggests that Condition 6 of the statement of Theorem 3.2 can be realized.

Step 5. *Diameters of* $g(G\times 0)$ *and* $g(G\times 1)$. We have the following inequalities:

$$\text{diameter}\, G > \text{diameter}\, S - 2\varepsilon_4\,,$$

$$\text{diameter}\big(H_1 g_i^{-1}(G) = g(G\times 0)\big) > \text{diameter}\, S - 2\varepsilon_4 - 20\varepsilon_1\,,$$

$$\text{diameter}\big(\pi H_1 g_i^{-1}(G) = g(G\times 1)\big) > \text{diameter}\, S - 2\varepsilon_4 - 4\varepsilon_3 - 26\varepsilon_1\,.$$

Hence we suppose that the diameters of $g(G\times 0)$ and $g(G\times 1)$ approximate the diameter of S.

Step 6. *Pulling closed curves off sections.* There is no assurance that $H_1(S_i+S_j)\cdot g(G\times[0,1]) = g(G\times 0)+g(G\times 1)$ because of the simple closed curves (such as J shown in Figure 4) that lie in the intersections of the sections of F with $H_1(S_i+S_j)$. We extend the isotopy H_t $(0\leqslant t\leqslant 1)$ to a piecewise linear isotopy H_t $(0\leqslant t\leqslant 2)$ so that $F\cdot H_2(S_i+S_j)\subset G_i+G_j$.

Let ε_5 be a positive number such that if J is a simple closed curve in $F\cdot H_1(S_i)$ or $F\cdot H_1(S_j)$ that misses the corners of F, then J bounds a disk in $H_1(S_i)$ or $H_1(S_j)$ of diameter less than ε_5. Since S contains no vertical interval, we can make ε_5 small by restricting ε_1 and ε_4. In fact we can set $\varepsilon_5 = \varepsilon_4 + 20\varepsilon_1$.

By putting 2-spheres about these simple closed curves in $F\cdot H_1(S_i)$ and $F\cdot H_1(S_j)$ that miss corners of F, we find from Theorem 7.5 that we can extend H_t $(0\leqslant t\leqslant 1)$ to a piecewise linear isotopy H_t $(0\leqslant t\leqslant 2)$ so that

1. $\varrho(H_1, H_2) < 9\varepsilon_5$,
2. $F\cdot H_2(S_i)\subset G_i$, $F\cdot H_2(S_j)\subset G_j$, and
3. $H_t(1\leqslant t\leqslant 2) = H_1$ on components of $H_1^{-1}(G_i)$ and $H_1^{-1}(G_j)$ containing

$$H_1^{-1}\big(g(G\times 0)+g(G\times 1)\big)\,.$$

We note that

$$\varrho(H_2, I) < 9\varepsilon_1 + 9\varepsilon_5\,.$$

One reason we did not suppose that $H_t(1\leqslant t\leqslant 2)$ equals H_1 on $H_1^{-1}(G_i)$ (or $H_1^{-1}(G_j)$) is that G_i (or G_j) may not be connected and one of the small simple closed curves (such as J shown in Figure 4) in a section of F may separate two points of it from each other in $H_1(S_i)$ (or $H_1(S_j)$).

We now have

$$H_2(S_i+S_j)\cdot g(G\times[0,1]) = g(G\times 0)+g(G\times 1)\,.$$

Since we shall define H_3 so that $H_3(S_i+S_j) = H_2(S_i+S_j)$, we find that Conditions 2 and 7 of the statement of Theorem 3.2 can be realized.

Step 7. *The homeomorphism* H_3. The reason for not using H_2 for the H_3 promised by Theorem 3.2 is that we are not sure that $g_j H_2^{-1} g(a\times 1) = a$ for each point a of G. Other important requirements other than Condition 5 can be met. The difficulty is that even though

$$\pi H_2 g_i^{-1} \text{ is close to } H_2 g_j^{-1} \text{ on } G\,,$$

they are not necessarily equal.

We shall extend H_t $(0\leqslant t\leqslant 2)$ to an isotopy H_t $(0\leqslant t\leqslant 3)$ so that

1. $H_t(2\leqslant t\leqslant 3) = H_2$ on S_i,
2. $H_t(S_j)$ $(2\leqslant t\leqslant 3)$ is the same point set as $H_2(S_j)$, and
3. $\pi H_3 g_i^{-1} = H_3 g_j^{-1}$ on G.

This last condition will insure that Condition 5 of the statement of Theorem 3.2 can be attained.

Consider an arbitrary positive number ε_6. We show in the next step that for $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$, suitably chosen, there is an isotopy F_t $(0\leqslant t\leqslant 1)$ on S such that

1. F_t moves no point more than ε_6, and
2. $F_1 = g_j H_2^{-1}\pi H_2 g_i^{-1}$ on G.

Then $H_2 g_j^{-1} F_t g_j H_2^{-1}$ is an isotopy on $H_2(S_j)$ such that

$$\varrho(H_2 g_j^{-1} F_t g_j H_2^{-1}, I) < \varepsilon_6 + 18\varepsilon_5 + 20\varepsilon_1\,.$$

Since $H_2(S_j)$ is a polyhedral 2-sphere, for each open set U in E^3 containing it, the isotopy $H_2 g_j^{-1} F_t g_j H_2^{-1} = F'_t$ can be extended to an isotopy on E^3 so that

1. $\varrho(F'_t, I) < \varepsilon_6 + 18\varepsilon_5 + 20\varepsilon_1$ and
2. $F'_t = I$ outside U.

In particular, we suppose $F'_t = I$ on $H_2(S_i)$.

We define

$$H_{t+2} = F'_t H_2$$

and note that

$$\varrho(H_3, I) < \varepsilon_6 + 27\varepsilon_5 + 29\varepsilon_1\,.$$

Hence, we see that Condition 1 of the statement of Theorem 3.2 can be attained.

Furthermore

$$\begin{aligned} H_3 g_j^{-1} &= H_2 g_j^{-1} F_t g_j H_2^{-1} H_2 g_j^{-1} \\ &= H_2 g_j^{-1} g_j H_2^{-1} \pi H_2 g_i^{-1} \\ &= \pi H_2 g_i^{-1} = \pi H_3 g_i^{-1} \text{ on } G\,. \end{aligned}$$

This shows that for each point a of G

$$g(a\times 1) = \pi H_1 g_i^{-1}(a) = \pi H_3 g_i^{-1}(a) = H_3 g_j^{-1}(a)\,.$$

Hence Condition 5 of the statement of Theorem 3.2 can be attained.

Step 8. *An isotopy on* S. A 2-sphere S has the property that for each positive number ε_6 there is a positive number ε_7 such that if h is a homeomorphism of S onto itself that moves no point by more than ε_7, there is an isotopy F_t $(0 \leqslant t \leqslant 1)$ on S such that $F_0 = I$, $F_1 = h$, and $\varrho(F_t, I) < \varepsilon_6$. See [10].

We have that G is a stable graph on S, each component of $S-G$ is of diameter less than ε_4, and for each point a of G,

$$\varrho\big(g_j H_2^{-1}\pi H_2 g_i^{-1}(a), a\big) < \varepsilon_1 + (9\varepsilon_1 + 9\varepsilon_5) + (3\varepsilon_1 + 2\varepsilon_3) + (9\varepsilon_1 + 9\varepsilon_5) + \varepsilon_1$$
$$= 18\varepsilon_5 + 2\varepsilon_3 + 23\varepsilon_1 .$$

It follows from Theorem 9.5 that for ε_1 and $18\varepsilon_5 + 2\varepsilon_3 + 23\varepsilon_1$ sufficiently small, $g_j H_2^{-1} f H_2 g_i^{-1}$ on G may be extended to a homeomorphism h of S onto itself such that $\varrho(h, I) < \varepsilon_7$. Then

$$F_1 = g_j H_2^{-1}\pi H_2 g_i^{-1} \text{ on } G .$$

5. The intersection of 2-spheres with cylinders. Let L be a vertical line. It may be that $S \cdot L$ contains infinitely many points. In this case, the number of components of $S_i \cdot L$ may increase with i. In this and the next section we seek to control these intersections.

Question. There is no known method for slightly adjusting an arbitrary 2-sphere with an isotopy of E^3 onto itself so as to make the intersection of the adjusted surface with a fixed line finite. The lack of such a method led to complications in [3] and [13]. It would be convenient to know if each 2-sphere is pierced by a tame arc.

Let L be a vertical line and Δ be a triangle in a horizontal plane P such that $P \cdot L$ is the center of Δ. Let $\Delta(x)$ denote the triangle with center at $P \cdot L$, diameter x, edges parallel to the edges of Δ, and vertices on the rays from $P \cdot L$ through the vertices of Δ. We use $C(x)$ to denote the vertical triangular cylinder such that $P \cdot C(x) = \Delta(x)$.

We suppose that c is so small that only one component of $S - C(c)$ has a large diameter and this component $U(c)$ lies in $\operatorname{Ext} C(c)$. For each $0 < x \leqslant c$, we use $U(x)$ to denote the component of $S - C(x)$ containing $U(c)$ and $K(x)$ to denote the collection of components X of $\overline{U}(x) \cdot C(x)$ such that $C(x) - X$ has two unbounded components.

THEOREM 5.1. *If* $0 < x \leqslant c$, *each element of* $K(x)$ *separates* $U(x)$ *from a point of* $S \cdot L$ *in* S.

Proof. Assume X is an element of $K(x)$ that separates no point of $S \cdot L$ from $U(x)$ in S. We show in six steps that the assumption that such an X exists leads to a contradiction.

1. If X does not separate any point of $S \cdot L$ from $U(x)$ in S, there is a continuum Y in $S - X$ that contains $S \cdot L$. Let X^+ be the component of $S - U(x)$ containing X. Then X^+ is a component of $\big(S - U(x)\big) + Y$ and there is a simple closed curve J in S separating X^+ from Y in $S - \big[\big(S - U(x)\big) + Y\big]$. This simple closed curve J bounds a disk D in $S - Y$.

2. Let x_1 be a positive number less than one half the distance from L to $C(x)$ and x_2 be a positive number so small that if p_1, p_2 are two points of D such that $\varrho(p_1, p_2) < x_2$, then $p_1 + p_2$ lies on an arc in D of diameter less than x_1. The existence of such a number x_2 follows from the uniform local connectedness of disk D.

3. Since each unbounded component of $C(x) - X$ is topologically equivalent to the interior of a circle minus its center, there is a simple closed curve J in $C(x)$ such that $C(x) - J$ has two unbounded components and $J \subset V(X, \frac{1}{3}x_2)$ where $V(A, \varepsilon)$ denotes the set of all points whose distances from A are less than ε. We call $V(A, \varepsilon)$ the ε neighborhood of A.

4. The simple closed curve J cannot be shrunk to a point without hitting L — that is, there is no map of a disk E into $E^3 - L$ that takes $\operatorname{Bd} E$ homeomorphically onto J. If there were such a map, one could find by using a projection from L that there is a map f_1 of E into $C(x)$ that takes $\operatorname{Bd} E$ homeomorphically onto J; there is a retraction f_2 of $C(x)$ onto J; then $f_1^{-1} f_2 f_1$ is a map of E onto $\operatorname{Bd} E$ that leaves each point of $\operatorname{Bd} E$ fixed. Rather than using the set of impossible maps to arrive at a contradiction to the assumption that J can be shrunk to a point without hitting L, certain linking arguments could have been used instead.

5. Since $J \subset V(X, \frac{1}{3}x_2)$, there is a map g of J into D such that $\varrho\big(p, g(p)\big) < x_1 + x_2 < \varrho\big(L, C(x)\big)$. To get such a g one could consider points $p_1, p_2, \ldots, p_n = p_1$ on J such that $\varrho(p_i, p_{i+1}) < \frac{1}{3}x_2$; points $g(p_1)$, $g(p_2), \ldots, g(p_n) = g(p_1)$ on X such that $\varrho\big(p_i, g(p_i)\big) < \frac{1}{3}x_2$; and extend g to an arc $p_i p_{i+1}$ of J onto an arc in D of diameter less than x_1.

6. Finally we show that the false assumption that X separates no point of $S \cdot L$ from $U(x)$ in S leads to the contradiction that J can be shrunk to a point without hitting L (violating Step 4). Let E be the disk $x^2 + y^2 \leqslant 4$ in the x, y plane and f be a homeomorphism of $\operatorname{Bd} E$ onto J. Then f can be extended to $1 \leqslant x^2 + y^2 \leqslant 4$ so that for each point (x, y) of $\operatorname{Bd} E$, f takes the interval from (x, y) to $(\frac{1}{2}x, \frac{1}{2}y)$ linearly onto the interval from $f(x, y)$ to $gf(x, y)$. Since $g(J) \subset D$, the map f can be extended to map $x^2 + y^2 \leqslant 1$ into D. The map f we have described shrinks J to a point in $E^3 - L$ and contradicts Step 4.

THEOREM 5.2. *If* $0 < x \leqslant c$, $X \in K(x)$, *and* $\varepsilon > 0$, *there are simple closed curves* J_1, J_2 *on* S *in an* ε *neighborhood of* X *such that* X *separates* J_1 *from* J_2 *on* S *and neither* J_1 *nor* J_2 *can be shrunk to a point in* $E^3 - L$.

Proof. We suppose $\varepsilon < \varrho(L, C(x))$. Let D be an open subset of S containing X such that $\bar{D} \subset V(X, \varepsilon)$ and $\bar{D}-D$ is the sum of a finite collection of simple closed curves $J_1, J_2, \ldots, J_n$ in different components of $S-X$. We show that the assumption that each J_i with the possible exception of J_1 can be shrunk to a point on a set R_i in E^3-L leads to a contradiction.

It follows from Steps 3 and 4 of the proof of Theorem 5.1 that there is a simple closed curve J on $C(x)$ such that J cannot be shrunk to a point in E^3-L but J can be shrunk in E^3-L to a closed set Y in D. But the set Y in turn can be shrunk to a point in $\bar{D}+R_2+\ldots+R_n$. This shrinking violates the condition that J cannot be shrunk to a point in E^3-L.

THEOREM 5.3. *For each fixed positive number $x \leqslant c$, $K(x)$ has only a finite number of elements.*

Proof. Cover $S\cdot L$ with a finite collection $V_1, V_2, \ldots, V_n$ of connected open subsets of S such that each V is of diameter less than $\varrho(L, C(x))$. Then each element of $K(x)$ separates a V_i from $U(x)$ in S but no two elements of $K(x)$ separates the same V_i from $U(x)$ in S. Hence, there are not more than n elements in $K(x)$.

THEOREM 5.4. *For each element X of $K(c)$ and each $0 < x < c$, X separates some element of $K(x)$ from $U(c)$.*

Proof. Assume otherwise. Then there is a continuum Y in $S-X$ containing all elements of $K(x)$ and a disk D in $S-Y$ such that $X \subset D$, $\mathrm{Bd}\,D \subset U(c)$. Let D' be the component of $D-C(x)$ containing $\mathrm{Bd}\,D$. It may be that $D\cdot C(x)$ contains a component that separates $C(x)$ into two unbounded sets but since D contains no element of $K(x)$, $\bar{D}'\cdot C(x)$ contains no component that separates $C(x)$ into two unbounded sets.

Following the proof of Theorem 5.3, we find that there is a simple closed curve J in $C(c)$ that cannot be shrunk to a point in E^3-L but J can be shrunk in $E^3-C(x)$ to a closed set on $\bar{D}'$. Then there is a map g of the disk $x^2+y^2 \leqslant 4$ into E^3 such that g takes the circle $x^2+y^2=4$ homeomorphically onto J,

$$g(1 \leqslant x^2+y^2 \leqslant 4) \subset \mathrm{Ext}\,C(x), \quad g(x^2+y^2=1) \subset \bar{D}', \quad g(x^2+y^2 \leqslant 1) \subset D.$$

Let Z be the set of all points p of the disk $x^2+y^2 \leqslant 4$ such that there is an arc A in $x^2+y^2 \leqslant 4$ from p to $\mathrm{Bd}(x^2+y^2 \leqslant 4)$ such that $g(A)$ misses $C(x)$. Let $f=g$ on $\bar{Z}$. The map f of $\bar{Z}$ into $E^3-\mathrm{Int}\,C(x)$ can be extended to take $x^2+y^2 \leqslant 4$ into $E^3-\mathrm{Int}\,C(x)$ since $C(x)\cdot f(\bar{Z})$ contains no component that separates $C(x)$ into two unbounded sets. Then f shrinks J to a point in E^3-L and this contradicts the definition of J.

COROLLARY 5.5. *If $0 < x \leqslant y \leqslant c$, the number of elements in $K(y)$ is less than or equal to the number of elements in $K(x)$.*

THEOREM 5.6. *Suppose that f is a homeomorphism of the plane set $(1 \leqslant x^2+y^2 \leqslant 4)$ into E^3-L such that $f(x^2+y^2=4)$ cannot be shrunk to a point in E^3-L. If a component of $C(x)\cdot f(1 \leqslant x^2+y^2 \leqslant 4)$ separates $f(x^2+y^2=1)$ from $f(x^2+y^2=4)$ in $f(1 \leqslant x^2+y^2 \leqslant 4)$, this component also separates $C(x)$ into two unbounded sets.*

Proof. The proof is a modification of the last paragraph of the proof of Theorem 5.4. Let D' be the component of $f(1 \leqslant x^2+y^2 \leqslant 4)-C(x)$ containing $f(x^2+y^2=4)$. If no component of $\bar{D}'\cdot C(x)$ separates $C(x)$ into two unbounded sets, there is a map f' of $(x^2+y^2 \leqslant 4)$ into $D'+C(x)$ such that f' agrees with f on $(x^2+y^2=4)$. This contradicts the condition that $f(x^2+y^2=4)$ cannot be shrunk to a point in E^3-L.

The preceding argument also gives the following result.

THEOREM 5.7. *If $0 < x < c$ and J is a simple closed curve in $U(x)-U(c)$ that cannot be shrunk to a point in E^3-L, then J separates an element of $K(x)$ from $U(c)$.*

THEOREM 5.8. *Each subinterval $[a, b]$ of $[0, c]$ contains a subinterval $[a_1, b_1]$ such that if x, y are elements of $[a_1, b_1]$, then $K(x), K(y)$ have the same number of elements.*

Proof. Suppose $K(a), K(b)$ have m, n elements respectively. If we consider $m-n+1$ mutually exclusive subintervals of $[a, b]$, it follows from Corollary 5.5 that one of them, $[a_1, b_1]$ satisfies the conclusions of Theorem 5.8.

THEOREM 5.9. *The interval $[a_1, b_1]$ of Theorem 5.8 contains a subinterval $[a_2, b_2]$ such that if x, y are elements of $[a_2, b_2]$, the elements of $K(x)$ and $K(y)$ may be ordered $X(x)_1, X(x)_2, \ldots, X(x)_r$ and $X(y)_1, X(y)_1, \ldots, X(y)_r$ so that $X(x)_i$, $X(y)_i$ separate the same subset of $S\cdot L$ from $U(c)$.*

Proof. If $a_1 \leqslant x \leqslant y \leqslant b_1$, it follows from Theorems 5.4 and 5.8 that if $X(x)$ is an element of $K(x)$, then there is one and only one element $X(y)$ of $K(y)$ that separates $X(x)$ from $U(c)$ in S. These are corresponding elements and are given the same subscripts.

Let $V_1, V_2, \ldots, V_r$ be connected subsets of $S\cdot \mathrm{Int}\,C(a_1)$ covering $S\cdot L$. If an element of $K(x)$ separates V_1 from $U(c)$ in S, so does the corresponding element of $K(y)$ if $a_1 \leqslant x \leqslant y \leqslant b_1$. Hence some subinterval $[a', b']$ of $[a_1, b_1]$ has the property that if x, y are elements of $[a', b']$, an element of $K(x)$ separates V_1 from $U(c)$ in S if, and only if, the corresponding elements of $K(y)$ does. A subinterval of $[a', b']$ has this property with respect to V_2. By taking subintervals for as many times as there are elements in $V_1, V_2, \ldots, V_r$, we arrive at a subinterval $[a_2, b_2]$ of $[a_1, b_1]$ such that if x, y are elements of $[a_2, b_2]$, an element of $K(x)$ separates a V_i from $U(c)$ in S if, and only if, the corresponding element of $K(y)$ does.

THEOREM 5.10. *If X', X'' are corresponding elements of $K(a_2), K(b_2)$ respectively of Theorem 5.9 and J is a simple closed curve in S that separates X' from X'' in S, then J cannot be shrunk to a point in E^3-L.*

Proof. Since X', X'' separates the same subset of $S \cdot L$ from $U(c)$ in S, the component U' of $S-(X'+X'')$ between X' and X'' contains no point of L.

Let X''' be the element of $K\big((a_2+b_2)/2\big)$ corresponding to X', X''. It follows from Steps 3 and 4 of the proof of Theorem 5.1 that there is a simple closed curve J' on $C\big((a_2+b_2)/2\big)$ such that J' cannot be shrunk to a point in E^3-L but it can be shrunk in E^3-L to a closed set Y in U' very near X'''. Since U' is topologically equivalent to the plane set $(1 \leqslant x^2+y^2 < 4)$, Y can be shrunk into J in U'. But J cannot be shrunk to a point in E^3-L or else J' could be shrunk to a point by way of Y and J.

THEOREM 5.11. *For each positive number ε there is a subinterval $[a_3, b_3]$ of $[a_2, b_2]$ of Theorem 5.9 such that if $x \in [a_3, b_3]$ and $X(x), X(a_3), X(b_3)$ are corresponding elements of $K(x), K(a_3), K(b_3)$, then the component of $S-\big(X(a_3)+X(b_3)\big)$ between $X(a_3)$ and $X(b_3)$ is in an ε neighborhood of $X(x)$ and is between $C(a_2)$ and $C(b_2)$.*

Proof. Pick an element $X(a_2)$ of $K(a_2)$ for preliminary consideration. For each element x of $[a_2, b_2]$ we let $X(x)$ denote the element of $K(x)$ corresponding to $X(a_2)$. The $X(x)$'s are linearly ordered on S in the sense that if $x<y<z$, $X(y)$ separates $X(x)$ from $X(z)$ on S. Using this linear ordering and the fact that there are uncountably many x's between a_2 and b_2 we find a subinterval $[a_2', b_2']$ of $[a_2, b_2]$ so that if C is a continuum in S separating $X(a_2')$ from $X(b_2')$, C lies between $C(a_2)$ and $C(b_2)$ while the component of $S-\big(X(a_2')+X(b_2')\big)$ between $X(a_2)$ and $X(b_2)$ is in an ε neighborhood of C.

If $[a_2', b_2']$ is successively shortened by an iteration of the above process in considering the various other elements of $K(a_2)$, we arrive at an interval $[a_3, b_3]$ satisfying the conclusion of the theorem.

6. The intersections of cylinders and approximating 2-spheres.

In the preceding section we considered the intersection of S with certain vertical triangular cylinders $C(x)$. The $C(x)$'s we use in this section are the same as those used in Section 5. In the present section we find that if S' is a close approximation to S, its intersection with certain $C(x)$'s has some properties in common with the intersection of S with these $C(x)$'s.

Suppose $0<x<c$ and h is a homeomorphism of S onto a 2-sphere S' such that h moves no point as far as $\varrho\big(U(c), C(x)\big)$. We use $U'(x)$ to denote the component of $S'-C(x)$ containing $h\big(U(c)\big)$ and $K(x, S')$ to

denote the collection of all components of $\overline{U}'(x) \cdot C(x)$ that separate $C(x)$ into two unbounded pieces.

THEOREM 6.1. *Suppose $[a_3, b_3]$ is the subinterval promised by Theorem 5.11, $a_3<a_4<b_4<b_3$, and h is a homeomorphism of S onto a 2-sphere S' such that*

$$\varrho(h, I) < \min\Big(\varrho\big(C(a_3), C(a_4)\big), \varrho\big(C(b_4), C(b_3)\big)\Big).$$

Then if $x \in [a_4, b_4]$, $K(x, S')$ has the same number of elements as $K(x)$ and there is a correspondence between the elements of $K(x)$ and $K(x, S')$ such that an element of $K(x)$ separates a point p of $S \cdot L$ from $U(c)$ if and only if the corresponding element of $K(x, S')$ separates $h(p)$ from $h\big(U(c)\big)$.

Proof. Since h is so near I,

$$h\big(U(b_3)\big) \subset U'(b_4) \subset h\big(U(a_3)\big).$$

Let X, X'' be corresponding elements of $K(b_3), K(a_3)$. Applying Theorems 5.2 and 5.7 to an annulus on S' between $h(X)$ and $h(X'')$, we find that for each element x of $[a_4, b_4]$, an element X' of $K(x, S')$ separates $h(X)$ from $h(X'')$ in S. Then X separates a point p of $S \cdot L$ from $U(c)$ on S if and only if X' separates $h(p)$ from $h\big(U(c)\big)$ on S.

We finish the proof of Theorem 6.1 by showing that all elements of $K(x, S')$ are of the sort described above. Suppose there is an element X' of $K(x, S')$ such that $h^{-1}(X')$ does not separate from $U(c)$ any element of $K(a_3)$. It follows from Theorem 5.2 that there is a simple closed curve J in S' such that J cannot be shrunk to a point in E^3-L, X separates J from $h\big(U(c)\big)$, and J is so near X that $h^{-1}(J) \subset U(a_3)-U(c)$ and J can be shrunk to $h^{-1}(J)$ in E^3-L. But Theorem 5.7 gives the contradiction that $h^{-1}(J)$ separates an element of $K(a_3)$ from $U(c)$ in S.

THEOREM 6.2. *If X, X' are corresponding elements of $K(x), K(x, S')$ of Theorem 6.1, then each lies in a 2ε neighborhood of the other where ε is as given in Theorem 5.11. If ε is taken to be sufficiently small, no element of $K(x)+K(x, S')$ separates X from X' on $C(x)$.*

Proof. Let $X(a_3), X(b_3)$ be elements of $K(a_3), K(b_3)$ corresponding to X of $K(x)$. Since $h^{-1}(X')$ separates $X(a_3)$ from $X(b_3)$ as shown in proof of Theorem 6.1, each of $X, h^{-1}(X')$ lies in an ε neighborhood of the other as shown in Theorem 5.11, and $\varrho(h, I) < \frac{1}{2}\varepsilon$, each of X, X' lies in a $\frac{3}{2}\varepsilon$ neighborhood of the other.

If X_1, X_2 are two elements of $K(b_3)$, the subsets of S separated from $U(c)$ by X_1, X_2 are a finite distance apart and for $x \in [a_4, b_4]$, the corresponding elements of $K(x)$ are at least this far apart. Decreasing the size of ε of Theorem 5.11 does not make noncorresponding elements close together. Hence, for ε sufficiently small, the last sentence in the statement of Theorem 6.2 is satisfied.

Applications of the theorems of Sections 5 and 6 to vertical triangular cylinders about the corners of a fence give the following result.

THEOREM 6.3. *Suppose S is a 2-sphere in E^3 that contains no vertical interval; $S_1, S_2, \ldots$ is a sequence of polyhedral 2-spheres in* $\mathrm{Int}\, S$ *such that* $S_i \subset \mathrm{Int}\, S_{i+1}$ *and* $H(S, S_i) < 1/i$; *F is a fence; and ε is a positive number. Then there are mutually exclusive vertical triangular cylinders $C_1, C_2, \ldots, C_n$ about the corners of F that intersect S such that*

1. *no corner of any C_r lies in F,*
2. *each C_r is in general position with respect to each S_i,*
3. *for i sufficiently large, $S_i - \sum C_r$ has a component U_i such that each component of $S_i - U_i$ is of diameter less than ε,*

Furthermore, if $K(r, j)$ denotes the collection of components of $C_r \cdot \mathrm{Bd}\, U_j$ that separate C_r into two unbounded pieces, there is an integer k such that for i, j greater than k, there is a 1-1 correspondence between the elements of $K(r, i)$ and $K(r, j)$ such that

1. *the distance between corresponding elements is less than ε,*
2. *and no element of $K(r, i) + K(r, j)$ separates two corresponding elements from each other on C_r.*

7. Isotopies near 2-spheres. A polyhedral 2-sphere S may have feelers that wander in and out of other 2-spheres $M_1, M_2, \ldots, M_n$. In this section we learn how to pull these feelers back without moving points too far.

THEOREM 7.1. *Suppose S, M are polyhedral 2-spheres in E^3 in relative general position; D is a disk in M such that* $D \cdot S = \mathrm{Bd} D$; *$E$ is a disk in S bounded by* $\mathrm{Bd} D$; *V_D, V_E are interiors of polyhedral 3-cells such that these interiors contain* $\mathrm{Int} D$, $\mathrm{Int} E$ *respectively; C is the polyhedral cube bounded by $D+E$; and $\varepsilon > 0$. Then there is a piecewise linear isotopy H_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself such that*

1. $H_1(E) = D$,
2. $H_1 = I$ *(Identity) outside* $C + V_D + V_E$,
3. $H_1(C) \subset D + V_D$,
4. *H_t moves no point which is outside $V_E + C$ by as much as ε,*
5. $\varrho\big(H_s(x), H_t(x)\big) < \varepsilon$ *if s, t are two values of $[0, 1]$ and x is a point such that neither $H_s(x), H_t(x)$ belong to $C + V_E$.*

Proof. Figure 5 shows how we use H_t to pull back a feeler E of S.

It follows from the extension of Alexander's theorem [1] due to Moise [16] and Graeub [12] that there is a piecewise linear homeomorphism h of E^3 onto itself that takes $D+E$ onto the surface of a tetrahedron $abcp$ whose base abc is $h(D)$. Let q be the center of abc and r, s

be points such that $rpqs$ is a straight line interval (with points r, p, q, s in the order indicated) such that

$$abcr \subset h(C + V_E),$$

$$\mathrm{Int}\, abcs \subset h(V_D), \quad \text{and} \quad s \text{ is very close to } q.$$

Let x_t be the point of pq whose distance from p is t times the length of pq. Let F_t be the homeomorphism of E^3 onto itself fixed outside $abcr + abcs$, that takes p to x_t and is linear on the tetrahedra $abpr$, $bcpr$, $acpr$, $abps$, $bcps$, $acps$. Then $H_t = h^{-1} F_t h$.

Conditions 1, 2, 3 of the hypothesis of the theorem are met without restricting s to be very close to q but a sufficiently stringent enforcement of this restriction causes Conditions 4 and 5 to be satisfied.

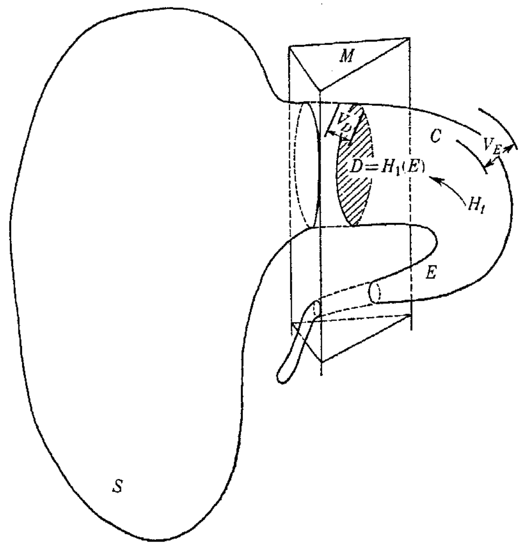

Fig. 5

THEOREM 7.2. *Suppose K is a polyhedral cube in E^3 and D, E are two polyhedral disks such that* $D \cdot \mathrm{Bd} K = \mathrm{Bd} D = \mathrm{Bd} E = E \cdot \mathrm{Bd} K$, $\mathrm{Int} D \subset \mathrm{Int} K$, $\mathrm{Int} E \subset \mathrm{Int} K$. *Then there is a piecewise linear isotopy H_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself that is the identity except on K and such that* $H_1(E) = D$.

Proof. With the exception of the piecewise linear part of the conclusion, this theorem was essentially proved by Alexander in [2]. It has been used as a lemma since but we include a short proof for completeness.

Let E' be a polyhedral disk such that $\operatorname{Bd} E' = \operatorname{Bd} D$, $\operatorname{Int} E' \subset \operatorname{Int} K$, $E' \cdot (D+E) = \operatorname{Bd} E'$. It follows from Theorem 7.1 that there is an isotopy $H_t (0 \leqslant t \leqslant \frac{1}{2})$ of E^3 onto itself so that H_t is the identity except on K and $H_{1/2}(E) = E'$. Another application of Theorem 7.1 shows that $H_t (0 \leqslant t \leqslant \frac{1}{2})$ can be extended to $H_t (0 \leqslant t \leqslant 1)$ so that H_t is the identity except on K and $H_1(E) = D$.

THEOREM 7.3. *Suppose $M_1, M_2, \ldots, M_n$ are mutually exclusive polyhedral 2-spheres in E^3 each of diameter less than ε.*

Suppose S is a polyhedral 2-sphere in general position with respect to each of the M_i's and U is a component of $S - \sum M_i$ of diameter more than 2ε such that each component of $S - U$ is of diameter less than ε.

Then for each continuum X in $E^3 - \big(\sum M_i + (S-U)\big)$ that intersects U there is an isotopy $H_t (0 \leqslant t \leqslant 1)$ of E^3 onto itself such that

1. *H_t moves no point of $\overline{U} + X$,*
2. *for each component E_j of $S - U$, $H_1(\operatorname{Int} E_j)$ lies in some $\operatorname{Int} M_i$,*
3. *H_1 moves no point of S by as much as 3ε,*
4. *H_t moves no point of S by as much as 6ε,*
5. *H_1 moves no point of E^3 by as much as 9ε,*
6. *H_t moves no point of E^3 by as much as 12ε.*

Proof. Some of the coefficients ε given in Conditions 3, 4, 5, 6 may be a bit extravagant but they are good enough for our purposes when we apply Theorem 7.3 in Theorem 3.2. We divide the proof into seven steps.

Step 1. *Preliminary simplifications.* Let $J_1, J_2, \ldots, J_n$ be the simple closed curves in $\operatorname{Bd} \overline{U}$ and E_i be the disk in $S - U$ bounded by J_i.

With no loss of generality we suppose that each M_j contains one of the J_i's since we can discard from consideration any that does not.

We suppose that no M_j contains two J_i's because if M_1 contains $J_1 + J_2$, we can split it and get two mutually exclusive polyhedral 2-spheres M', M'' such that $J_1 \subset M'$, $J_2 \subset M''$, $M_1 \cdot S \subset M' + M'' \subset M_1 + \operatorname{Int} M_1$. Hence we suppose that $J_i \subset M_i$.

If $\operatorname{Int} E_i \subset \operatorname{Int} M_i$, we consider E_i as joined to U and ignore both E_i and M_i. There is no need for the isotopy to move E_i.

Step 2. *Description of H_t.* The isotopy promised by Theorem 7.3 is given by iterated applications of Theorem 7.1 used to reduce the number of components of $S \cdot \sum M_i$. Suppose there are $n+k$ such components.

If $\operatorname{Int} E_j \not\subset \operatorname{Int} M_j$, consider a disk D in M_j such that $D \cdot S = \operatorname{Bd} D \subset S - \overline{U}$. Let E be the disk in $S - \overline{U}$ bounded by $\operatorname{Bd} D$ and C be the polyhedral cube bounded by $D + E$.

We take V_D and V_E to be open sets containing D and E respectively such that each point of V_D is very near D and each point of V_E is very near E. We shall discuss how close this should be later when we use this restriction to show that an isotopy we describe does not move points too far.

We apply Theorem 7.1 to get an isotopy H_t $(0 \leqslant t \leqslant 1/2k)$ pulling D to E and fixed at each point of $S - E$ and at each point outside $C + V_D + V_E$. The V's we described in the preceding paragraph are slightly larger than those mentioned in Theorem 7.1 but we use these larger ones since we shall need them in the next paragraph when we shove points across D. We suppose the isotopy H_t $(0 \leqslant t \leqslant 1/2k)$ satisfies Conditions 1, 2, 3, 4, 5 of Theorem 7.1 (with $1/2k$ substituted for 1 in determining range of t) where the size of the ε mentioned in Conditions 4 and 5 is to be given later. It is not the same as the ε mentioned in the statement of Theorem 7.3.

Moving nothing except in V_D, we extend H_t $(0 \leqslant t \leqslant 1/2k)$ to H_t $(0 \leqslant t \leqslant 1/k)$ by shoving D to one side of M_j so that nothing moves far (far is explained later) and $H_{1/k}(S)$ is a polyhedral 2-sphere in general position with respect to the M_i's, $H_{1/k}(S) \subset (S-E) + V_D$, and $H_{1/k}(S) \cdot \sum M_i$ is a proper subset of $S \cdot \sum M_i$.

Using $H_{1/k}(S)$ instead of S, the process is continued to reduce the number of components of the intersection of $\sum M_i$ and the resulting image of S. Taking a disk D in some M_j such that $D \cdot H_{1/k}(S) = \operatorname{Bd} D \subset \big(H_{1/k}(S) - \overline{U}\big)$ and the disk E in $H_{1/k}(S) - \overline{U}$ bounded by $\operatorname{Bd} D$, we consider neighborhoods V_D, V_E about D and $H_{1/k}(E)$ and extend H_t $(0 \leqslant t \leqslant 1/k)$ to H_t $(0 \leqslant t \leqslant 2/k)$ so as to shove E across D as before. In general we suppose that H_t is extended so that $H_{j/k}(S) \cdot \sum M_i$ is the sum of at most $n+k-j$ components of $S \cdot \sum M_i$.

It is clear that if the V's are chosen so as not to intersect $\overline{U} + X$, then H_t $(0 \leqslant t \leqslant 1)$ satisfies Condition 1 of the conclusion of the theorem and $H_1(S)$ satisfies Condition 2. We need to place restrictions on the V's in order to show that Conditions 3, 4, 5, 6 are satisfied.

Step 3. *Distance H_t moves points of S.* Let us consider the points of the E_i's and see how far they move under H_t. Let E_i^+ be the sum of E_i and all the M_j's intersecting it. If we suppose that each point of V_D is within δ of D, then each point of $H_{j/k}(E_i)$ is within δ of E_i^+. Since diameter E_i^+ is less than 3ε, we may restrict the V_D's and suppose that each $H_{j/k}(E_i)$ is of diameter less than 3ε and $\varrho\big(p, H_{j/k}(p)\big) < 3\varepsilon$ if $p \in S$. Hence Condition 3 is satisfied.

The cube C used at any stage is near the appropriate E_i^+ so we may suppose that for each point x of E^3,

$$\big(H_{j/k}(x), H_t(x)\big) \leqslant 3\varepsilon \quad \text{if} \quad j/k \leqslant t \leqslant (j+1)/k .$$

The preceding two paragraphs show that by restricting the V_D's and V_E's, we can keep H_t from moving any point of S by more than 6ε. Hence Condition 4 is satisfied.

Step 4. *Traps to be avoided.* Now let us turn our attention to Conditions 5 and 6 as applied to points of E^3-S. We must control our isotopies so as not to move a point too far. A trap to be avoided is illustrated by Figure 6. The point p_1 is near the leftmost sticker and is pulled into the left cube when the sticker is pulled in. If it goes into p_2 near another sticker, this point may in turn be pulled to p_3 when this sticker is removed. Care must be taken so that the point p_1 does not move to p_2, then to p_3, then to p_4, then to p_5, etc. and hence move more than 12ε. This is accomplished by restricting the sizes of the V_D's and V_E's used.

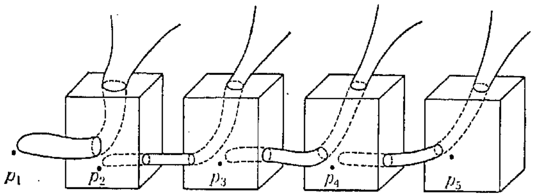

Fig. 6

Points of the V_E's offer special difficulties as suggested in the preceding paragraph so we first consider a point p of E^3-S such that for $j=r, r+1, \ldots$, or, $k-1$, the point $H_{j/k}(p)$ is not a point of the V_E about the disk E in $H_{j/k}(S)$ used in extending H_t past $H_{j/k}$. We let $s=r/k$ and show that with suitable restrictions on the V's, $\varrho\big(H_s(p), H_t(p)\big)<6\varepsilon$ $(s\leqslant t\leqslant 1)$.

Step 5. *Diameter of $H_s(p)^+$.* Let $H_s(p)^+$ denote the sum of $H_s(p)$ and all M_j's such that each arc from $H_s(p)$ to U in $E^3-H_s(\sum E_i)$ intersects M_j. As a help in showing that $\varrho\big(H_s(p), H_t(p)\big)<6\varepsilon$, we show in the next four paragraphs that diameter $H_s(p)^+<3\varepsilon$.

If no M_j separates $H_s(p)$ from U in $E^3-H_s(\sum E_i)$, then $H_s(p)^+$ is the point $H_s(p)$ and its diameter is less than 3ε. Hence we suppose that some M_j separates $H_s(p)$ from U in $E^3-H_s(\sum E_i)$.

We use $H_s(E_i)^+$ to denote the sum of $H_s(E_i)$ and all M_j's that intersect it. We suppose $H_s(E_i)^+$ is so close to E_i^+ that it is of diameter less than 3ε.

If $H_s(p)\in M_j$, let rq be an arc on M_j irreducible from $r=H_s(p)$ to $H_s(\sum E_i)$. Suppose $q\in H_s(E_j)$. Then $H_s(p)^+\subset H_s(E_j)^+$ and is therefore of diameter less than 3ε. A similar argument shows that $H_s(p)^+$ is of diameter less than 3ε if $H_s(p)\subset \operatorname{Int} M_i$.

If $H_s(p)$ lies in the exterior of each M_i, it follows from the unicoherence of $E^3-\sum M_i$ that there is an $H_s(E_j)$ accessible from $H_s(p)$ in $E^3-\big(\sum M_i+H_s(\sum E_i)\big)$ such that $H_s(E_j)+\sum M_i$ separates $H_s(p)$ from U in E^3. It follows from the unicoherence of $E^3-H_s(E_j)$ that $H_s(E_j)^+$ separates $H_s(p)$ from U in E^3. Since $H_s(p)^+\subset H_s(p)+H_s(E_j)^+$, diameter $H_s(p)^+\leqslant$ diameter $H_s(p)+H_s(E_j)^+=$ diameter $H_s(E^+)<3\varepsilon$. This completes the proof that for each point p of E^3-S, diameter $H_s(p)^+<3\varepsilon$.

Step 6. *Distance moved by a point p whose image is never in V_E again.* Again we suppose that if $j\geqslant r$, $H_{j/k}(p)$ does not lie in the V_E used in extending $H_{j/k}$. We show that if $s=r/k$, then $\varrho\big(H_s(p), H_t(p)\big)<6\varepsilon$ if $s\leqslant t\leqslant 1$. We can accomplish this by showing that $\varrho\big(H_s(p), H_{j/k}(p)\big)<3\varepsilon$ if $s=r/k\leqslant j/k$ since it has already been demonstrated that $\varrho(H_{j/k}, H_t)<3\varepsilon$ $(j/k\leqslant t\leqslant (j+1)/k)$.

Since $H_s(p)$ does not belong to the V_E used in extending H_s, as t moves from $s=r/k$ to $(r+1)/k$, $H_s(p)$ does not move far unless it is in the C used at this stage. If it is in the C, $H_s(p)$ is moved close to D which lies in $H_s(p)^+$. In any case, by a proper choice of the ε of Theorem 7.1, we can cause each $H_{(r+1)/k}(p)$ to be very close to its $H_s(p)^+$.

If M_j belongs to $M_{(r+1)/k}(p)^+$, it belongs to $H_{r/k}(p)^+$. Hence, as t takes on the values $r/k, (r+1)/k, (r+2)/k, \ldots, 1$, $H_t(p)$ remains close to $H_s(p)^+$. We may suppose that $\varrho\big(H_s(p), H_{j/k}(p)\big)<3\varepsilon$ $(s=r/k\leqslant j/k)$ and $\varrho\big(H_s(p), H_t(p)\big)<6\varepsilon$ $(s\leqslant t\leqslant 1)$ if $H_{j/k}(p)$ is never in the V_E used in extending $H_{j/k}$ $(j=r, r+1, \ldots, k-1)$.

The preceding argument shows that H_t does not move p by as much as 6ε if for no j does $H_{j/k}(p)$ belong to the V_E used in extending $H_{j/k}$.

Step 7. *Distances moved by a point one of whose images is in a V_E.* At any stage we have disk E and can select a V_E each of whose points is very close to E. No point of E has been moved by as much as 6ε and $H_{j/k}$ moves no point of E by as much as 3ε. Hence we may select V_E so that for each point p of V_E, $\varrho\big(p, H_{j/k}(p)\big)<3\varepsilon$ and $\varrho\big(p, H_t(p)\big)<6\varepsilon$ if $0\leqslant t\leqslant j/k$. We now show that this restriction on V_E insures that Conditions 5 and 6 are satisfied.

Suppose $H_{r/k}(p)$ lies in the V_E used in extending $H_{r/k}$ but if $r<j$, $H_{j/k}(p)$ does not belong to the V_E used in extending $H_{j/k}$. It follows from the restriction placed on V_E in the preceding paragraph that $\varrho\big(p, H_{r/k}(p)\big)<3\varepsilon$ and $\varrho\big(p, H_t(p)\big)<6\varepsilon$ for $0\leqslant t\leqslant r/k$. We found in Step 3 that $\varrho(H_{r/k}, H_t)<3\varepsilon$ if $r/k\leqslant t\leqslant (r+1)/k$ and in Step 6 that $\varrho\big(H_{(r+1)/k}(p), H_1(p)\big)<3\varepsilon$ and $\varrho\big(H_{(r+1)/k}(p), H_1(p)\big)<6\varepsilon$ if $(r+1)/k\leqslant t\leqslant 1$. These relations imply Conditions 5 and 6.

THEOREM 7.4. *If $E_1, E_2, \ldots, E_n$ are the disks of $S-U$ of Theorem 7. and $D_1, D_2, \ldots, D_n$ are mutually exclusive polyhedral disks such that $\operatorname{Bd} E_i = \operatorname{Bd} D_i$ and $\operatorname{Int} D_i \subset \operatorname{Int} M_i$, then the isotopy H_t $(0 \leqslant t \leqslant 1)$ may be chosen satisfying the conclusion of Theorem 7.3 and such that $H_1(E_i) = D_i$.*

Proof. The proof of this theorem follows the pattern of the proof of Theorem 7.3 except that we use $n+k$ rather than k steps in describing H_t, the extra steps being used to apply Theorem 7.2 and pull the E_i's onto the D_i's.

Theorem 7.4 may be extended by replacing S by a finite collection of mutually exclusive 2-spheres. We need the following result where there are two S's,

THEOREM 7.5. *Suppose $M_1, M_2, \ldots, M_n$ are mutually exclusive polyhedral 2-spheres in E^3 such that each is of diameter less than ε.*

Suppose S', S'' are mutually exclusive polyhedral 2-spheres each in general position with respect to the M_i's and U', U'' are components of $S'-\sum M_i$, $S''-\sum M_i$ respectively each of diameter more than 2ε and such that each component of $S'-U'+S''-U''$ is of diameter less than ε.

Suppose $D_1, D_2, \ldots, D_m$ are mutually exclusive disks such that $U'+U''+ +\sum D_i$ is the sum of two polyhedral 2-spheres and each $\operatorname{Int} D_j$ lies in $\sum \operatorname{Int} M_i$.

Then for each continuum X in $E^3-\left(\sum M_i + S'-U'+S''-U''\right)$ that intersects $U'+U''$ there is an isotopy H_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself such that

1. *H_t moves no point of $\overline{U}'+\overline{U}''+X$,*
2. *$H_1(S'+S'') = U'+U''+\sum D_i$,*
3. *H_1 moves no point of $S'+S''$ by as much as 3ε,*
4. *H_t moves no point of $S'+S''$ by as much as 6ε,*
5. *H_1 moves no point of E^3 by as much as 9ε,*
6. *H_t moves no point of E^3 by as much as 12ε.*

Proof. The proof of Theorem 7.5 follows the pattern of the proof of Theorem 7.3 so we only give a broad outline of it.

As in the proof of Theorem 7.3 we subdivide the M_i's so that each of those that needs to be considered contains one and only one simple closed curve in $\operatorname{Bd}\overline{U}'+\operatorname{Bd}\overline{U}''$, and the interior of this M_i contains the appropriate $\operatorname{Int} D_j$.

In a finite sequence of applications of Theorem 7.1 each followed by pushing a disk E to one side of a disk D, we obtain an isotopy of E^3 onto itself that reduces the number of components of $(S'+S'')\cdot\sum M_i$ to the number of components of $\operatorname{Bd}\overline{U}'+\operatorname{Bd}\overline{U}''$. Then Theorem 7.2 is applied to push $S'+S''$ onto $U'+U''+\sum D_i$.

For each component F of $S'-U'+S''-U''$, the image of F at each stage is near the sum of F and the M_i's that F intersects. This insures that Condition 3 is satisfied.

At no stage is any point moved more than 3ε. This insures that Condition 4 is satisfied.

If at a certain stage the image of a point p fails to belong to an open set V_E about the disk E used at this certain stage, the image is not moved much unless it moves near an M_i that separates p from $\overline{U}'+\overline{U}''$ in $E^3-(S'-U'+S''-U'')$. This insures that H_t does not move too far those points whose images fail to belong to V_E's.

By choosing the V_E's so that points in them have not been moved too far already, we insure that the isotopy we describe satisfies Conditions 5 and 6.

THEOREM 7.6. *Suppose S; $S_1, S_2, \ldots$; F; $C_1, C_2, \ldots, C_n$; and the $K(r,j)$'s are taken as in Theorem 6.3, and ε is taken so small and k so large that if $k<i<j$, noncorresponding elements of $K(r,i)$, $K(r,j)$ are farther apart than 2ε. If J_s is an element of $K(r,s)$ that separates on C_r two corresponding elements J_i, J_j of $K(r,i)$, $K(r,j)$ and $i,j,s>k$, then $i<s<j$ and J_i, J_s, J_j are corresponding elements.*

Proof. We let U_i, U_s, U_j be the large components of $S_i-\sum C_r$, $S_s-\sum C_r$, $S_j-\sum C_r$ and find from the techniques of Theorems 7.3 and 7.5 (we are not interested here in preventing points from moving far) that there is an isotopy on E^3 that is fixed on $U_i+U_s+U_j$ that takes S_i, S_s, S_j onto polyhedral 2-spheres S_i', S_s', S_j' such that each of $S_i'-\overline{U}_i$, $S_s'-\overline{U}_s$, $S_j'-\overline{U}_j$ misses $\sum C_r$.

Since J_s separates J_i from J_j on C_r, then J_i, J_s, J_j are corresponding elements because noncorresponding elements are not within 2ε of each other.

Since neither J_i nor J_j separates the other from J_s on C_r, neither S_i' nor S_j' separates the other from S_s' in E^3. Therefore S_s' separates S_i' from S_j' and $i<s<j$.

The preceding argument also gives the following result.

THEOREM 7.7. *If in Theorem 7.6, J_t is an element of $K(r,t)$ corresponding to J_i, J_j and $k<i<t<j$, then J_t is between J_i and J_j on C_r.*

8. Special disks with respect to cylinders. Each polygonal simple closed curve J on C bounds a polygonal disk D which lies except for $J = \operatorname{Bd} D$ in $\operatorname{Int} C$. Suppose L is a vertical line in $\operatorname{Int} C$. If J bounds a disk in C, we may choose such a D which misses L; if $C-J$ has two unbounded components, we can pick D so that D intersects L in just one point.

Suppose D', D'' are two such disks bounded by J and P is a plane containing L such that P contains no vertex of some rectilinear triangulation of $D'+D''$. Each component of $P\cdot D'$ with a point p on J is an arc with only its end points on J. The same goes for components of $P\cdot D''$. However, the arc in $P\cdot D'$ containing p may not have the same other end point as the arc in $P\cdot D''$ containing p. We wish to define a special property such that if D', D'' have this special property, then if an arc which is a component of $P\cdot D'$ shares an end with a component of $P\cdot D''$, then their other ends are identical also.

Special property for a disk. If J is a polygonal simple closed curve on a vertical triangular cylinder C and L is a vertical line in $\operatorname{Int} C$, then a polygonal disk D is said to have the *special property with respect to* J, C, L provided:

1. $\operatorname{Bd} D = J$,
2. $\operatorname{Int} D \subset \operatorname{Int} C$,
3. $\operatorname{Diameter} D = \operatorname{diameter} J$,

4a. $D\cdot L = 0$ if J bounds a disk in C,

4b. $D\cdot L$ contains only one point if $C-J$ has two unbounded components,

5. if P is a plane containing L but no vertex of a rectilinear triangulation of D and ab is an arc in $P\cdot C$ with end points on J and interior in the component of $C-J$ that does not reach below some horizontal plane, then some component of $D\cdot P$ is an arc with end points at a, b.

THEOREM 8.1. *Suppose J is a polygonal simple closed curve on a vertical triangular cylinder C, L is a vertical line in* $\operatorname{Int} C$, *and D', D'' are two polyhedral disks each with the special property with respect to J, C, L. If P is a plane containing L but no vertex of some triangulation of $D'+D''$ and ab is a component of $P\cdot D'$ intersecting J, then there is a component of $P\cdot D''$ which is an arc with the same end points as ab.*

Proof. Since P contains no vertex of some triangulation of $D'+D''$, each component of $P\cdot D'$ (of $P\cdot D''$ also) that intersects J is an arc with only its end points on J.

For the case where J bounds a disk in C, Condition 5 of the definition of the special property of a disk causes two points of $J\cdot P$ to belong to an arc in $P\cdot D$ if and only if they belong to an arc in $P\cdot D''$.

If $C-J$ has two unbounded components, $J\cdot P$ has two points p, q such that each pair of points of $(J\cdot P)-(p+q)$ belong to an arc in $P\cdot D'$ if and only if they belong to an arc in $P\cdot D''$ according as they do or do not belong to an arc in $P\cdot C$ whose interior belongs to the upper component of $C-J$. Since p, q are the only two points of $J\cdot P$ left to consider, there is an arc from p to q in $P\cdot D'$ and also an arc from p to q in $P\cdot D''$.

THEOREM 8.2. *If J is a polygonal simple closed curve on a vertical triangular cylinder C, and L is a vertical line in* $\operatorname{Int} C$, *then there is a polygonal disk D that has the special disk property with respect to J, C, L.*

If $P_1, P_2, \ldots, P_n$ is a finite collection of planes each containing L but no point at which J is broken, such a disk D may be selected with a rectilinear triangulation with no vertex on $P_1+P_2+\ldots+P_n$ and no edge intersecting L.

Proof. The proof of the case where J bounds a disk E in C is easy. Here we obtain D by pushing $\operatorname{Int} E$ in slightly toward L.

Suppose $C-J$ has two unbounded components U^+ and U^- where U^+ is above some horizontal plane. The selection of D is immediate if J lies in a plane so we suppose that the convex hull of J is a polyhedral 3-cell K.

Let E be the disk on $\operatorname{Bd} K$ bounded by J and missing U^-. Although E satisfies conditions 1, 3, 4, 5 of the definition of the special property, $\operatorname{Int} E$ may not lie in $\operatorname{Int} C$. Also $L\cdot E$ may be on an edge of every triangulation of E. This latter difficulty may be taken care of by adjusting E slightly near $L\cdot E$. The former exception may be removed by shoving certain points of $\operatorname{Int} E$ near C slightly toward L. Care is taken to see that no "corner" of the resulting disk D lies on any P_i.

THEOREM 8.3. *Suppose $J_1, J_2, \ldots, J_m$ are m mutually exclusive polygonal simple closed curves on a vertical triangular cylinder C, L is a vertical line in* $\operatorname{Int} C$, *and $P_1, P_2, \ldots, P_n$ are n planes each containing L but none containing a point where any J is broken. Then there are m mutually exclusive polyhedral disks $D_1, D_2, \ldots, D_m$ such that each D_i has the special property with respect to J_i, C, L and D_i has a triangulation with no vertex on $P_1+P_2+\ldots+P_n$ and no edge intersecting L.*

Proof. Suppose $J_1, J_2, \ldots, J_m$ are ordered so that if $C-J_i$ has two unbounded components U^+, U^- with U^- the lower one, then $i \leqslant j$ if J_j either bounds a disk in U^- or fails to intersect U^-. The disks $D_1, D_2, \ldots, D_m$ may be obtained by an iteration of the process described in the proof of Theorem 8.2 where we define D_1, then $D_2, \ldots$, and finally D_n.

THEOREM 8.4. *Suppose S; $S_1, S_2, \ldots$; F; $C_1, C_2, \ldots, C_n$; and the $K(r,j)$'s are as in Theorem 6.3 and 7.6 and ε, k are as in Theorem 7.6.*

Suppose U_i $(i=1,2,\ldots)$ is the large component of $S-\sum C_r$ and S_i' is a 2-sphere in general position with respect to F formed by replacing each component D of S_i-U_i by a disk E which lies except for its boundary in some $\operatorname{Int} C_r$ *and has the special property with respect to* $\operatorname{Bd} D$, C_r, L_r *where L_r is the center of C_r.*

For each point p of $S_i'\cdot L_r$, let $\pi_j(p_i)$ be the corresponding point of $S_j'\cdot L_r$ as determined by the correspondence between $K(r,i)$ and $K(r,j)$.

There is an integer k' such that if $i, j > k'$, V is a section of F, and $p_i q_i$ is an arc of $V \cdot S_i'$ between two corner points of F, then there is an arc in $V \cdot S_j'$ between $\pi_j(p_i)$ and $\pi_j(q_i)$.

Proof. Suppose $p \in S_1' \cdot L_r$ such that $\pi_i(p) = p_i$. Since for each integer j there is an arc in $V \cdot S_j'$ from $\pi_j(p)$ to a corner point of F, there is a point q of $S_1' \cdot L_r$ such that for infinitely many j's, there is an arc in $V \cdot S_j'$ from $\pi_j(p)$ to $\pi_j(q)$. The truth of Theorem 8.4 will follow if we show that if $k < i < s < j$ and arcs in $V \cdot S_i'$ and $V \cdot S_j'$ from $\pi_i(p)$ to $\pi_i(q)$ and $\pi_j(p)$ to $\pi_j(q)$ imply that there is an arc in $V \cdot S_s'$ from $\pi_s(p)$ to $\pi_s(q)$.

We find from Theorem 8.1 that whether or not there is an arc in $V \cdot S_s'$ from $\pi_s(p)$ to $\pi_s(q)$ is not determined by what E's we use to replace the D's as long as they have the special property. Hence we suppose that the E's do not intersect. Theorem 8.3 shows that we can pick these E's so they do not intersect and the techniques used in Theorems 7.3 and 7.5 show that there is an isotopy on E^3 pulling $S_i + S_s + S_j$ onto $S_i' + S_s' + S_j'$. Here we do not need the full strength of Theorems 7.3 and 7.5 since, although we do not want to move points of $U_i + U_s + U_j$, we are not interested in preventing other points from moving far. It follows from Theorem 7.7 that $\pi_s(p)$ is between $\pi_i(p)$ and $\pi_j(p)$ while $\pi_s(q)$ is between $\pi_i(q)$ and $\pi_j(q)$.

The arc in $V \cdot S_s'$ from $\pi_s(p)$ is trapped on V between the arcs in $V \cdot S_i'$ from $\pi_i(p)$ to $\pi_i(q)$ and the arc in $V \cdot S_j'$ from $\pi_j(p)$ to $\pi_j(q)$ so it can lead only to $\pi_s(q)$.

9. Finite graphs on a 2-sphere. Recall that $T(1)$ is the 2-sphere with center at the origin and radius 1. We prove some theorems about stable graphs on $T(1)$ and extend these to theorems about graphs on arbitrary 2-spheres.

THEOREM 9.1. *Suppose G' is a finite graph on $T(1)$ such that each component of $T(1) - G'$ is of diameter less than $\varepsilon < \frac{1}{3}$. Then G' contains a finite graph G such that each component of $T(1) - G$ is an open 2-cell of diameter less than 3ε and no two of the closures of these open 2-cells meet in a disconnected set.*

Proof. An advantage of working on $T(1)$ rather than on an arbitrary 2-sphere is that the diameter of a small set is equal to the diameter of its boundary.

For each component U of $S - G'$, let $J(U)$ denote the simple closed curve in $\overline{U}$ bounding the large component of $S - \overline{U}$ and $D(U)$ denote the disk in S containing U and bounded by $J(U)$. The diameter of $D(U)$ is the same as the diameter of U.

We note that if two $D(U)$'s have an interior point in common, one contains the other. Hence there is a collection of $D(U)$'s covering S such that the interiors of these $D(U)$'s are mutually exclusive. If no two of these $D(U)$'s intersected in a disconnected set, we could use the sum of their boundaries for G.

We wish to adjust the above mentioned $D(U)$'s so that the intersection of two of the adjusted disks that have a point in common meet in a connected set. We are willing for the diameters of the adjusted disks to be more than ε.

The finite graph shown in Figure 7 is not stable because a homeomorphism of the graph onto itself interchanging arcs axb and ayb or czd and cwd could not be extended to the plane. However, if D_1 and D_5 are combined (by removing $\mathrm{Int}\, axb$) and if D_6 is added to D_3 (by removing $\mathrm{Int}\, czd$) the resulting graph is stable. We prove the theorem by considering a scheme to combine such disks.

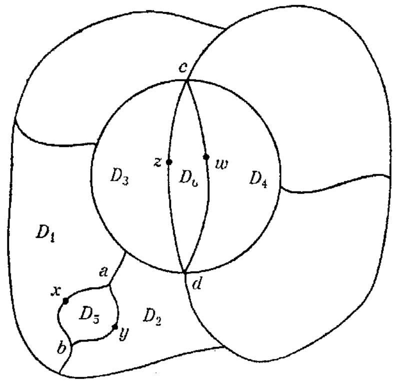

Fig. 7

We finish the proof of the theorem by combining certain of the $D(U)$'s. Suppose that this combining has proceeded until we have disks $D_1, D_2, \ldots, D_n$, $E_1, E_2, \ldots, E_m$ covering $T(1)$ such that the interiors of these disks are mutually exclusive, each is the sum of $D(U)$'s described earlier, diameter $D_i < \varepsilon$, diameter $E_i < 3\varepsilon$, and no one of the E's meets one of the other disks in a disconnected set. If we can eliminate the D's and make all the disks E's, the sum of the boundaries of the E's will be the required finite graph G.

If J is a simple closed curve on $T(1)$ of diameter less than 1, we use $D(J)$ to denote the smaller disk in $T(1)$ bounded by J. Let E_{m+1} be the sum of all disks R of $D_1, D_2, \ldots, D_n$, $E_1, \ldots, E_m$ such that there is a simple closed curve J in $T(1)$ such that $R \subset D(J)$ and

$J\cdot(\mathrm{Bd}D_1+\mathrm{Bd}D_2+\ldots+\mathrm{Bd}E_m)\subset\mathrm{Bd}D_n$. In general, J need not lie in $\mathrm{Bd}D_1+\mathrm{Bd}D_2+\ldots+\mathrm{Bd}E_m$. For example, if $D_n=D_4$ of Figure 4, the simple closed curve J that shows that D_6 (which has been renamed an E) is to be combined with D_4 lies in $D_4+\mathrm{Ind}D_3$. Then E_{m+1} is a disk of diameter less than 3ε, and it has swallowed up some of the D's and perhaps some of the E's, but it does not intersect any of the remaining disks in a disconnected set.

A continuation of this process changes all the D's into E's.

The following extension of Theorem 9.1 follows from use of the fact that each 2-sphere is the image of $T(1)$ unter a uniformly continuous homeomorphism.

THEOREM 9.2. *If S is a 2-sphere and $\varepsilon>0$, there is a positive number δ such that if G' is a finite graph on S such that each component of $S-G'$ is of diameter less than δ, then G' has a finite subgraph G such that each component of $S-G$ is an open 2-cell of diameter less than ε and no two of the closures of these open 2-cells meet in a disconnected set.*

THEOREM 9.3. *Suppose $D_1, D_2, \ldots, D_n$ are disks whose sum is a 2-sphere S and the intersection of two of these disks with a point in common is either a point or an arc. Then $\sum\mathrm{Bd}D_i$ is connected and no $\mathrm{Bd}D_j$ separates it.*

Proof. If $\sum\mathrm{Bd}D_i$ were not connected, a simple closed curve in $S-\sum\mathrm{Bd}D_i$ would separate $\sum\mathrm{Bd}D_i$ in S. But this simple closed curve would not lie in any D_j.

If $\mathrm{Bd}D_j$ separates $\sum\mathrm{Bd}D_i$, there is a simple closed curve J in $S-(\sum\mathrm{Bd}D_i-\mathrm{Bd}D_j)$ such that $(\sum\mathrm{Bd}D_i)-\mathrm{Bd}D_j$ contains points p, q lying in different components of $S-J$. It follows from the unicoherence of $S-\mathrm{Int}D_j$ that there is an arc A in J irreducible with respect to separating p from q in $S-\mathrm{Int}D_j$. But A would intersect D_j in only two points and would lie in a D_i such that $D_i\cdot D_j$ is not connected.

THEOREM 9.4. *Suppose G is a finite graph on $T(1)$ such that each component of $T(1)-G$ is an open 2-cell of diameter less than $\varepsilon<\frac{1}{3}$ and no two of the closures of these open 2-cells meet in a disconnected set. If h is a homeomorphism of G onto a finite graph in $T(1)$ such that h moves no point by more than ε, h can be extended to a homeomorphism of $T(1)$ onto itself that does not move any point by as much as 3ε.*

Proof. Let $D_1, D_2, \ldots, D_n$ be the disks which are the closures of the components of $T(1)-G$. Then for each i, h can be extended from $\mathrm{Bd}D_i$ to map D_i onto the smaller disk in $T(1)$ bounded by $h(\mathrm{Bd}D_i)$. This extension moves no point by as much as 3ε. We now show that the combined map of h on all the D_i's is a homeomorphism of $T(1)$ onto itself.

If $h(\mathrm{Int}D_i)$ intersects $h(\mathrm{Int}D_j)$, $i\neq j$, then one of $h(D_i)$, $h(D_j)$ contains the other. Assume $h(D_j)\subset h(D_i)$. Since $h(G-\mathrm{Bd}D_i)$ is connected, $h(D_i)$ contains $h(G)$. This contradicts the facts that diameter $h(D_i)<3\varepsilon$, diameter $G>5\varepsilon$, diameter $h(G)>3\varepsilon$.

The preceding paragraph showes that h sends no two points into the same point. Since $T(1)$ is compact and h is continuous, h is a homeomorphism. Since no proper subset of $T(1)$ is homeomorphic with $T(1)$, h takes $T(1)$ onto itself.

THEOREM 9.5. *For each 2-sphere S and each positive number ε, there is a positive number δ such that if G is a finite graph on S such that each component of $S-G$ is an open 2-cell of diameter less than δ but the closures of no two of the open 2-cells meet in a disconnected set and h is a homeomorphism of G into S that moves no point by more than ε, then h can be extended to a homeomorphism of S onto itself that does not move any point by as much as 3ε.*

Theorem 9.5 follows from Theorem 9.4 and the fact that S is the image of $T(1)$ under a uniformly continuous homeomorphism.

The following theorem shows that we have been considering stable graphs in Theorem 9.1 to 9.5.

THEOREM 9.6. *A finite graph G on a 2-sphere S is stable if there is a finite collection of disks $D_1, D_2, \ldots, D_n$ covering S such that $\mathrm{Int}D_i\cdot D_j=0$ if $i\neq j$, $\sum\mathrm{Bd}D_i=G$, and if two of the D's have a point in common, their intersection is connected.*

Proof. We denote that $\mathrm{Bd}D_i$ does not separate G or else some D_j intersects D_i in a disconnected set. A homeomorphism h of G into a 2-sphere S' may be extended to a homeomorphism of S onto S' by taking D_i onto the disk on S' bounded by $h(\mathrm{Bd}D_i)$ and containing no point of $h(G-\mathrm{Bd}D_i)$.

10. Freeing surfaces of intervals. In the proof of Theorem 2.1 we supposed that there is no loss of generality in supposing that a 2-sphere S contains no vertical interval. We justify this supposition with Theorem 10.1.

A different proof of Theorem 10.1 was originally employed which made use of tame Cantor sets. It was decided to include this different proof in another paper devoted to properties of tame Cantor sets and substitute here a shorter proof suggested by M. K. Fort, Jr. and modeled after the proof of Theorem 4 in [10].

THEOREM 10.1. *If X is a closed 2-dimensional set in E^3 and ε is a positive number, there is a homeomorphism h of E^3 onto itself such that h moves no point by more than ε and $h(X)$ contains no straight line interval.*

Proof. Let H be the set of all homeomorphisms h_a of E^3 onto itself and such that each of $\varrho(h_a, I)$, $\varrho(h_a^{-1}, I)$ is finite. If H is metrized with the metric

$$D(h_1, h_2) = \varrho(h_1, h_2) + \varrho(h_1^{-1}, h_2^{-1}),$$

the resulting space is a complete metric space.

Let H_n be the set of all elements h_n such that $h_n(X)$ contains a straight line interval of length no less than $1/n$ and at a distance from the origin of no more than n. Then H_n is closed and $H_1+H_2+\ldots$ is an F_σ set.

We now show that no H_i contains an open subset of H by showing that for each element h_i of H_i, there is an element h_0 of $H-H_i$ very close to h_i. Let T be a triangulation of E^3 of very small mesh and such that the 2-skeleton of T contains no straight line interval of length more than $1/2i$ and no center of a 3-simplex of T intersects $h_i(X)$. Then h_0 can be taken as fh_i where f is a homeomorphism fixed on the 2-skeleton of T which moves parts of the 3-simplexes of T in a straight line directly away from the centers of these 3-simplexes. In fact, f can be taken so that $fh_i(X)$ is so close to the 2-skeleton of T as to contain no straight line interval of length $1/i$.

Since no H_i contains an open subset of H, it follows from the Baire category theorem that H contains an element h near I that does not belong to any H_i.

11. Extension of preceding results to surfaces other than 2-spheres. Theorem 2.2 gave a condition under which a 2-sphere in E^3 is tame. This result may be extended by the methods we have used to show the following results.

THEOREM 11.1. *A surface S in E^3 is tame if for each positive number ε there are surfaces S', S'' on different sides of S such that*

$$H(S, S') \leqslant \varepsilon, \quad H(S, S'') \leqslant \varepsilon.$$

THEOREM 11.2. *A surface S in E^3 is locally tame at a point p of S if there is a disk D with $p \in \operatorname{Int} D \subset S$ such that for each positive number ε, there are disks D', D'' on opposite sides of S such that*

$$H(D, D') \leqslant \varepsilon, \quad H(D, D'') \leqslant \varepsilon.$$

The original intention was to prove Theorems 11.1 and 11.2 in the present paper but the paper seems long enough already and their proofs will follow briefly from extensions of Theorem 2.2 to be given in another paper. This will be accomplished as follows.

It will be shown in [8] that a 2-sphere in E^3 is tame if its complement is uniformly locally simply connected. Theorem 11.2 will be

established by showing that if p is a point of a surface S satisfying the hypothesis of Theorem 11.2, then there is a disk E and a 2-sphere K such that $p \in \operatorname{Int} E \subset K$, $E \subset S$, and E^3-K is uniformly locally simply connected. Theorem 11.1 will follow from Theorem 11.2 and the fact that locally tame surfaces are tame.

The above theorems are true in 3-manifolds as well as in E^3.

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